INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
Advanced Graph Theory: End-Semester Examination 2019-20

Date of Examination: 25 November 2019
Duration: 3 Hours
Full Marks: 60
Subject No: CS60047
Subject: Advanced Graph Theory
Department/Center/School: COMPUTER SCIENCE AND ENGINEERING

Special instruction (if any): If you want to use any result which is not proved in class, you should prove it.

Answer any four questions.

1. (i) Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a simple undirected graph, whose vertices are labelled as $1, 2, 3, \ldots, 10$, and the degree of each vertex in $\mathcal{V}$ is even. How many such distinct (labelled) graphs with 10 vertices can be constructed?

   (ii) Suppose 100 students including Ishan are attending a gathering and each student has at least one friend present in the same gathering. When Ishan asks each of the other 99 students how many friends of theirs are present, each gives a different answer. How many of Ishans friends are present in the gathering?

   (iii) Let $\mathcal{G}$ be a simple undirected graph such that the degree of each vertex is greater than or equal to 3. Show that $\mathcal{G}$ has a cycle of even length.

[$8 + 4 + 3$ Marks]

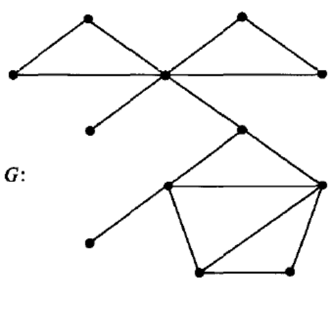
2. (i) If B_1, B_2 are two distinct blocks in an undirected graph $\mathcal{G}$, then show that

$$|\mathcal{V}(\mathcal{B}_1) \cap \mathcal{V}(\mathcal{B}_2)| = 1, \text{where } \mathcal{V}(\mathcal{B}_i) \text{ denotes the set of vertices in block } \mathcal{B}_i, i \in \{1, 2\}$$

   (ii) Show that if $\mathcal{G}$ is a simple undirected graph, its block-cutpoint graph $BC(\mathcal{G})$ is a tree.

   (iii) Construct the block-cutpoint graph for the graph shown below.

G:



[$5 + 5 + 5$ Marks]

3. (i) A graph is called a maximal planar graph if adding any new edge would make the graph non-planar. Show that a maximal simple planar graph has $3n - 6$ edges.

   (ii) For a planar graph $\mathcal{G}$, prove that $\mathcal{G}$ is bipartite if and only if its dual graph $\mathcal{G}^*$ is Eulerian.

[$8 + 7$ **Marks]**

4. (i) In a bipartite graph $\mathcal{G} = (\mathcal{A} \cup \mathcal{B}, \mathcal{E})$, a subset $\mathcal{F} \subseteq \mathcal{E}$ is called perfect 2-matching if every vertex in $\mathcal{A}$ has exactly 2 edges in $\mathcal{F}$ incident on it and every vertex in $\mathcal{B}$ has at most one edge in $\mathcal{F}$ incident on it. Prove, using Hall's Theorem, that the following is a necessary and sufficient condition for $\mathcal{G}$ to have a perfect 2-matching.

$$\forall S \subseteq \mathcal{A}, |\mathcal{N}(S)| \geqslant 2|S|, \text{ where } \mathcal{N}(S) = \{b \in \mathcal{B} : \exists a \in S, \{a, b\} \in \mathcal{E}\}$$

   (ii) Prove that a tree $\mathcal{T}$ has a perfect matching if and only if, for every vertex v of $\mathcal{T}$, the number of add components in $\mathcal{T} \setminus \{v\}$ is exactly 1.

[$8 + 7$ **Marks]**

5. (i) Prove or disprove: In a graph $\mathcal{G}$, if the chromatic number of every 2-connected subgraph of $\mathcal{G}$ is at most k, then the chromatic number of $\mathcal{G}$ is also at most k.

   (ii) Prove that, among all multipartite graphs on n vertices which avoids a clique $\mathcal{K}_\ell$ on ℓ vertices, Turán graph $\mathcal{T}^{\ell-1}(n)$ has a maximum number of edges.

[$8 + 7$ **Marks]**

Best of luck