1

1. (a) Replace

$$\sum_{(u,v)\in E} f_{u,v} - \sum_{(v,w)\in E} f_{v,w} = 0$$

By the pair of constraints

$$\sum_{(u,v)\in E} f_{u,v} - \sum_{(v,w)\in E} f_{v,w} \leq 0$$

$$-\sum_{(u,v)\in E} f_{u,v} + \sum_{(v,w)\in E} f_{v,w} \leq 0$$

(b) After the replacement as in part (a), the LP becomes:

maximize $\sum_{(s,v)\in E} f_{s,v}$ <u>Dual variabl</u>

subject to,

$\forall e \in E, \quad f_e \leq c_e$ μ_e

$\forall v \notin \{s,t\}, \quad \sum_{(u,v)\in E} f_{u,v} - \sum_{(v,w)\in E} f_{v,w} \leq 0$ b_v

$$-\sum_{(v,w)\in E} f_{u,v} + \sum_{(u,v)\in E} f_{v,w} \leq 0 \qquad b_v'$$

Thus our dual variables are:

- M_e for each edge e
- δ_v, δ'_v for each vertex $v \neq s, t$.

The dual is a minimization problem.

The objective is $\sum_{e \in E} c_e \cdot M_e + \sum_{v \neq s,t} 0 \cdot \delta_v + \sum_{v \neq s,t} 0 \cdot \delta'_v$

$$= \sum_{e \in E} c_e \cdot M_e .$$

The dual has one constraint for every primal variable $f_{u,v}$. The constraint depends on whether $u = s$ or t, $v = s$ or t.

- <u>If $(s,t) \in E$,</u> then there is a primal variable $f_{s,t}$. Since there is no flow conservation constraint for s or t, $f_{s,t}$ appears only in the capacity constraint of the edge (s,t): $f_{(s,t)} \leq c_{(s,t)}$.

  Thus the dual constraint corresponding to it will be

$$M_{(s,t)} \geq 1$$

(Note that the coefficient of $f_{(s,t)}$ in the primal objective is 1).

- $\underline{u = s,\ v \neq t}$. In this case $f_{u,v}$ appears

in,

i) The flow-conservation constraints of vertex v with coefficients, respectively $+1$ and -1 (two constraints after the replacement in part (a))

ii) The capacity constraint of edge (u,v) with coefficient $+1$.

iii) The primal objective function with ~~capacity~~ coefficient $+1$.

Thus the dual constraint corresponding to it is:

$$\delta_v - \delta'_v + \mu_{u,v} \geq 1.$$

- $\underline{u \neq s,\ v = t}$. In this case $f_{u,v}$ appears

in,

i) The flow-conservation constraints of vertex u with coefficients respectively -1 and $+1$.

ii) The capacity constraint of edge (u,v) with coefficient $+1$.

iii) Does not appear in the primal objective function (i.e. coefficient $=0$).

Thus the dual constraint corresponding to it is:

$$-\delta_u + \delta'_u + \mu_{u,v} \geq 0$$

- $u \neq s,\ v \neq t$. In this case $f_{u,v}$ appears in,

i) The flow-conservation constraints of u with coefficients -1 and $+1$ respectively.

ii) The flow-conservation constraints of v with coefficients $+1$ and -1 respectively.

iii) The capacity constraint of edge (u,v) with coefficient $+1$.

It does not appear in the primal objective function (coefficient is 0). Thus the dual constraint corresponding to it is:

$$-\delta_u + \delta'_u + \delta_v - \delta'_v + \mu_{u,v} \geq 0.$$

Thus, following is the dual L' of L:

minimize $\sum_{e \in E} c_e . \mu_e$

subject to.

$\mu_{(s,t)} \geq 1$ (if $(s,t) \in E$),

$\delta_v - \delta'_v + \mu_{s,v} \geq 1$ ($\forall (s,v) \in E$, $v \neq t$)

$-\delta_u + \delta'_u + \mu_{u,t} \geq 0$ ($\forall (u,t) \in E$, $u \neq s$)

$-\delta_u + \delta'_u + \delta_v - \delta'_v + \mu_{u,v} \geq 0$

($\forall (u,v) \in E$, $u \neq s$, $v \neq t$).

$\mu_e, \delta_v \geq 0$ $\forall e \in E$, $v \in V \setminus \{s,t\}$ (Non-negativity constraint).

(c) Define $\alpha_v := \delta_v - \delta'_v$ for each $v \neq s,t$.

Then the dual becomes

minimize $\sum_{e \in E} c_e . \mu_e$

subject to:

$$\mu_{s,t} \geq 1 \quad \text{if } (s,t) \in E$$

$$\alpha_v + \mu_{s,v} \geq 1 \quad (\forall (s,v) \in E, v \neq t)$$
$$-\alpha_u + \mu_{u,t} \geq 0 \quad (\forall (u,t) \in E, u \neq s)$$
$$-\alpha_u + \alpha_v + \mu_{u,v} \geq 0$$
$$(\forall (u,v) \in E, u \neq s, v \neq t)$$

$\mu_e \geq 0 \ \forall e \in E$ (Non-negativity constraint for μ_e).

Note, that α_v, being the difference between two non-negative quantities, can potentially become negative. So we allow α_v to take any real value.

Now, every feasible assignment of L' induces a feasible assignment of this LP (just assign α_v the difference between the assignments of μ_v and μ'_v). Also, in every feasible assignment of this LP, α_v is assigned a real number r. But any real number can be expressed as difference between two non-negative -

numbers. For example,

if $r \geq 0$,

assign $\mu_e := r$ and $\mu'_e := 0$

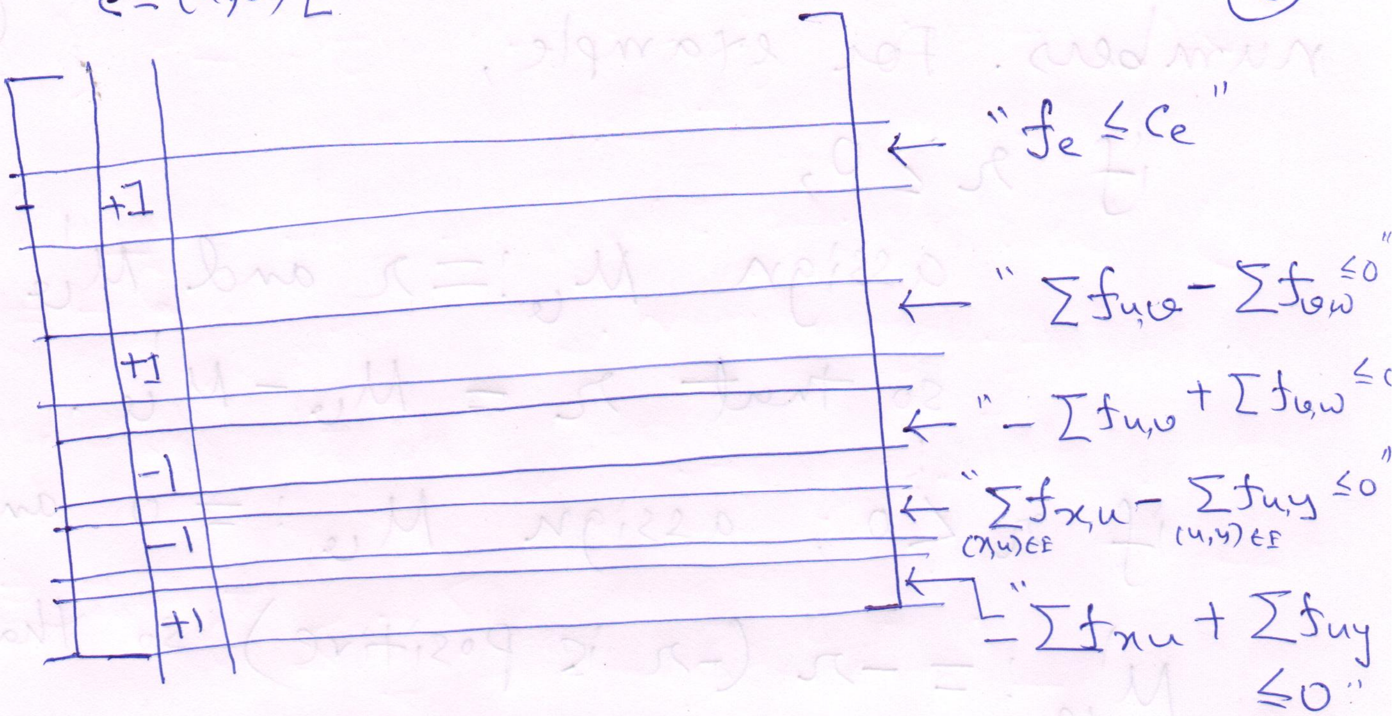
so that $r = \mu_e - \mu'_e$.

if $r < 0$, assign $\mu_e := 0$ and $\mu'_e := -r$ ($-r$ is positive) so that $r = \mu_e - \mu'_e$. The objective function, being dependent only on μ_e's, is unaffected.

(d). Consider the coefficient matrix for L (after doing the replacements of part (a)).

The columns correspond to edges. ~~For each edge, there~~ The rows correspond to constraints.

$e = (u,v)$ $[u \neq s, v \neq t]$



The column corresponding to an edge (u,v) $u \neq s, v \neq t$, has 5 non-zero entries in the rows that correspond to the constraints in which f_{uv} appears. Refer to the figure above. Similarly, check the entries of the columns corresponding to edges (s,v), (u,t) and (s,t) (if $(s,t) \in E$). Assume that this matrix is not totally unimodular (towards a contradiction).

Let A be a smallest square submatrix such that $\det(A) \notin \{+1,-1,0\}$.

- ~~Case 1~~: A has a ~~row~~ column of all 0's. Then $det(A) = 0$ which is a contradiction.

- Case 2: A has a row or a column having exactly one +1 or -1. Then we can expand along that row/column and express $det(A)$ as $\pm 1 \cdot det(A')$ for a submatrix A' of A. Thus $det(A')$ is $\pm det(A)$ and hence $det(A') \notin \{+1, -1, 0\}$. But we assumed that A is the smallest such matrix. This is a contradiction.

- Case 3: A has two rows that correspond to flow-conservation constraints of the same vertex. Recall that after the replacement in part (a), each flow-conservation constraint (equality) is replaced by two constraints. Note that these two rows are

Negations of each other. Thus ~~A~~ det $(A') = 0$.

Case 4. The only remaining possibility is that A' consists only of rows corresponding to the flow conservation constraints. (rows corresponding to capacity constraints have exactly one 1). ~~[illegible]~~ Multiply the rows in A' that correspond to the second kind of flow conservation constraints (i.e., the second of the two inequalities in part (a)) by -1. Verify that now each column of A' has one $+1$ and one -1 (assuming that A' does not have a column with a single non-zero entry). Thus, the sum of all the rows of A' is ~~a~~ the all 0 row. Hence, the rows of A' are not linearly independent, and $\det(A') = 0$.

B e) Let $(A, V \setminus A)$ be a s-t cut.

$$\mathrm{cap}(A, V\setminus A) = \sum_{\substack{(u,v)\in E \\ u\in A,\, v\notin A}} c_{u,v}.$$

Assign dual variables as follows:

$$M_{u,v} = \begin{cases} 1 & \text{if } u\in A,\ v\notin A,\ (u,v)\in E \\ 0 & \text{otherwise.} \end{cases}$$

$$\alpha_v = \begin{cases} 1 & \text{if } v\in A, \\ 0 & \text{if } v\notin A. \end{cases}$$

Verify that this satisfies all the dual constraints (do it yourself). The value of the dual objective under this assignment is $\sum_{(u,v)\in E} c_{u,v} \cdot M_{v,u}$

$$= \sum_{\substack{u\in A,\, v\notin A, \\ (u,v)\in E}} c_{u,v} = \mathrm{cap}(A, V\setminus A)$$

Thus, $\mathrm{Val}(L') \leq$ size of a mincut ——— (1).

Now, consider an integral optimal point $(M_e^*, \delta_v^*, \delta_v'^*)$ of L'. Let $\alpha_v^* := \delta_v^* - \delta_v'^*$.

Let $E' = \{e \in E : M_e^* > 0\}$. Remove edges in E' from G. Call the resultant graph G'. We will show that ~~s-t~~ there is no $s-t$ path in G'. Towards a contradiction, assume that $s - u_1 - u_2 - \cdots - u_k - t$ is a path in G'. Dual constraints give us:

$$M_{s,u_1} \geq 1 - \alpha_{u_1},$$

$$M_{u_i,u_{i+1}} \geq \alpha_{u_i} - \alpha_{u_{i+1}},$$

$$M_{u_k,t} \geq \alpha_{u_k}.$$ Thus we have

$$M_{s,u_1} + M_{u_1,u_2} + \cdots + M_{u_{k-1},u_k} + M_{u_k,t}$$
$$\geq (1 - \alpha_{u_1}) + (\alpha_{u_1} - \alpha_{u_2}) + \cdots + (\alpha_{u_{k-1}} - \alpha_{u_k}) + \alpha_{u_k}$$
$$= 1$$

But, since the path $s, u_1, \cdots, u_n, t$ is in G', we have that

$$M_{s,u_1} = 0,\ M_{u_1,u_2} = 0,\ \cdots,\ M_{u_n,t} = 0.$$

This is a contradiction.

Define $A := \{v : v \text{ is reachable from } s \text{ in } G'\}$.

Let $(u,v) \in E$ be such that $u \in A$, $v \notin A$. Then we claim that $M^*_{(u,v)} \neq 0$. Towards a contradiction assume that $M^*_{(u,v)} = 0$. Then (u,v) is present in G'. Since $u \in A$, u is reachable from s in G'. Since (u,v) is in G', v is also reachable from s in G'. So $v \in A$, which is a contradiction. Thus, $M^*_{(u,v)} \neq 0$. Since $M^*_{(u,v)}$ is ~~pos~~ non-negative and integral, this implies that $M^*_{(u,v)} \geq 1$.

Thus, for each edge (u,v) such that $u \in A, v \notin A$,

$$1 \le \mu^*_{(u,v)} \text{ ——— } (2)$$

Thus,

$$Cap(A, V \setminus A) = \sum_{\substack{u \in A, v \notin A, \\ (u,v) \in E}} 1 \cdot C_{u,v}$$

$$\le \sum_{\substack{u \in A, v \notin A, \\ (u,v) \in E}} \mu^*_{u,v} \cdot C_{uv} \quad (\text{from } (2))$$

$$\le \sum_{e \in E} \mu^*_e \cdot C_e \,. \quad (\text{since, } \mu^*_e, C_e \ge 0 \ \forall e \in E)$$

$= Val(L')$.

$\exists$ a cut $(A, V \setminus A)$ such that $Cap(A, V \setminus A) \le Val(L')$ ——— (3)

Max-flow min-cut theorem follows from (2) and (3).

2)

▶ Let $(A, V \setminus A)$ be a max-cut in G. Set the variables as follows:

$$\gamma_{(u,v)} = \begin{cases} 1 & \text{if } u \in A, v \in V \setminus A \\ 0 & \text{otherwise} \end{cases}$$

$$M_v = \begin{cases} 1 & \text{if } v \in A \\ 0 & \text{otherwise.} \end{cases}$$

Verify that the ILP constraints are satisfied.

Value of the program $= \sum_{e \in E} \gamma_e$

$$= \sum_{\substack{(u,v) \in E \\ u \in A, v \notin A}} \gamma_{u,v} = \text{size of a max-cut.}$$

▶ Let (γ^*, M^*) be an optimal assignment for the program. Since it is a 0-1 integer program, each variable takes 0-1 values.

Define $A := \{ v : M_v^* = 1 \}$.

Let $(u,v) \in E$ be a cut-edge, i.e., ~~Then~~ $u \in A$, $v \notin A$.

~~Then~~ $\cancel{\gamma_e}$ $\therefore M_u^* = 1$, $M_v^* = 0$.

Thus the right hand sides of the constraint says that

$$\gamma^*_{(u,v)} \leq \begin{cases} M_u^* + M_v^* = 1 \\ 2 - (M_u^* + M_v^*) = 1. \end{cases}$$

Thus, $\gamma^*_{(u,v)} = 0$ or 1 satisfies both constraints. ~~But since, it is a maximization program,~~ If $\gamma^*_{(u,v)} = 0$, we can increase its value to 1, satisfy all the constraints and increase the value of the objective. But since (γ^*, M^*) is optimal, the objective is ~~ma~~ already maximized under the current assignment. We conclude that

If $(u,v) \in E$, $u \in A$, $v \notin A$ then $\underline{\gamma^*_{(u,v)} = 1}$

Thus, size of the cut

$$= \sum_{\substack{(u,v)\in E,\\ u\in A,\, v\notin A}} 1 \;\le\; \sum_{\substack{(u,v)\in E,\\ u\in A,\, v\notin A}} r^*_{(u,v)}$$

$$\le \sum_{e\in E} r^*_e \quad (\text{since, } r^*_e \ge 0 \;\forall e)$$

$=$ Value of the program.

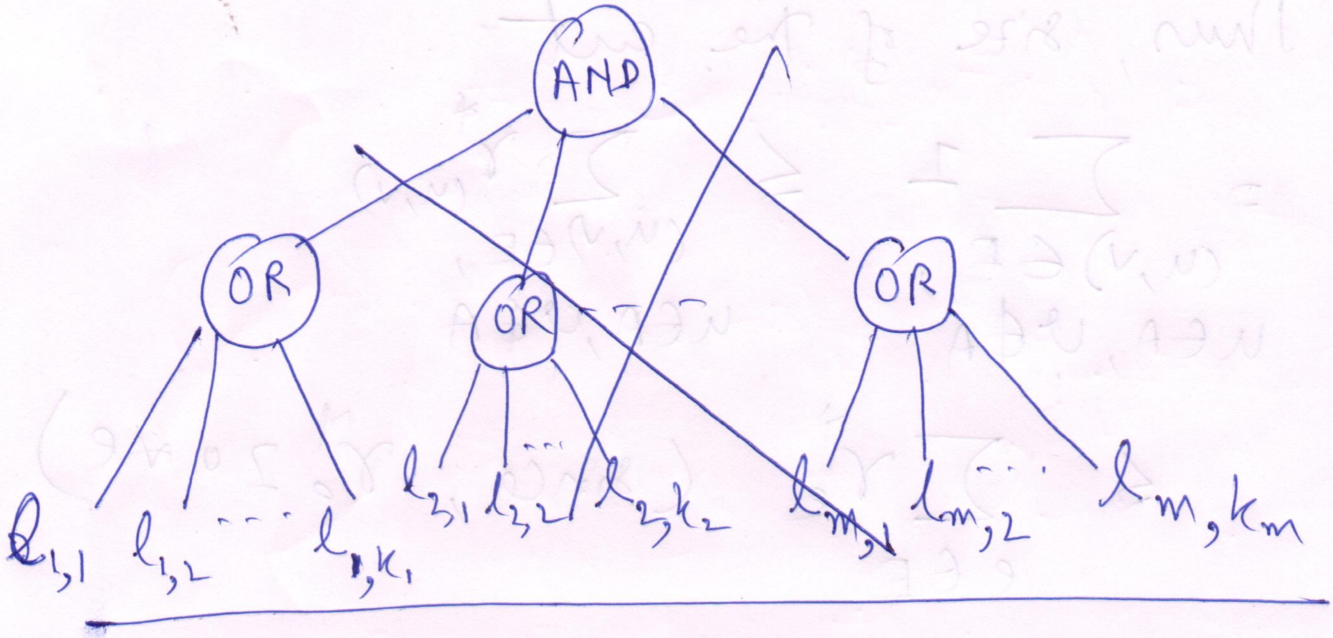
(9) A 3-SAT formula is AND of ORs and hence already a Boolean circuit.

$$\cancel{\phi = C_1 \wedge C_2 \wedge \dots \wedge C_m}$$

$$\cancel{C_i = l_{i,1} \vee l_{i,2} \vee \dots \vee l_{i,k_i}}.$$

~~OR of literals~~ $l_{i,j}$.

~~Thus, it is equivalent to the following Boolean circuit~~:

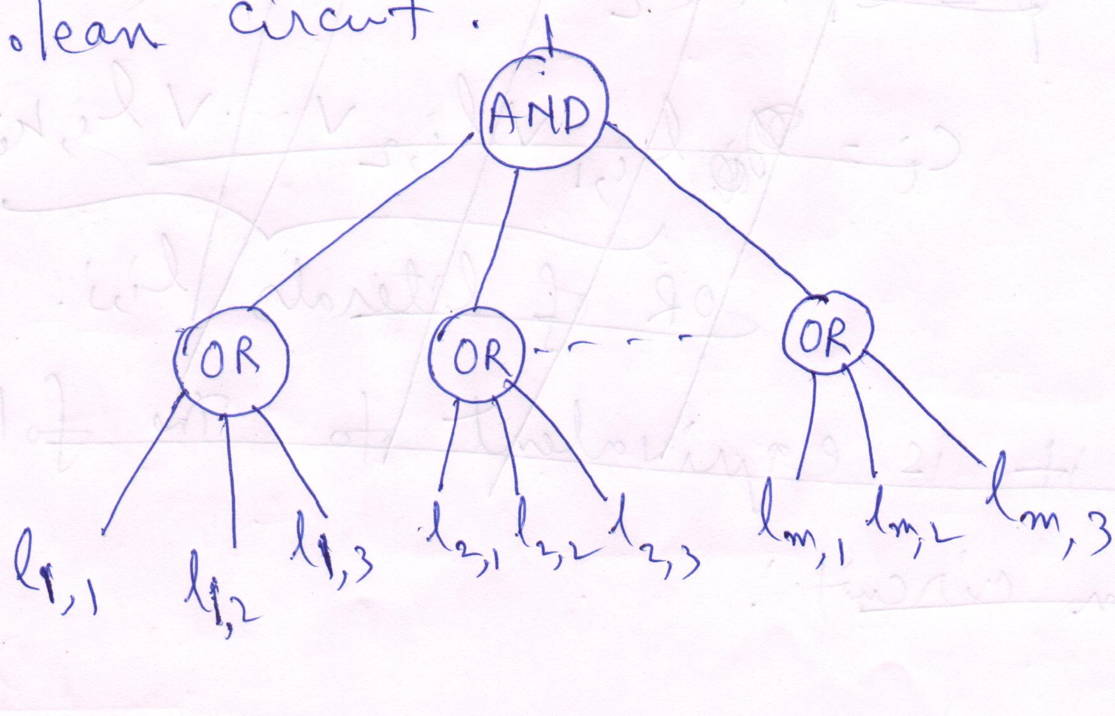


$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

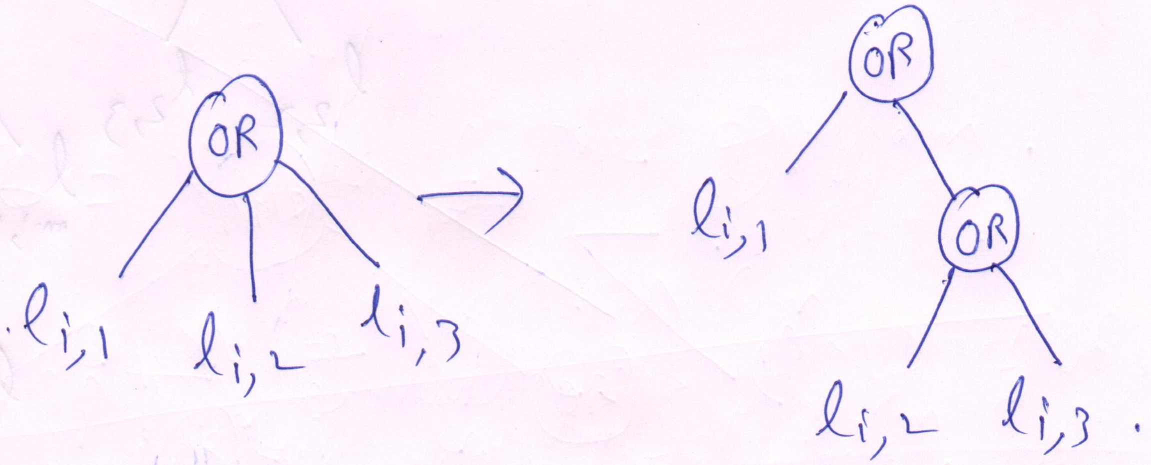
$$C_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}.$$

[$l_{i,1}$, $l_{i,2}$, $l_{i,3}$ are the literals in C_i].

Thus ϕ is equivalent to the following Boolean circuit.

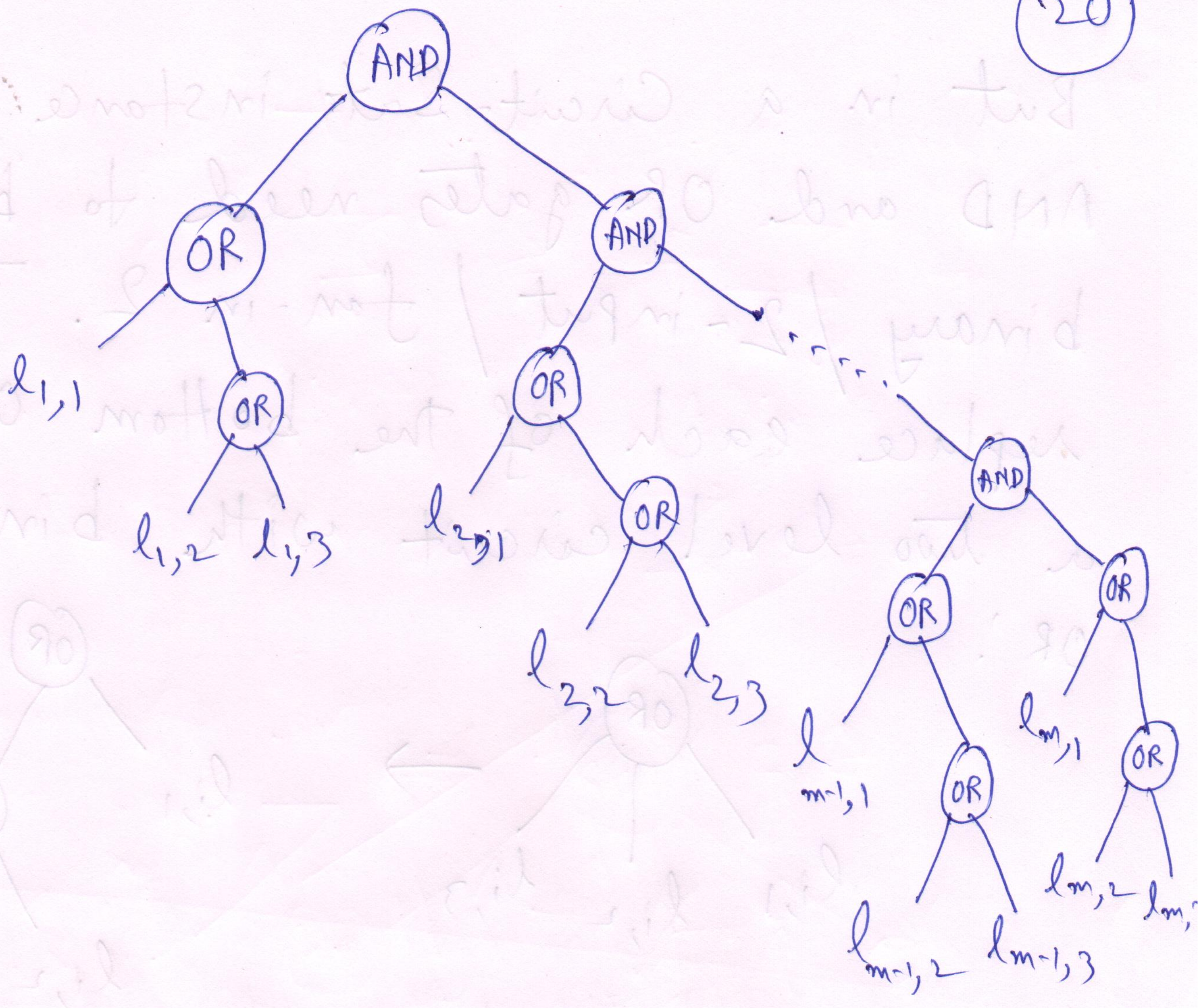


But in a Circuit-Sat instance, the AND and OR gates need to be binary / 2-input / fan-in 2. Thus replace each of the bottom ORs by a two level circuit with binary OR:



Finally, replace the top AND by ~~the~~ follo ~~Boolean~~ wing Boolean circuit.

PTO →


AND
OR
AND
$l_{1,1}$
OR
OR
AND
$l_{1,2}$
$l_{1,3}$
$l_{2,1}$
OR
OR
OR
$l_{2,2}$
$l_{2,3}$
$l_{m-1,1}$
OR
$l_{m,1}$
OR
$l_{m-1,2}$
$l_{m-1,3}$
$l_{m,2}$
$l_{m,3}$