Network Centrality
Part 2

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CS 60017
Autumn 2021

Recap: Betweenness centrality

$$C_B(i) = \sum_{j<k} g_{jk}(i)/g_{jk}$$

Where g_{jk} = number of geodesics (shortest paths) connecting *jk*
g_{jk} (i) = the number of these geodesics that actor *i* is on.

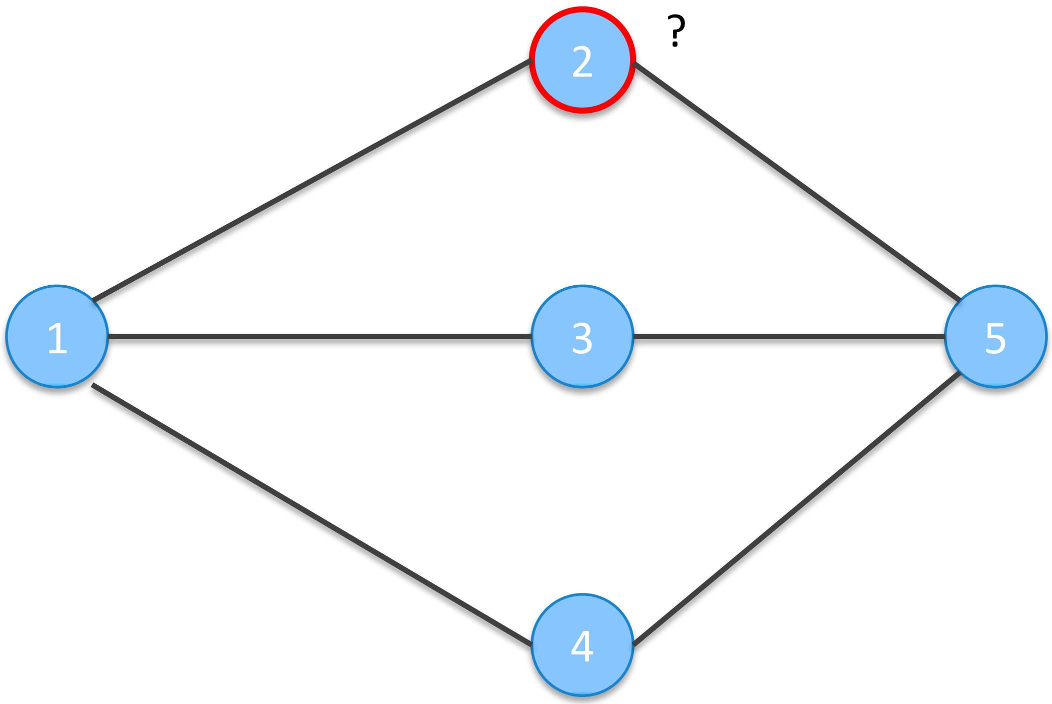
Can be normalized by:

$$C'_B(i) = C_B(i)/[(n-1)(n-2)/2]$$

number of pairs of vertices
excluding the vertex itself

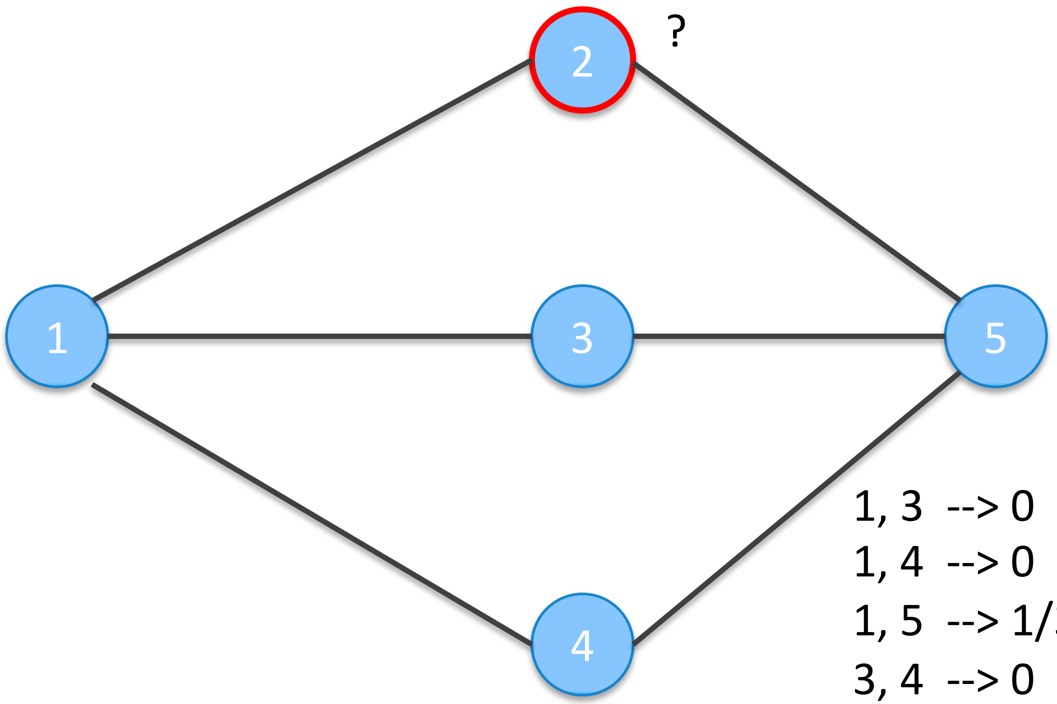
Calculate Betweenness centrality

- Non-normalized



Calculate Betweenness centrality

- Non-normalized



1, 3 --> 0
1, 4 --> 0
1, 5 --> 1/3
3, 4 --> 0
3, 5 --> 0
4, 5 --> 0

Total = 1/3
Normalized = 1/3 x 1/(4C2) = 1/18

CENTRALITY IN LARGE DIRECTED GRAPHS (WEB GRAPH)

Requirements for Web search

- Results of Web search need to consider
  - Relevance to query
  - <u>Importance / authoritativeness</u>
  - Location / time of query
  - Recency of page
  - … and many others

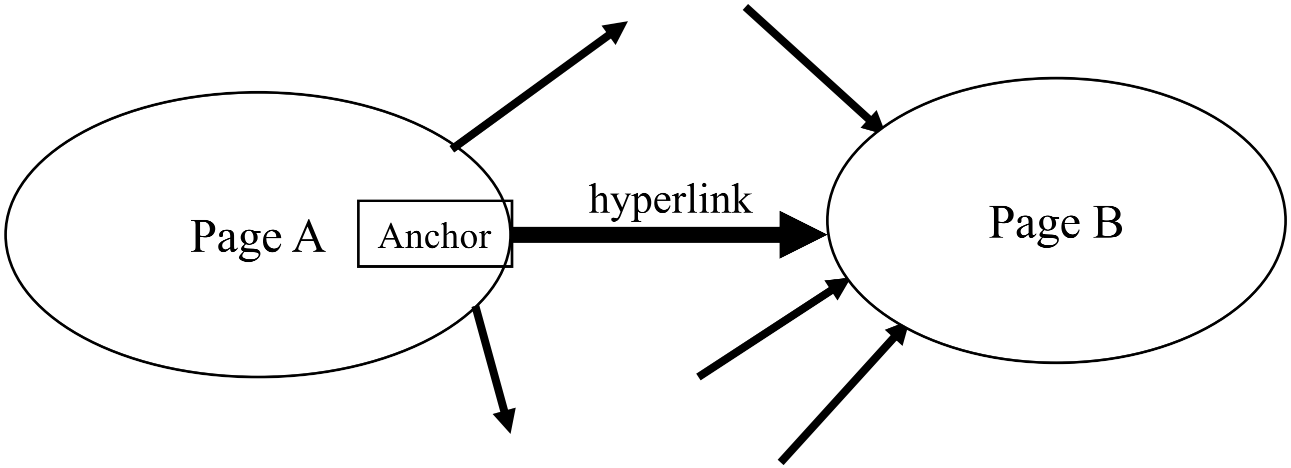
- Initial days of the Web: only relevance to query was used to rank webpages
  - Ranking algorithms easily spammed by manipulating the text on spam webpages

Need to consider authoritativeness

- Importance / authoritativeness – centrality on the Web graph (webpages are nodes, hyperlinks are directed edges)

- An edge from node p to node q denotes endorsement
  - Node p endorses/recommends/confirms the authority/centrality/importance of node q
  - May not be always true (e.g., all pages on a website linking to the Copyright page) but mostly true
  - Use the graph of recommendations to assign an authority value to every node

The Web as a Directed Graph



Hypothesis 1: A hyperlink between pages denotes a conferral of authority (quality signal)

Hypothesis 2: The text in the anchor of the hyperlink on page A describes the target page B

How to compute node centrality on Web?

- First attempt: indegree of webpages used to rank pages according to importance
    - Easily gamed by spammers creating their own webpages

- Subsequent better algorithms: HITS and PageRank

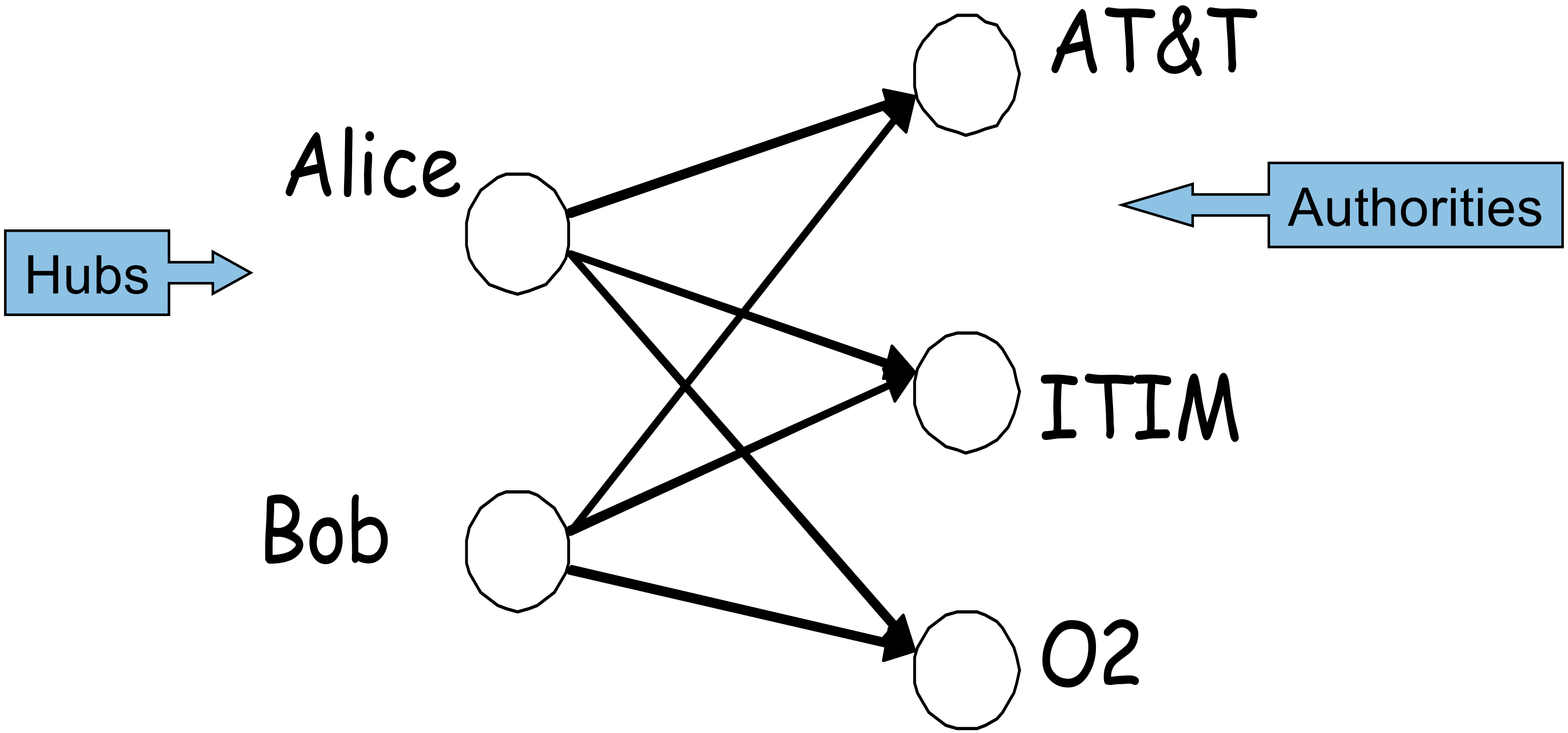
HITS ALGORITHM

HITS algorithm

- Hyperlink-Induced Topic Search, by Jon Kleinberg

- Two types of important pages on the Web
  - Authority: has authoritative content on a topic
  - Hub: pages which link to many authoritative pages, e.g., a directory or catalog
  - A good hub is one which links to many good authorities
  - A good authority node is one which is pointed to by many good hubs

The hope



Mobile telecom companies

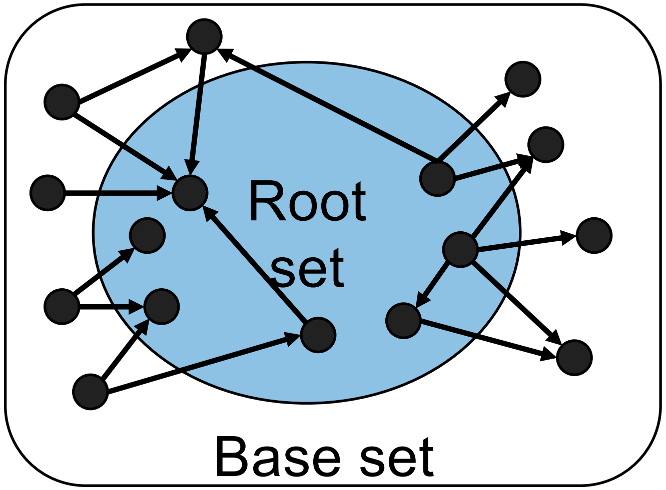
HITS

- HITS computes two scores for each page *p*
  - Authority score: sum of hub scores of all pages which point to *p*
  - Hub score: sum of authority scores of all pages which *p* points to

- Iterative algorithm
  - The definitions of hubs and authorities are “circular” in nature
  - A series of iterations run, until the scores of all pages converge

HITS run on a query-dependent sub-graph

- Meant to run on a (sub)set of pages that are relevant to a given query
  - Top N pages relevant to query retrieved based on content → called the root set
  - Add to the root set all pages that are linked from it or that links to it → base set
  - Sub-graph of all nodes in base set → focused sub-graph

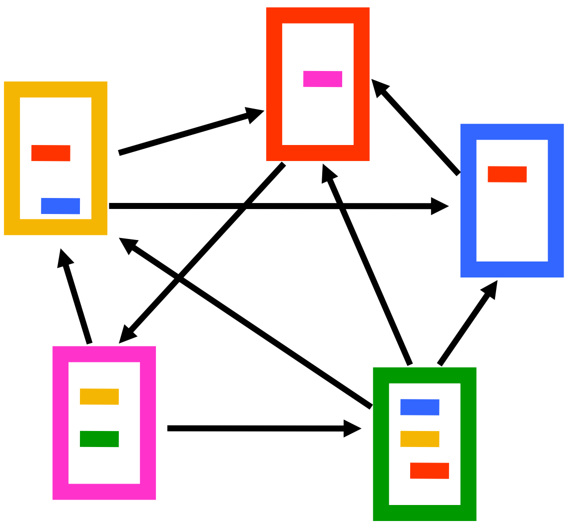


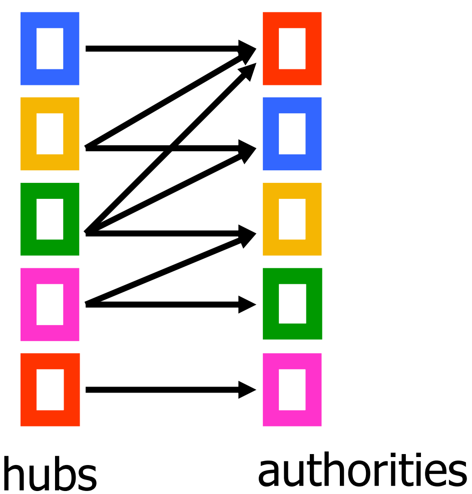
HITS run on a query-dependent sub-graph

- Why is the root set not sufficient?

- Motivation of building base set
  - A good authority page may not contain the query term
  - Hubs describe authorities through the anchor text / text surrounding hyperlinks

Visualization: hubs & authorities



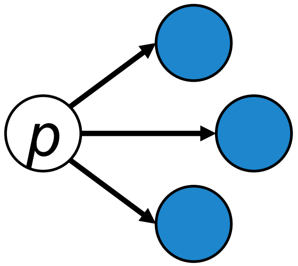


HITS Algorithm

Find focused sub-graph G of pages relevant to given query

for each page p in G:

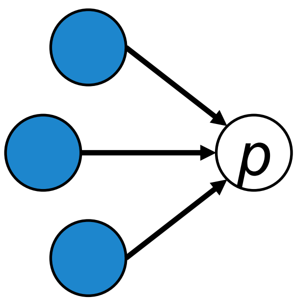
p.auth ← 1, p.hub ← 1



do until convergence

for each page p in G

p.hub ← Σ r.auth for all pages r which p links to
p.auth ← Σ q.hub for all pages q which link to p



Normalize hub and auth scores for all pages

Check convergence of scores

Output pages with highest authority scores and hub scores

Normalization of scores

- Scores need to be normalized after each iteration

- Different normalization schemes proposed
  - Normalize so that score vectors sum to 1
  - Normalization factor F: square root of sum of squares of current scores of all pages; divide score of each page by F at the end of each iteration

Checking for convergence

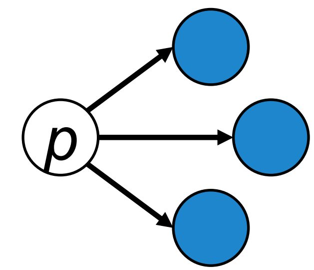
- Various convergence criteria used
  - Fixed number of iterations
  - Iterate until scores do not change appreciably from one iteration to the next (compute difference of score vectors from previous and current iterations)
  - Iterate until rankings of pages do not change

HITS Algorithm (again)

Find focused sub-graph G of pages relevant to given query

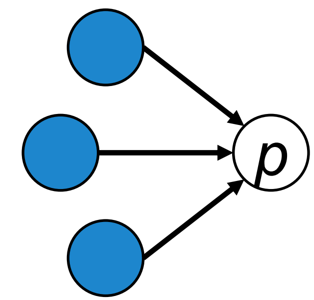
for each page p in G:

p.auth ← 1, p.hub ← 1



do until convergence

for each page p in G



p.hub ← Σ r.auth for all pages r which p links to
p.auth ← Σ q.hub for all pages q which link to p

Normalize hub and auth scores for all pages

Check convergence of scores

Matrix version of HITS

- Matrices / vectors
  - A: adjacency matrix of web graph. (u, v)-th element is 1 if page u links to page v
  - h: vector of hub scores of all pages
  - a: vector of authority scores of all pages

- In Matrix notation, each step of the algorithm becomes
  - $h \leftarrow A \times a$
  - $a \leftarrow A^T \times h$
  - Normalize h and a

Convergence of HITS

- HITS will converge (h/a in iteration t = h/a in iteration (t + 1))
  Recall:

  $h \leftarrow A \times a$

  $a \leftarrow A^T \times h$

- Thus at stationary state:

  $c' \, . \, h = A \times A^T \times h$

  $c'' \, . \, a = A^T \times A \times a$

c' and c'' are normalizations in the algorithm

Convergence of HITS

- HITS will converge (h/a in iteration t = h/a in iteration (t + 1))
  Recall:

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- Thus at stationary state:

  $c' \,.\, h = A \times A^T \times h$

  $c'' \,.\, a = A^T \times A \times a$

c' and c'' are normalizations in the algorithm

In stationary state / convergence authority/hub scores are eigenvectors of A^TA and AA^T with largest eigenvalue

HITS – summary

- HITS is guaranteed to converge

- Reasonably efficient for large Web-scale graphs, since updates involve local operations only

- Still, not very popularly used. Why?

HITS – summary

- HITS is guaranteed to converge

- Reasonably efficient for large Web-scale graphs, since updates involve local operations only

- Still, not very popularly used. Why?
  - Easy for a spam page to obtain high hub score (just by following many authorities)
  - Hubs often transit to authorities
  - Search engines themselves become hubs

PAGERANK ALGORITHM

PageRank

- By Larry Page and Sergey Brin
- Problem in measuring importance by indegree
  - Not all in-links are same
  - How important are those pages which link to page *p*?
- PageRank of a page
  - A measure of the 'authority value' of the page
  - Independent of query
  - One of many factors used by Google to rank pages

Idea of PageRank

- Good authorities should be pointed to by other good authorities
  - PR_v of page (node) *v* is a function of the sum of PRs of all those pages which point to *v*
- Each node *u* distributes its authority value equally among all those nodes to which *u* points
  - If page *u* links to 4 pages, *u* contributes $PR_u/4$ to the PR of each of those 4 pages

$$PR_v = \sum_{u \to v} \frac{1}{d_{out}(u)} PR_u$$

Equations for PR (here $w_v \sim PR_v$)

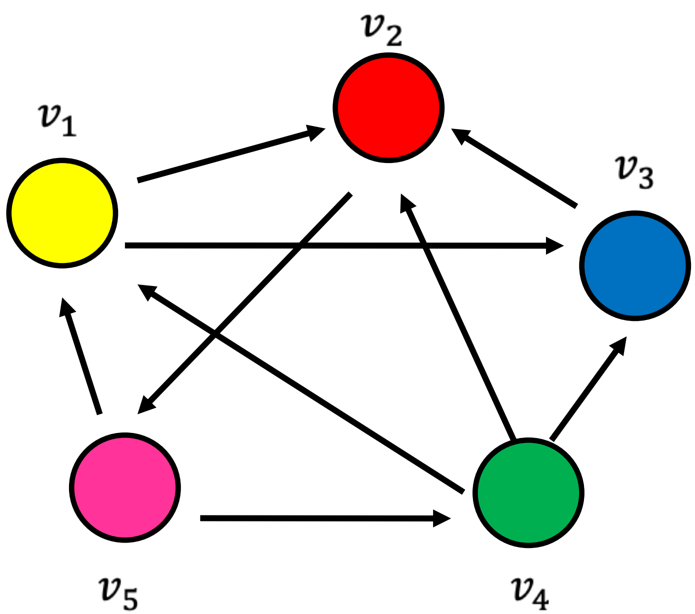
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

Iterative algorithm used to solve such a system of equations (multiple iterations until convergence)

PageRank computation

/* initialization */

for all nodes u in G: $d(u) \leftarrow 1/N$, where N = #nodes

for all nodes u in G: $PR(u) \leftarrow d(u)$

/* iteration */

do until *PR* vector converges

    for all nodes *u* in G

      for all nodes *v* that links to *u*

        $t = \Sigma\, PR(v)$ / out-degree(v)

      $PR(u) \leftarrow \alpha * t + (1 - \alpha) * d(u)$ //α to be explained later

    normalize scores

    check for convergence

end

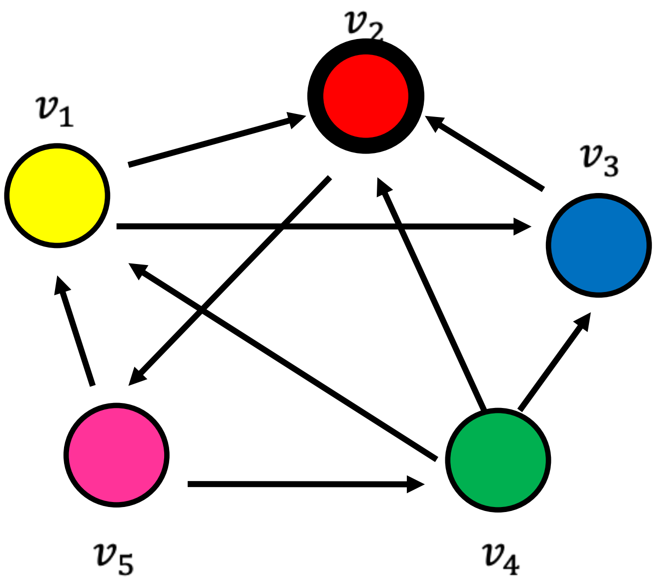
Theoretical basis of PageRank

- Random walks on a graph
  - Start from a node chosen uniformly at random with prob $\frac{1}{N}$
    - Move to the node
    - From the node your are in, pick one of the outgoing links uniformly at random
    - Move to the destination node of the chosen link
  - Repeat

- The “Random surfer model”
  - Users wander on the web, following hyperlinks
  - Nodes visited most frequently in this random walk are web-pages with higher PR

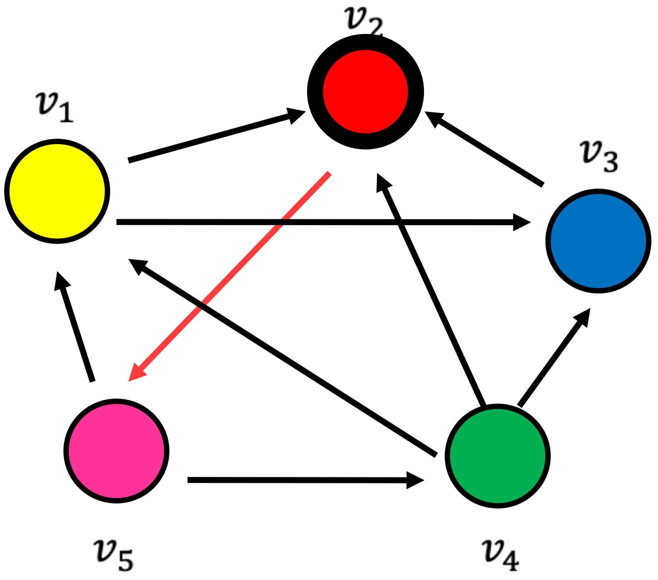
Example

- Step 0



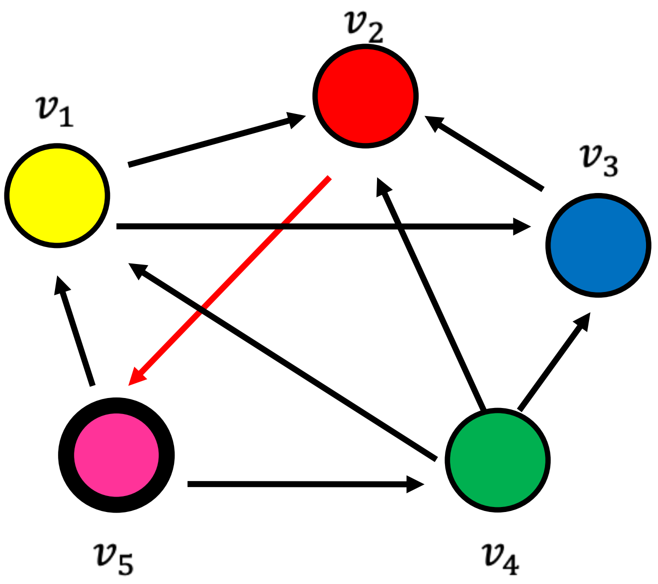
Example

- Step 0



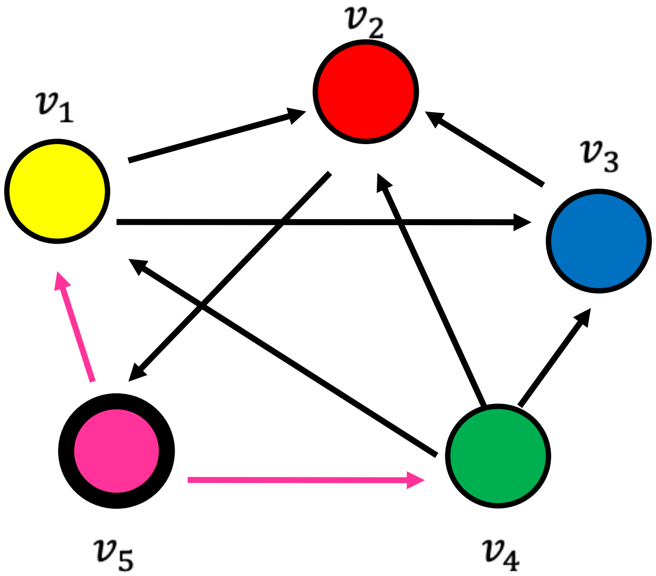
Example

- Step 1



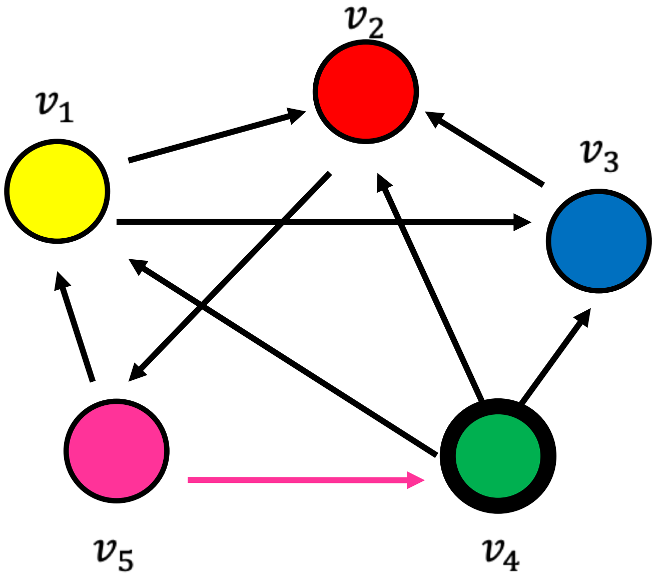
Example

- Step 1



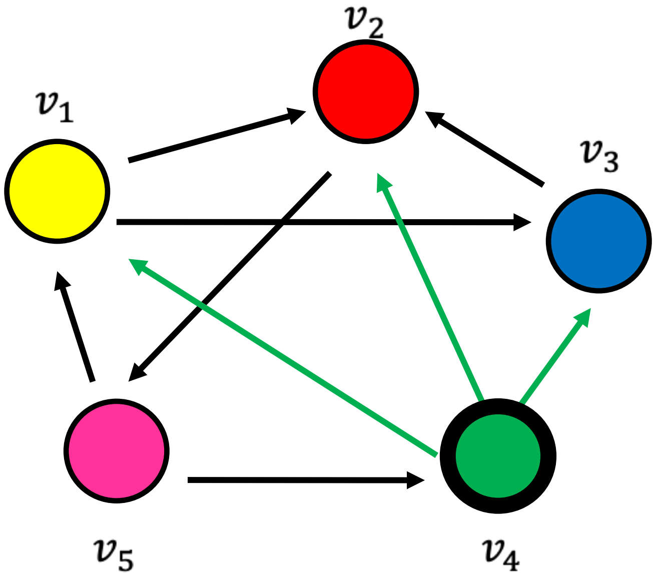
Example

- Step 2



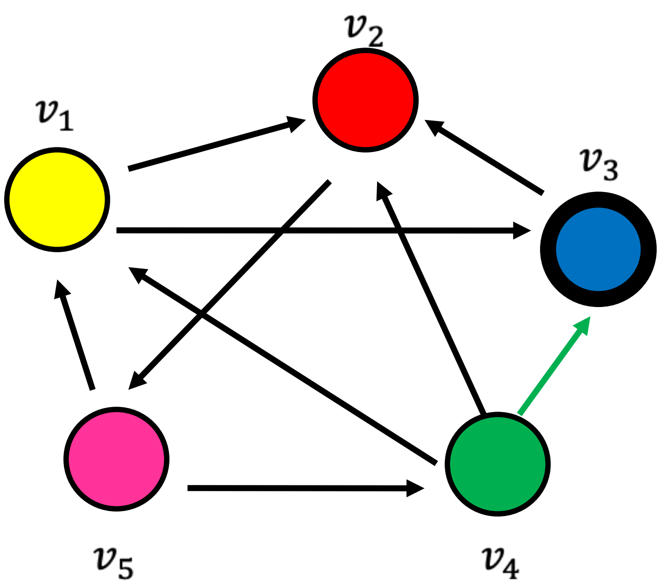
Example

- Step 2



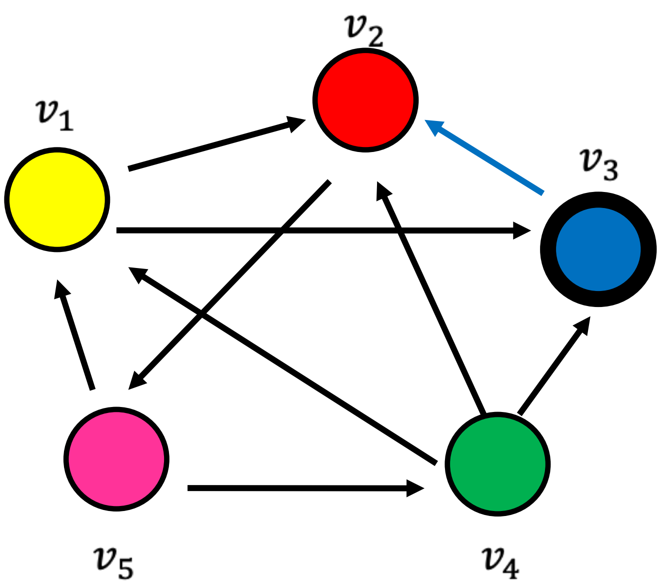
Example

- Step 3



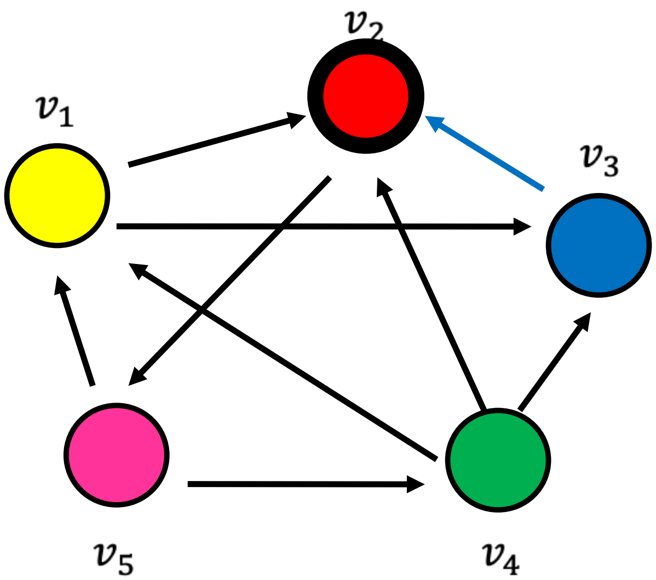
Example

- Step 3



Example

- Step 4…



Equations for Random Walk

- Question: what is the probability p_i^t of being at node i after t steps?

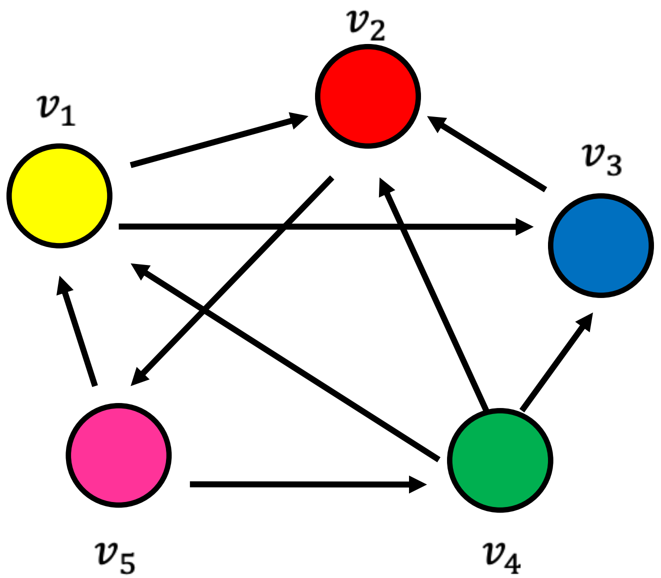
$$p_1^0 = \frac{1}{5} \qquad p_1^t = \frac{1}{3}p_4^{t-1} + \frac{1}{2}p_5^{t-1}$$

$$p_2^0 = \frac{1}{5} \qquad p_2^t = \frac{1}{2}p_1^{t-1} + p_3^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_3^0 = \frac{1}{5} \qquad p_3^t = \frac{1}{2}p_1^{t-1} + \frac{1}{3}p_4^{t-1}$$

$$p_4^0 = \frac{1}{5} \qquad p_4^t = \frac{1}{2}p_5^{t-1}$$

$$p_5^0 = \frac{1}{5} \qquad p_5^t = p_2^{t-1}$$



The equations are the same as those for the PageRank computation

Equations for PR (again)

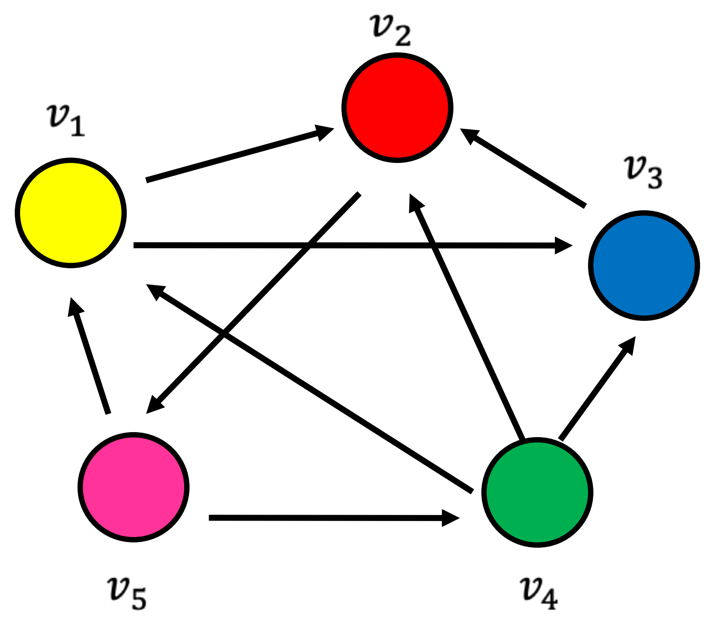
$w_1 = 1/3\ w_4 + 1/2\ w_5$

$w_2 = 1/2\ w_1 + w_3 + 1/3\ w_4$

$w_3 = 1/2\ w_1 + 1/3\ w_4$

$w_4 = 1/2\ w_5$

$w_5 = w_2$



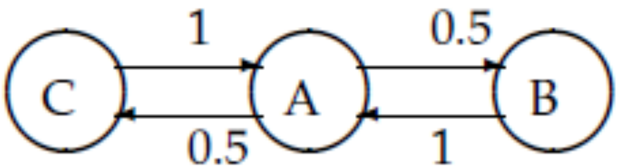
$$w_v = \sum_{u \to v} \frac{1}{d_{out}(u)} w_u$$

Iterative algorithm used to solve such a system of equations (multiple iterations until convergence)

Theoretical basis of PageRank

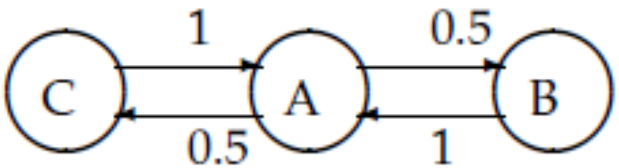
- The random walk defines a Markov chain
  - A discrete time stochastic process following Markov property (next state depends only on current state)
  - N states corresponding to the N nodes; chain is at one of the states at any given time-step
  - N x N transition probability matrix P : P_{ij} is the probability that state at next time-step is j, given current state is i
- $$\forall i, j, P_{ij} \in [0,1] \qquad \forall i, \sum_{j=1}^{N} P_{ij} = 1.$$

An example



$$\begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

An example



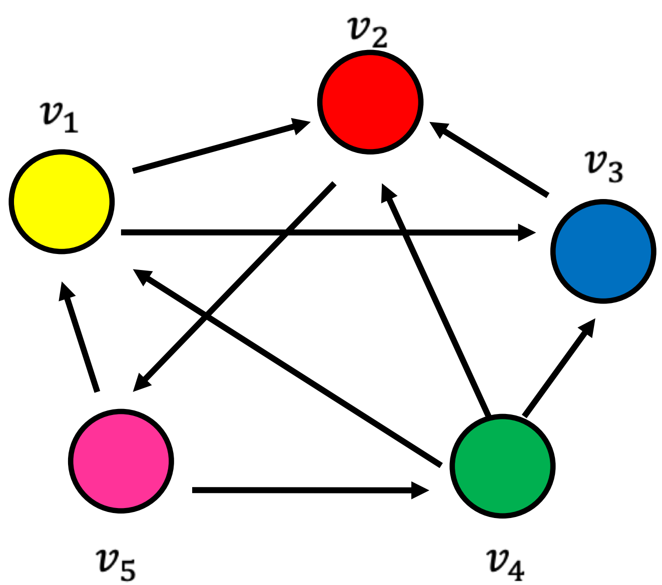
$$\begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- P is a stochastic matrix
    - Every element is in [0, 1]
    - Sum of every row is 1
    - Largest eigenvalue is 1
    - Has a principal left eigenvector corresponding to its largest eigenvalue

Another example

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



Transition matrix for random surfer

- How to derive the transition matrix for the random surfer on the Web graph?

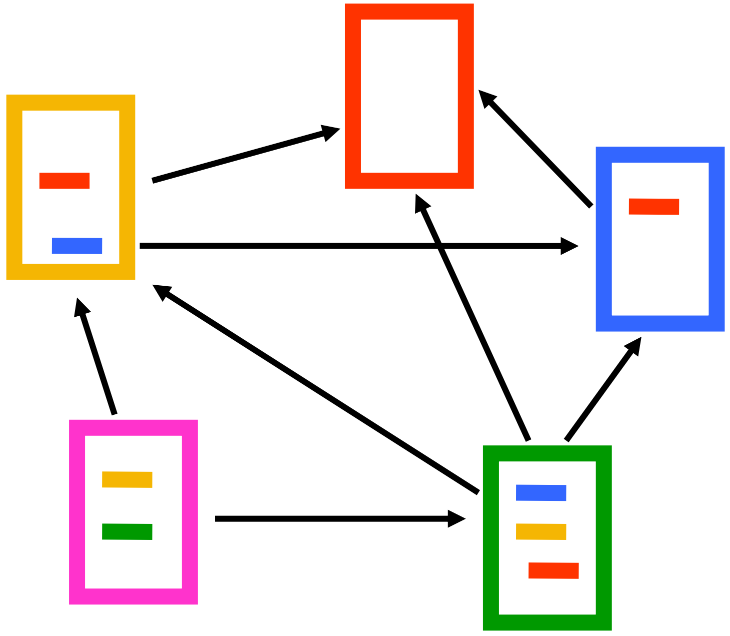
- Adjacency matrix of Web graph
  - $A_{ij} = 1$ if there is a hyperlink from page i to page j
  - $A_{ij} = 0$ otherwise

- Derive transition matrix P of Markov chain from A

Some practical challenges

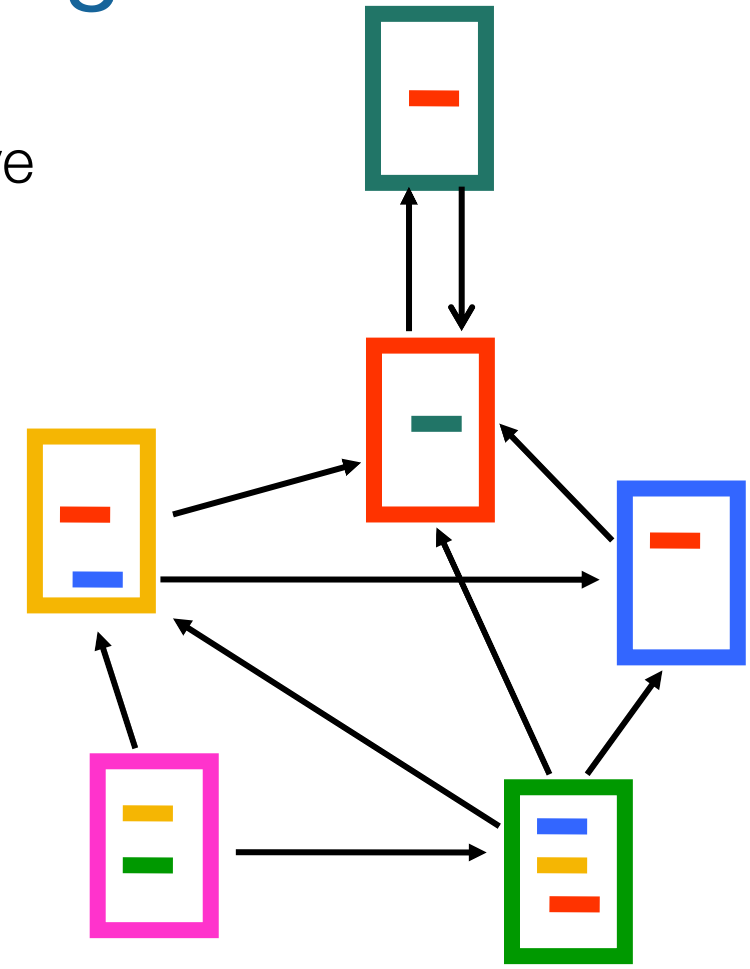
- Web graph (or any graph) can have
  - Dead-ends or sink nodes – nodes with no out-edges

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



Some practical challenges

- Web graph (or any graph) can have
  - Loops

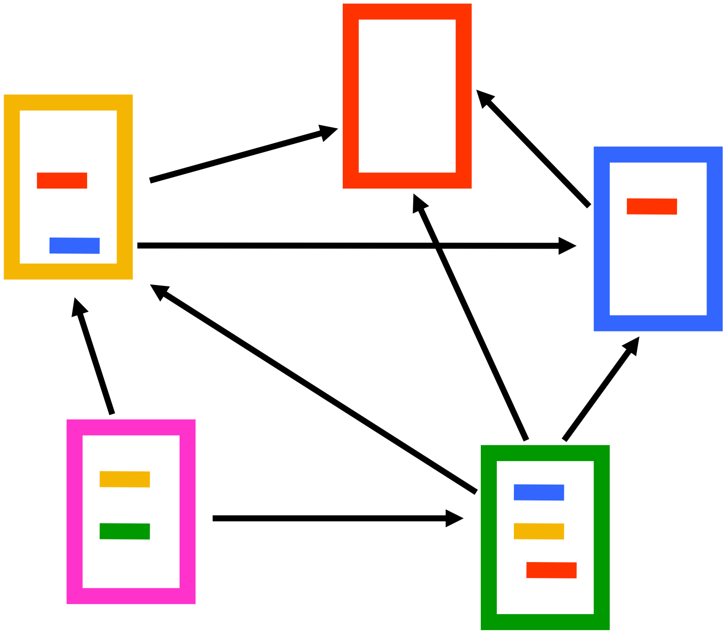


Transition matrix for random surfer

- Derive transition matrix P of Markov chain from A
  - If a row of A has no 1's, replace each element by $1/N$
  - For all other rows: divide each 1 by the number of 1's in the row
  - Multiply the resulting matrix by α
  - Add $(1-\alpha)/N$ to every entry of the resulting matrix

Dealing with sink nodes

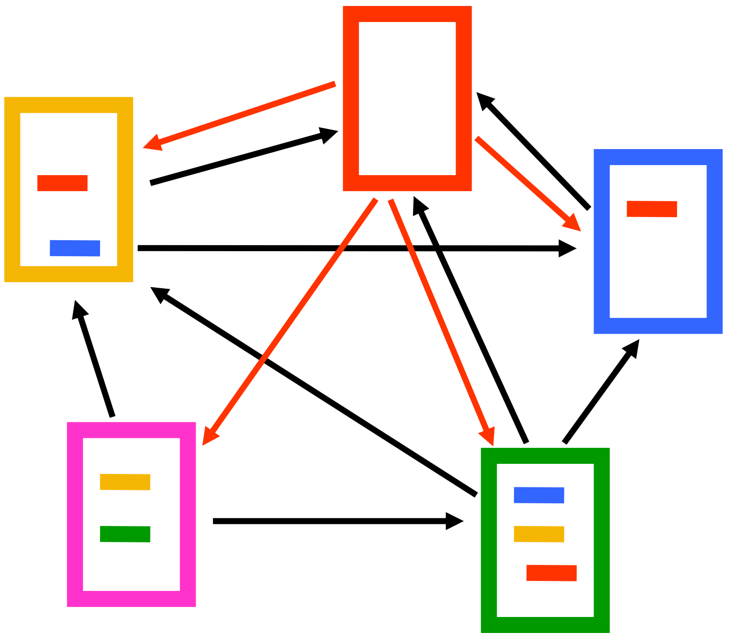
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



Dealing with sink nodes

As if synthetic edges are inserted from the sink node to every other node in the graph

$$P' = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ \boxed{1/5 \quad 1/5 \quad 1/5 \quad 1/5 \quad 1/5} \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



Dealing with loops

- As if synthetic edges are inserted to enable jump from any node to any other node in the graph
- Teleportation: jump to any random node with probability 1/N

$$P'' = \alpha \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

Why teleportation?

- Convergence of PageRank is guaranteed only if
  - The transition probability matrix P is irreducible, i.e., all transitions have a non-zero probability
  - In other words, if the graph (on which random surfing is taking place) is strongly connected

- To ensure convergence
  - To nodes with out-degree 0, add an outgoing edge to every node
  - Damp the walk by factor α, by adding a complete set of outgoing edges, with weight $(1-\alpha)/N$, to all nodes

Transition matrix for random surfer: Recap

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  - Add $(1-\alpha)/N$ to every entry of the resulting matrix

Given P, how to compute PageRank?

- Vector x (dimension N): probability distribution of surfer's position at any time
  - At $t = 0$: one entry in x is 1, rest are 0
  - At $t = 1$: xP
  - At $t = 2$: $(xP)P = xP^2$
  - …

- Steady-state $x = \Pi$ gives the PageRank scores
  - At steady-state: $\Pi P = \Pi$
  - In other words, at steady state: $\Pi P = 1.\Pi$

Given P, how to compute PageRank?

- Vector x (dimension N): probability distribution of surfer's position at any time
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  - At $t = 2$: $(xP)P = xP^2$
  - …

- Steady-state $x = \Pi$ gives the PageRank scores
- PageRank scores obtained as the principal left eigenvector of P (corresponding to eigenvalue 1)

PageRank computation

- Need to compute principal left eigenvector of a stochastic matrix

- Several numerical methods, e.g., power iteration

- Difficult to compute for matrices of the size of the Web graph; iterative method (already discussed) can be more efficient

Theoretical basis of PageRank: Recap

- Random surfer model
  - Start at a node, execute a random walk on Web graph
  - At each step, proceed from current node *u* to a randomly chosen node that *u* links to
  - Teleport: jump to any random node with probability 1/N
  - At a node with no outgoing links, teleport
  - At a node that has outgoing links
    - Follow standard random walk with probability α where $0<\alpha<1$
    - Teleport with probability $(1-\alpha)$
- Nodes visited more frequently in this random walk are web-pages with higher PR

PageRank computation: Recap

/* initialization */

for all nodes u in G: *d(u)* ← *1/N*, where *N* = #nodes

for all nodes u in G: *PR(u)* ← *d(u)*

/* iteration */

do until *PR* vector converges

    for all nodes *u* in G

        for all nodes *v* that links to *u*

            $t = \Sigma\, PR(v)$ / out-degree(v)

        $PR(u) \leftarrow \alpha * t + (1 - \alpha) * d(u)$

    normalize scores

    check for convergence

end

Practical challenges

- All links *u* → *v* do not signify a vote for *v*
  - E.g., links to a copyright page from all pages in a website

- Attempts to spam PageRank: link spam farms or link farms
  - A target page (whose PR the spammer wants to boost)
  - A number of boosting pages, which link to the target page, link to each other and also to external pages
  - Hijacked links – links accumulated from pages outside the link farm

Example link farm

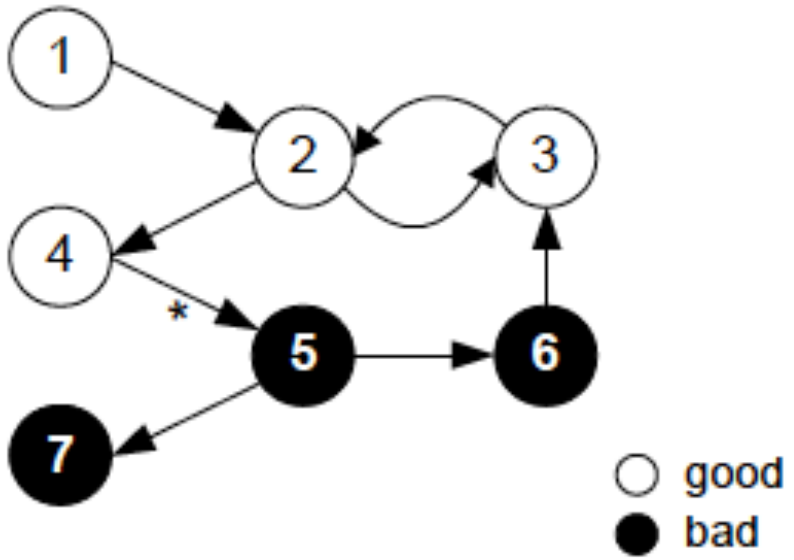

Figure 2: A web of good (white) and bad (black) nodes.

VARIATIONS OF PAGERANK

PageRank computation

/* initialization */

for all nodes u in G: $d(u) \leftarrow 1/N$, where N = #nodes

for all nodes u in G: $PR(u) \leftarrow d(u)$

/* iteration */

do until *PR* vector converges

    for all nodes *u* in G

      for all nodes *v* that links to *u*

        $t = \Sigma\, PR(v)$ / out-degree(v)

      $PR(u) \leftarrow \alpha * t + (1 - \alpha) * d(u)$

    normalize scores

    check for convergence

end

Biased PageRank

- Instead of using the uniform vector *d(u)* ← *1/N* for all nodes u, use a non-uniform preference vector:

$$d(u) = 1 / |S|, \text{ for all } u \in S$$
$$= 0 \text{ otherwise}$$

- Implication for random surfer:
    - With probability α, follow standard random walk
    - With probability (1-α), teleport to a node in S, where the particular node in S is chosen randomly

Biased PageRank

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  d(u) = 1 / |S|, for all u ε S

  = 0 otherwise

- Implication for random surfer:
  - With probability α, follow standard random walk
  - With probability (1-α), teleport to a node in S, where the particular node in S is chosen randomly
- Bias the ranks towards nodes that are closer to nodes with a larger value in the preference vector

Topic-sensitive PageRank [Haveliwala, WWW 2002]

- Webpages are classified into various topics (16 Open Directory Project high-level categories)
- Computes PageRank for a particular topic of interest
- For category c_j
  - T_j is the set of websites for category c_j
  - Modified teleportation function

$$v_{ji} = \begin{cases} \frac{1}{|T_j|} & i \in T_j, \\ 0 & i \notin T_j. \end{cases}$$

TrustRank [Gyongyi, VLDB 2004]

- Aims to rank trusted pages higher, and push untrusted pages down in the rankings
- Assumes
  - A way of knowing trusted nodes: oracle
  - Trusted (good) nodes will only link to other good nodes but this assumption is violated in the real Web
  - Bad nodes will link to other bad nodes and good nodes
- Run PageRank by biasing the preference vector towards a set of trusted nodes

TrustRank vs. PageRank

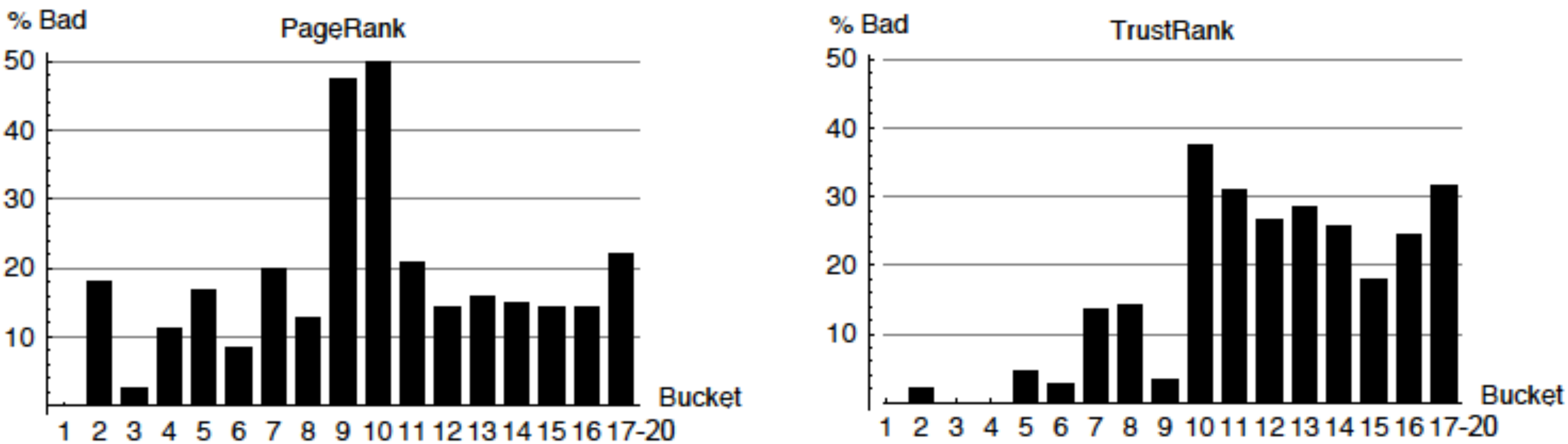

Figure 10: Bad sites in PageRank and TrustRank buckets.