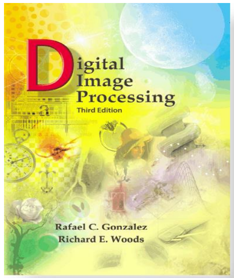
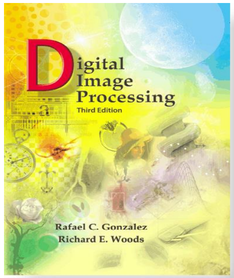


Filtering in Frequency Domain



Objectives :

- Understand the meaning of frequency domain filtering, and how it differs from filtering in the spatial domain.
- Be familiar with the concepts of sampling, function reconstruction, and aliasing.
- Understand convolution in the frequency domain, and how it is related to filtering.
- Know how to obtain frequency domain filter functions from spatial kernels, and vice versa.
- Be able to construct filter transfer functions directly in the frequency domain.
- Understand why image padding is important.
- Know the steps required to perform filtering in the frequency domain.
- Understand when frequency domain filtering is superior to filtering in the spatial domain.
- Be familiar with other filtering techniques in the frequency domain, such as unsharp masking and homomorphic filtering.
- Understand the origin and mechanics of the fast Fourier transform, and how to use it effectively in image processing.

Basics of Frequency Domain concepts for 1-D and 2-D functions

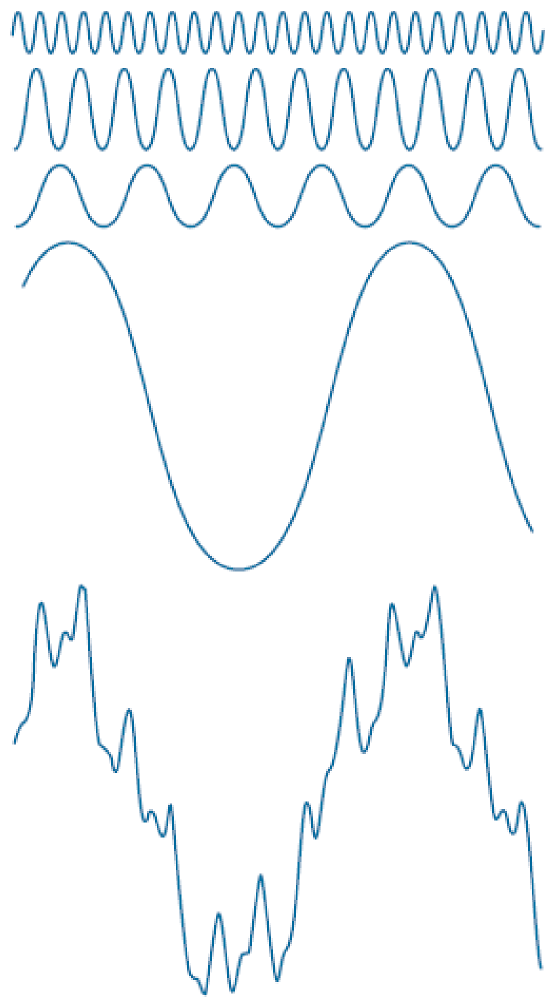
Fourier Series :

Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient.

Fourier Transform :

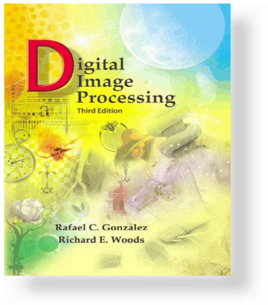
Functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function.

FIGURE 4.1
The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

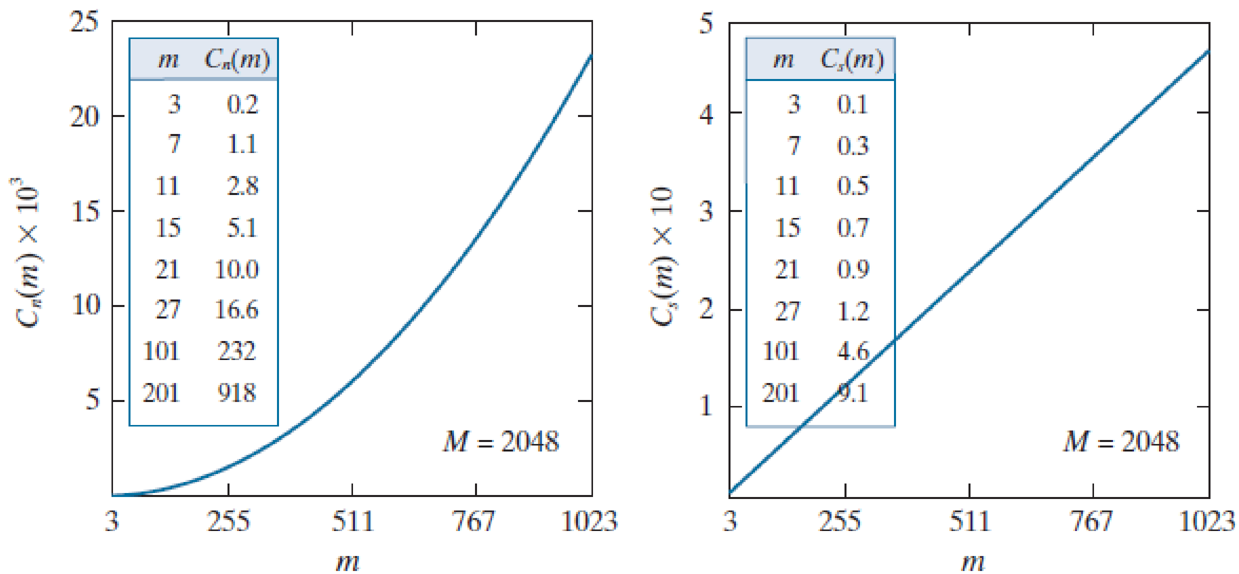
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$



Computational advantage of filtering in Frequency domain

a b

FIGURE 4.2
(a) Computational advantage of the FFT over non-separable spatial kernels.
(b) Advantage over separable kernels. The numbers for $C(m)$ in the inset tables are not to be multiplied by the factors of 10 shown for the curves.



m	$C_n(m)$
3	0.2
7	1.1
11	2.8
15	5.1
21	10.0
27	16.6
101	232
201	918

m	$C_s(m)$
3	0.1
7	0.3
11	0.5
15	0.7
21	0.9
27	1.2
101	4.6
201	9.1

$$C_n(m) = \frac{M^2 m^2}{2M^2 \log_2 M^2} = \frac{m^2}{4 \log_2 M}$$

$$C_s(m) = \frac{2M^2 m}{2M^2 \log_2 M^2} = \frac{m}{2 \log_2 M}$$

Impulse function and its properties (Continuous domain)

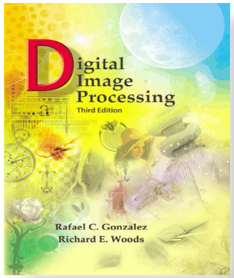
$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_0)\, dt = f(t_0)$$

$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$



Impulse function and its properties (Discrete domain)

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

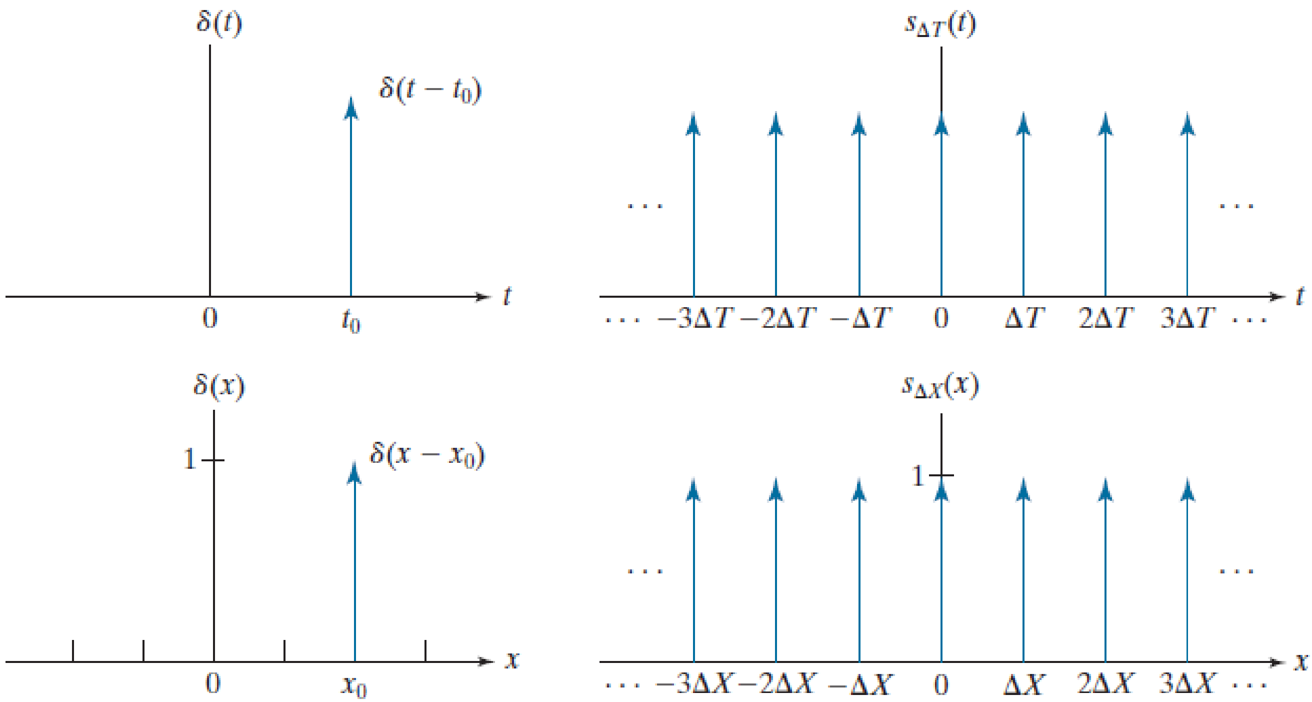
$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

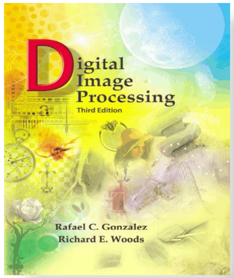
$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

a b
c d

FIGURE 4.3
(a) Continuous impulse located at $t = t_0$. (b) An impulse train consisting of continuous impulses. (c) Unit discrete impulse located at $x = x_0$. (d) An impulse train consisting of discrete unit impulses.





Fourier Transform (1D-Continuous Domain)

$$\Im\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

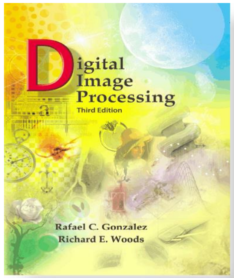
Fourier Transform Pair

$$f(t) \Leftrightarrow F(\mu)$$

$$\Im\{f(t)\} = F(\mu) \qquad F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)]dt$$

$$f(t) = \Im^{-1}\{F(\mu)\} \qquad f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$



Chapter 4

Filtering in the Freque

Fourier Transform (FT) of a Box function

$$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}dt$$

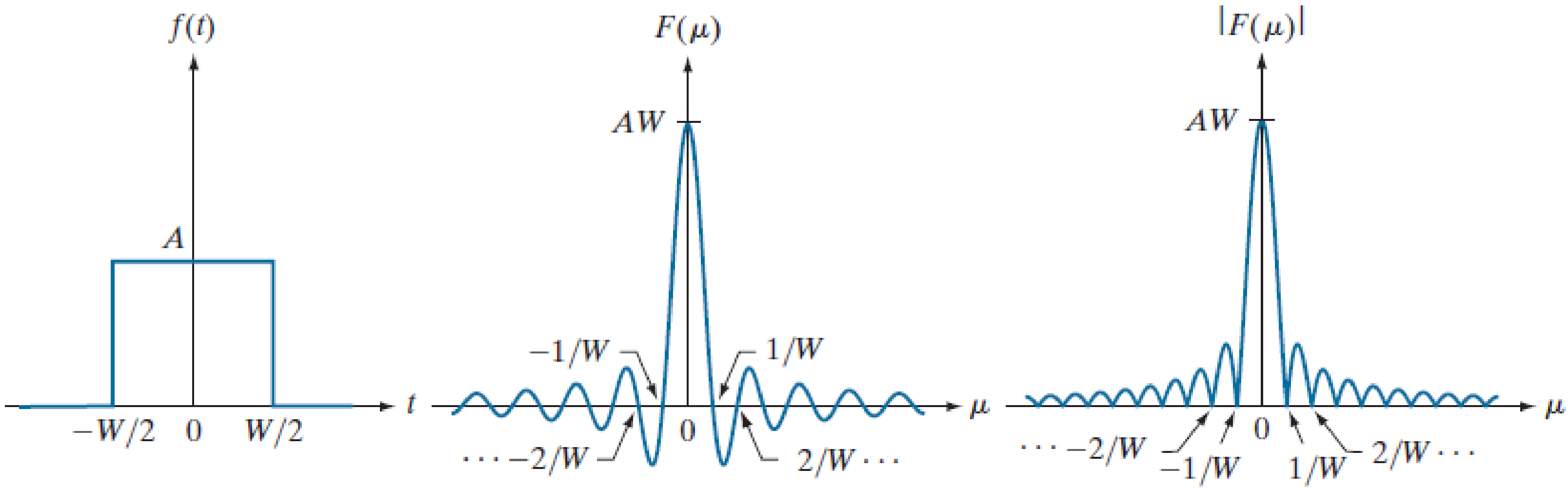
$$= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right]$$

$$= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right]$$

$$= AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

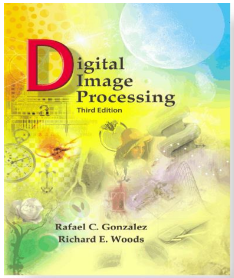
$$\mathrm{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)}$$

$$|F(\mu)| = AW\left|\frac{\sin(\pi\mu W)}{(\pi\mu W)}\right|$$



a b c

FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.



FT of an impulse and impulse train

$$\Im\{\delta(t)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t) dt = e^{-j2\pi\mu}$$

$$\Im\{\delta(t-t_0)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t-t_0) dt = e^{-j2\pi\mu t_0}$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt \qquad c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T} e^0 = \frac{1}{\Delta T}$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

$$S(\mu) = \Im\{s_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T} \Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

Convolution

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

$$(f \star h)(t) \Leftrightarrow (H \bullet F)(\mu)$$

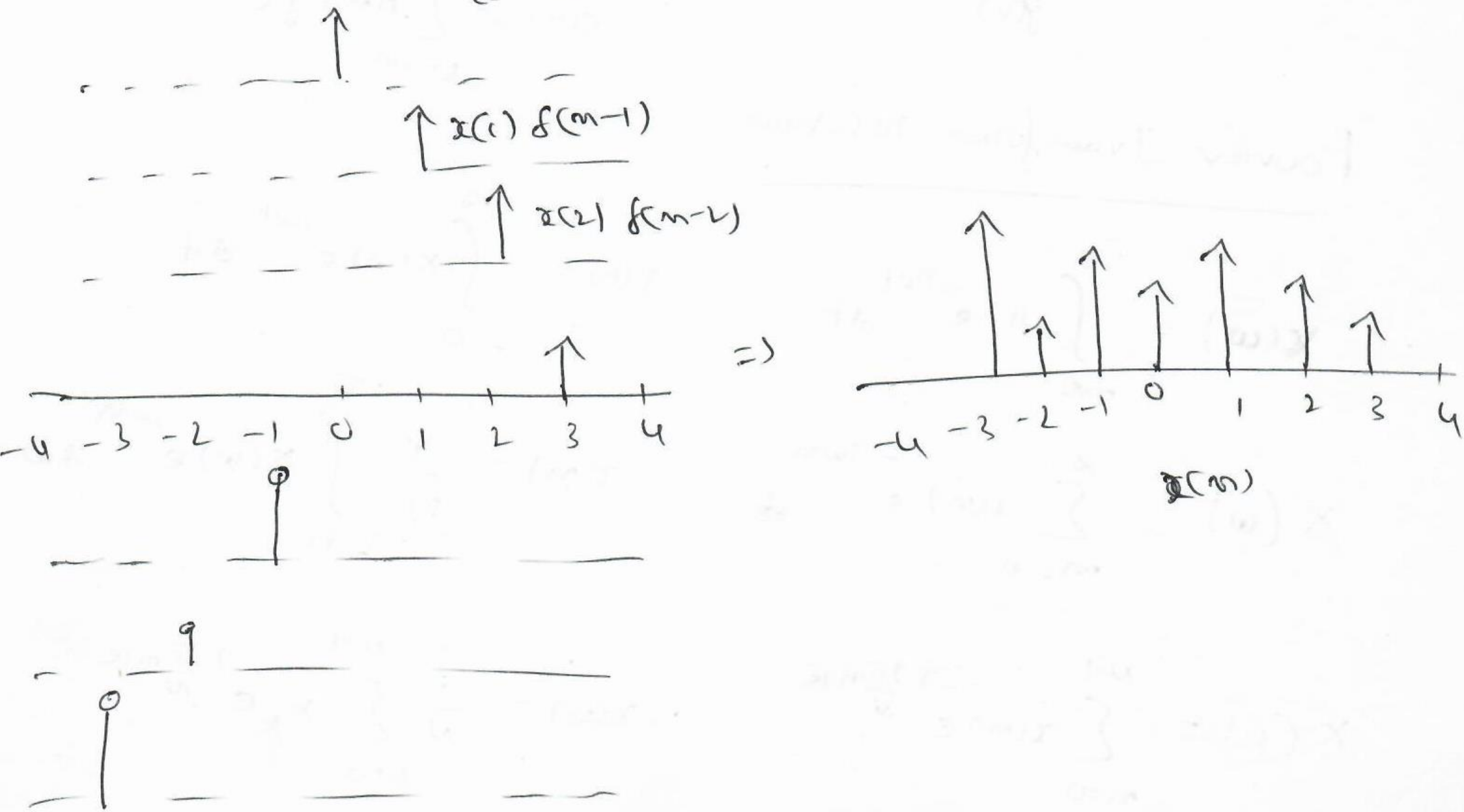
$$(f \bullet h)(t) \Leftrightarrow (H \star F)(\mu)$$

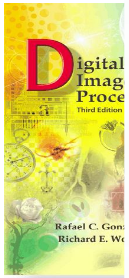
$$\begin{aligned}\Im\{(f \star h)(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\right] e^{-j2\pi\mu t}\,dt \\ &= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j2\pi\mu t}\,dt\right] d\tau\end{aligned}$$

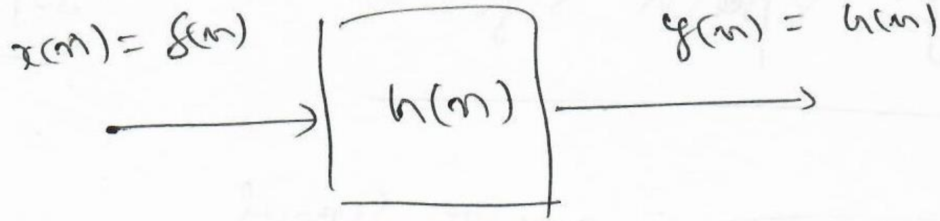
$$\begin{aligned}\Im\{(f \star h)(t)\} &= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right] d\tau \\ &= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,d\tau \\ &= H(\mu)F(\mu) \\ &= (H \bullet F)(\mu)\end{aligned}$$

$x(k)$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$







I/p   O/p

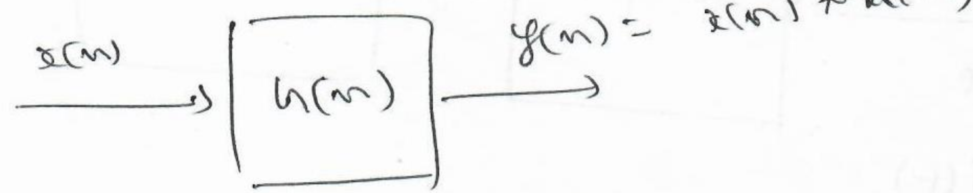
$\delta(n)$ $h(n)$

$\delta(n-k)$ $h(n-k)$

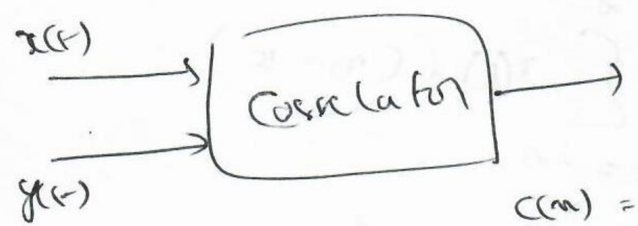
$x(k)\,\delta(n-k)$ $x(k)\,h(n-k)$

$\sum x(k)\,\delta(n-k)$ $\sum x(k)\,h(n-k) \Rightarrow x(n) * h(n)$

Convolution



Correlation



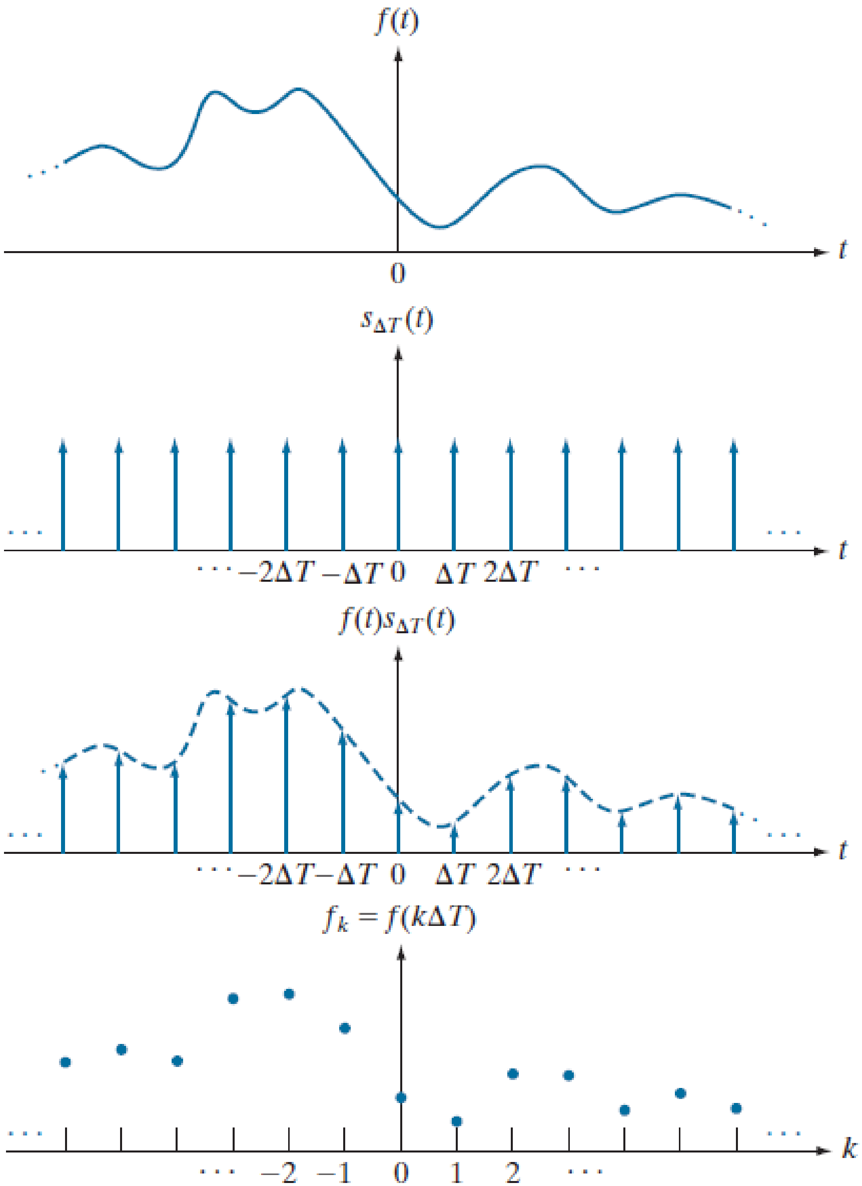
$\int x(t)\,y(t+\tau)$

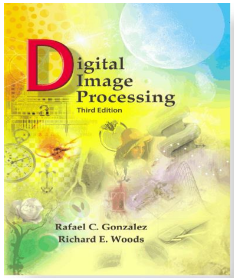
$$c(n) = \sum_{k=-\infty}^{\infty} x(n)\,y(n+k)$$

Sampling and FT of Sampled Signals

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt$$
$$= f(k\Delta T)$$





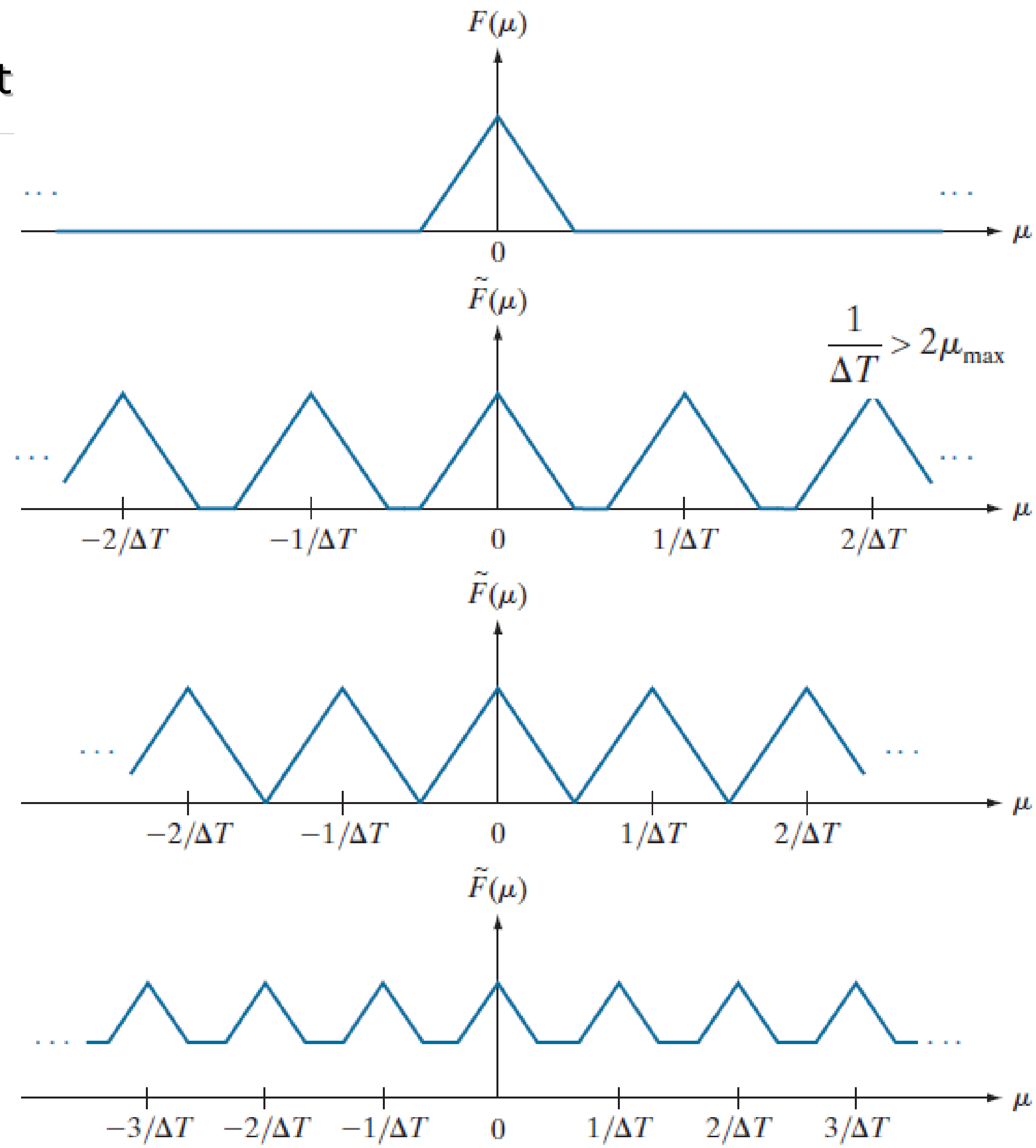
Filtering in t

FT of Sampled Signals

$$\tilde{F}(\mu) = \Im\left\{\tilde{f}(t)\right\} = \Im\left\{f(t)s_{\Delta T}(t)\right\}$$
$$= (F \star S)(\mu)$$

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$\tilde{F}(\mu) = (F \star S)(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau)d\tau$$
$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)d\tau$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)d\tau$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

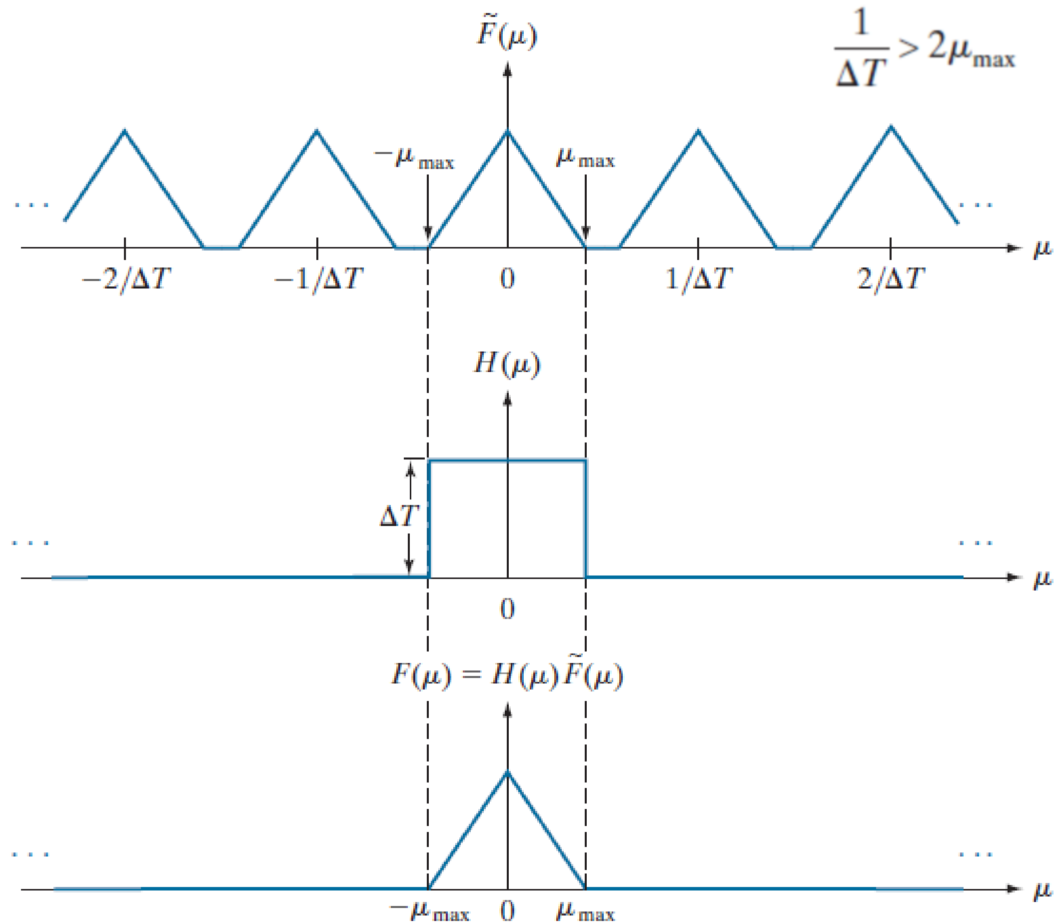


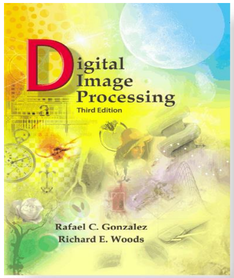
Reconstruction from Frequency Domain

$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

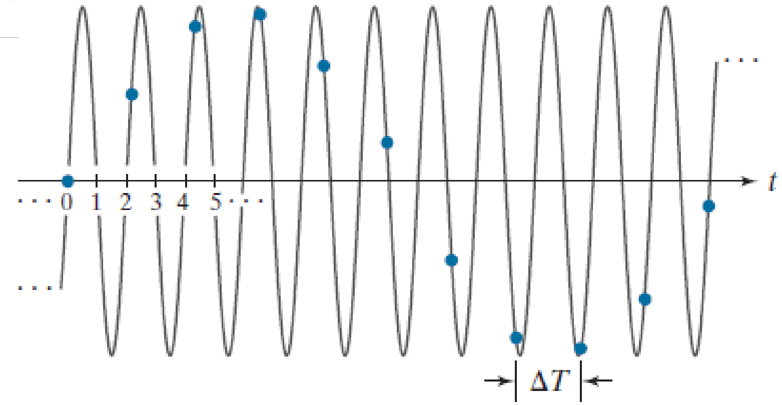
$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

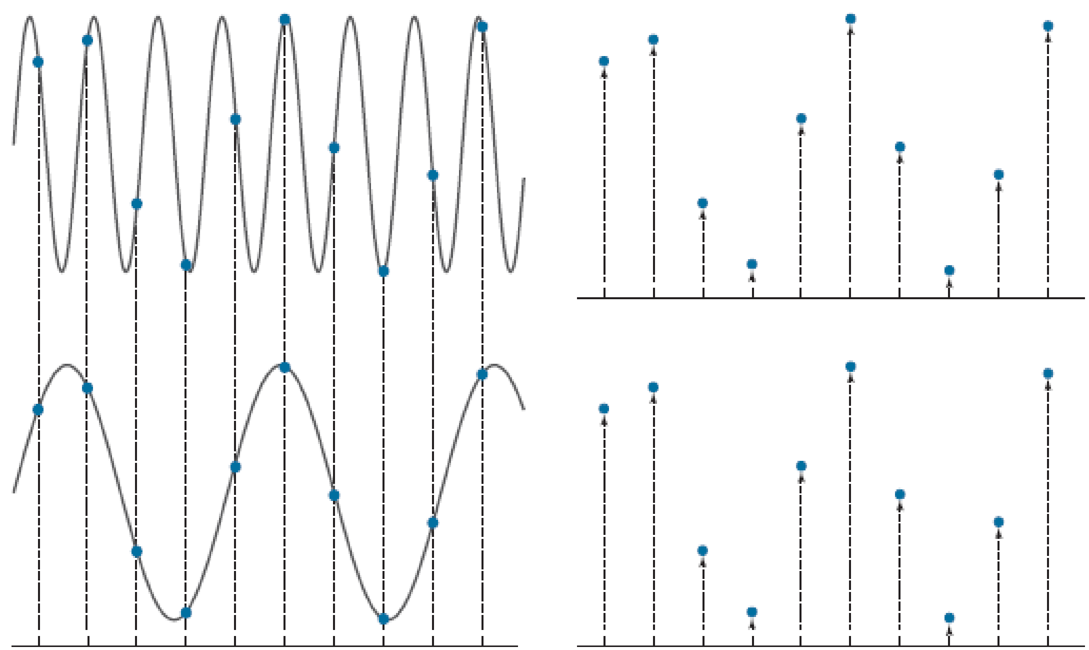
$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$





Aliasing





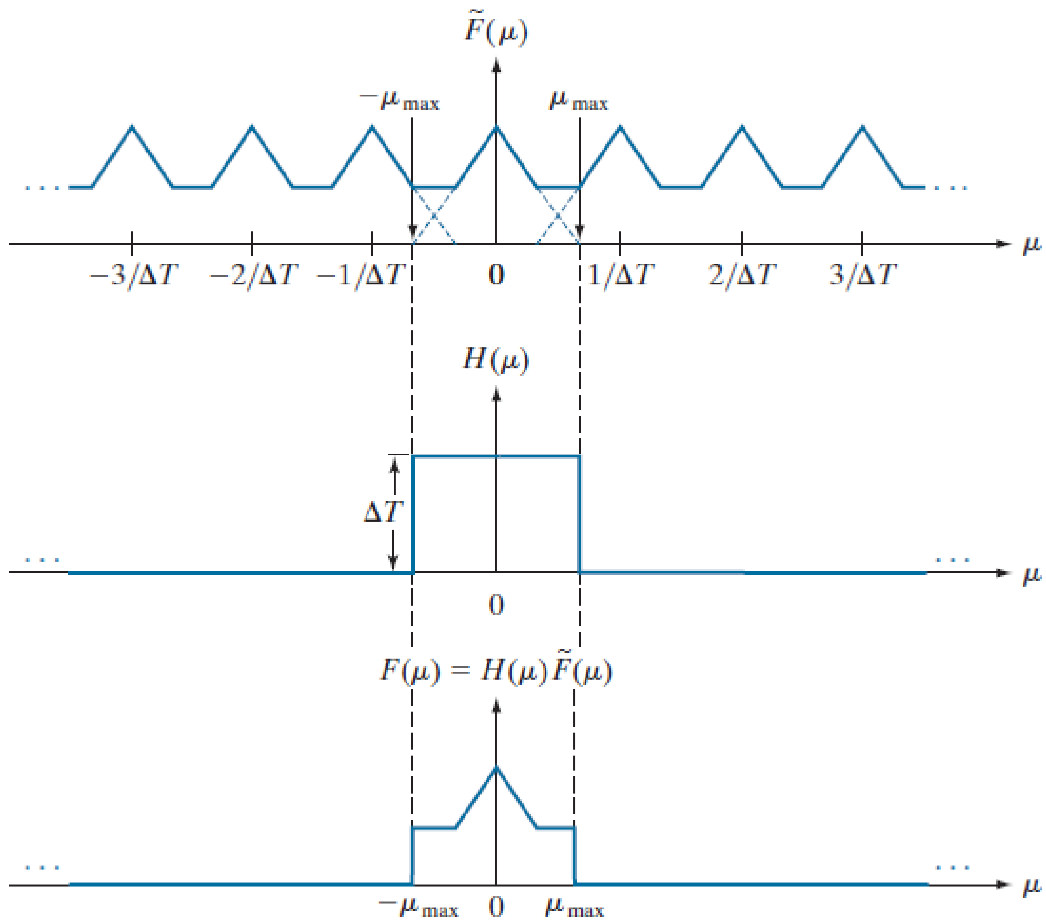
a b
c d

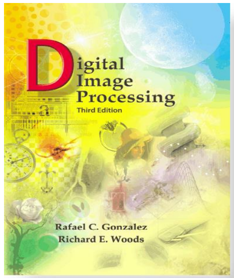
FIGURE 4.9 The functions in (a) and (c) are totally different, but their digitized versions in (b) and (d) are identical. Aliasing occurs when the samples of two or more functions coincide, but the functions are different elsewhere.

Recovery from the aliased signals

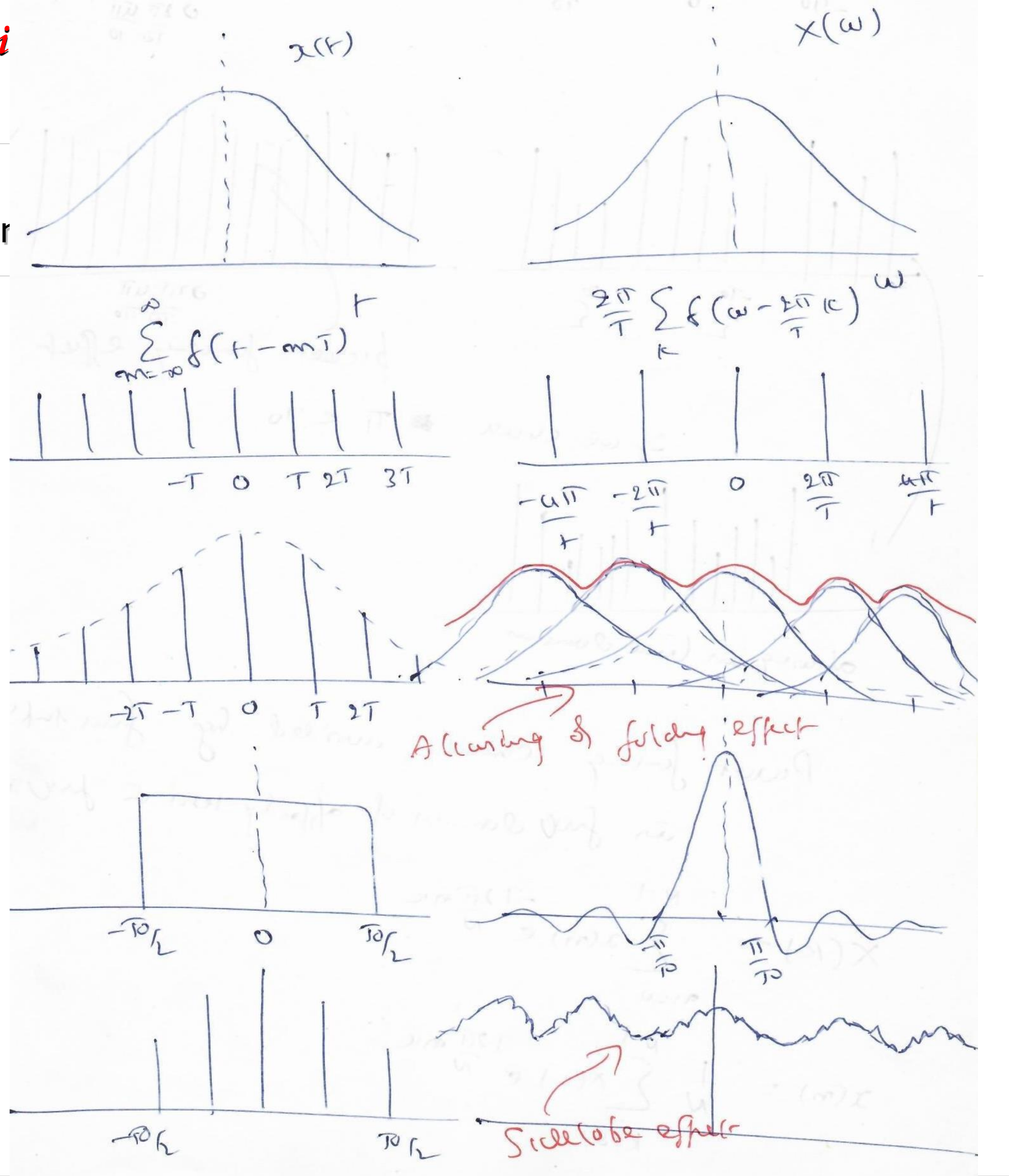
$$
\begin{aligned}
f(t) &= \Im^{-1}\left\{F(\mu)\right\} \\
&= \Im^{-1}\left\{H(\mu)\tilde{F}(\mu)\right\} \\
&= h(t) \star \tilde{f}(t)
\end{aligned}
$$

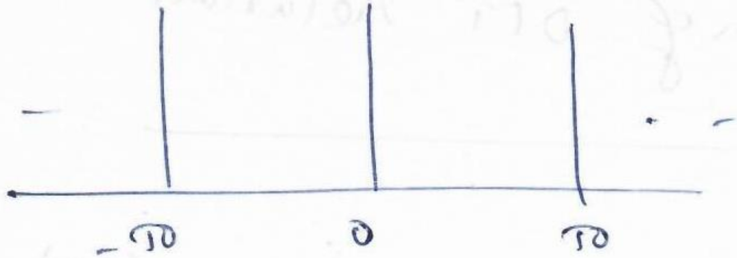
$$
f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\,\mathrm{sinc}\left[(t - n\Delta T)/\Delta T\right]
$$

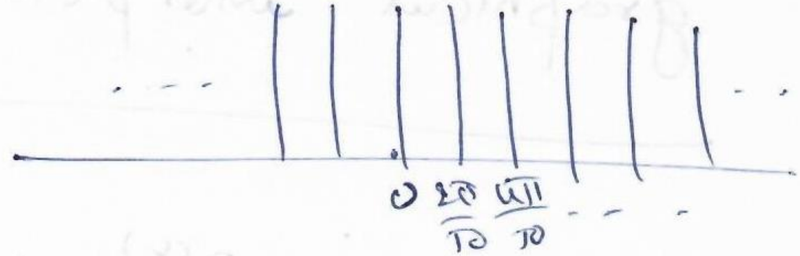


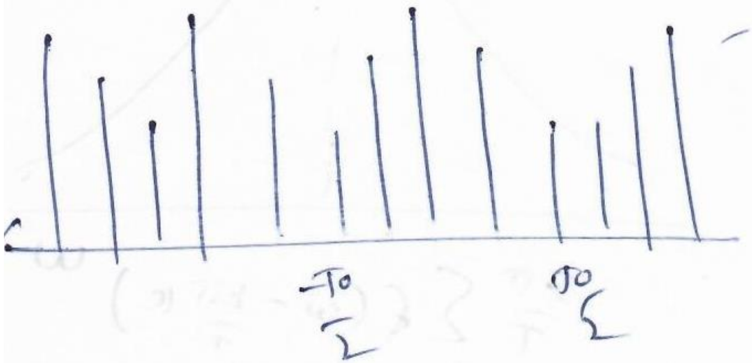


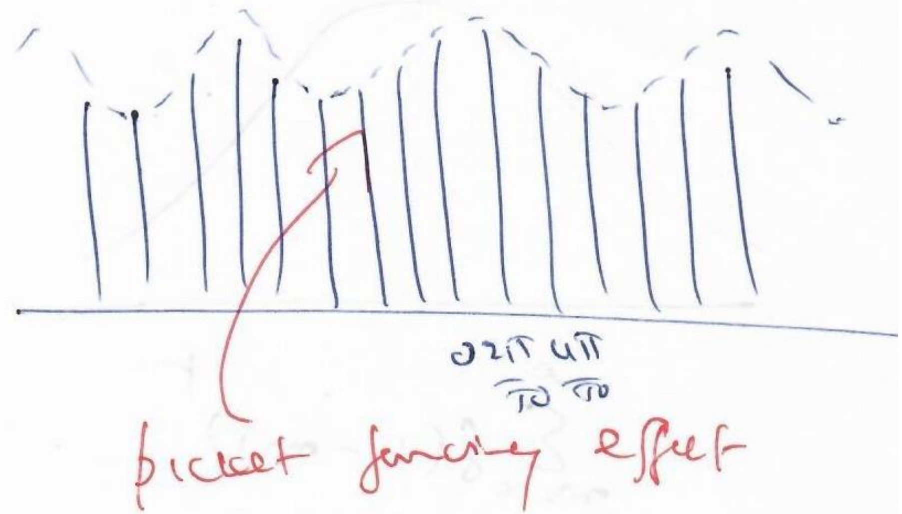
Graphical Interpretation of DFT relations



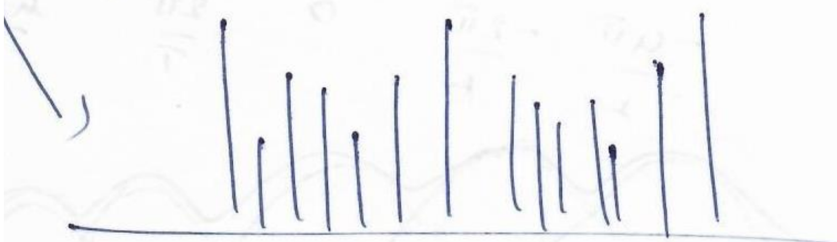




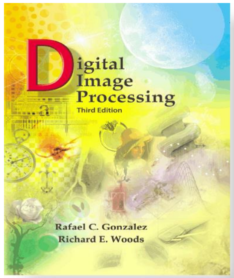




If we choose $T_1 < T_0$



aliasing in time domain



Discrete Fourier Transform (DFT) of 1-D data

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt$$

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

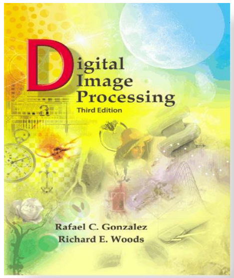
$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$



Relation between sampling (time) and frequency resolutions

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$F(u) = F(u + kM)$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

$$f(x) = f(x + kM)$$

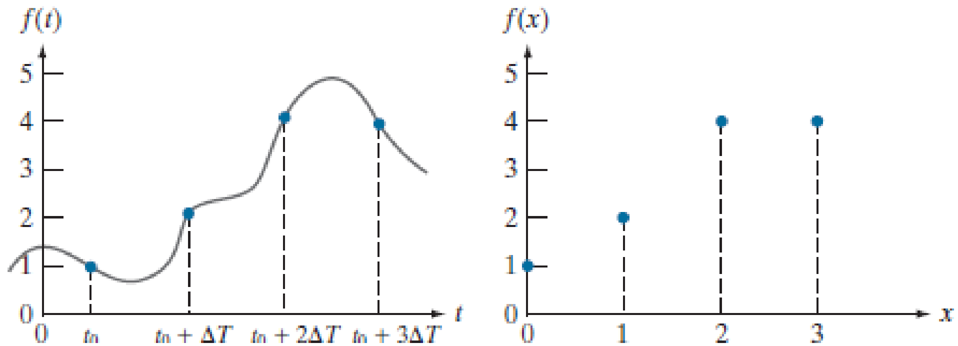
$$\{f(x)\}, x = 0, 1, 2, \ldots, M-1$$

$$T = M\Delta T$$

$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

$$R = M\Delta u = \frac{1}{\Delta T}$$

Example to illustrate the computation of 1-D DFT

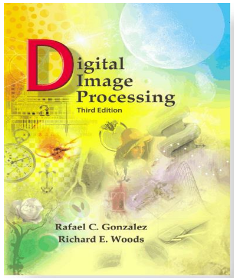


$$F(0)=\sum_{x=0}^{3} f(x)=[f(0)+f(1)+f(2)+f(3)]=1+2+4+4=11$$

$$F(1)=\sum_{x=0}^{3} f(x)e^{-j2\pi(1)x/4}=1e^{0}+2e^{-j\pi/2}+4e^{-j\pi}+4e^{-j3\pi/2}=-3+2j$$

$$F(2)=-(1+0j) \text{ and } F(3)=-(3+2j).$$

$$f(0)=\frac{1}{4}\sum_{u=0}^{3} F(u)e^{j2\pi u(0)}=\frac{1}{4}\sum_{u=0}^{3} F(u)=\frac{1}{4}[11-3+2j-1-3-2j]=\frac{1}{4}[4]=1$$



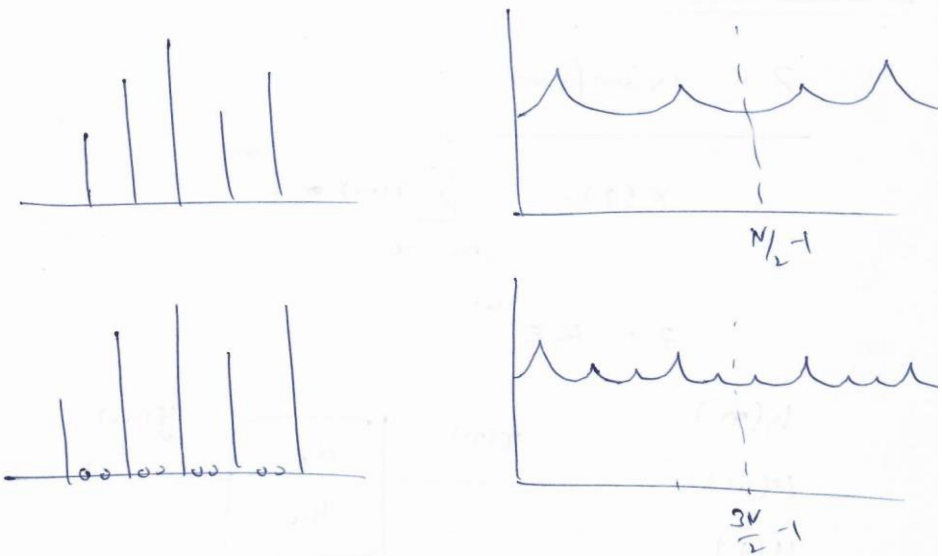
Filtering in t

Important DFT properties

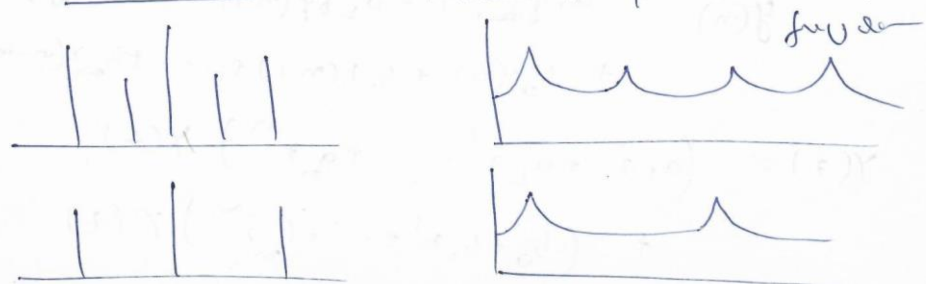
Symmetry & periodicity

$N^2 \longrightarrow N \log_2 N$ (FFT computation)

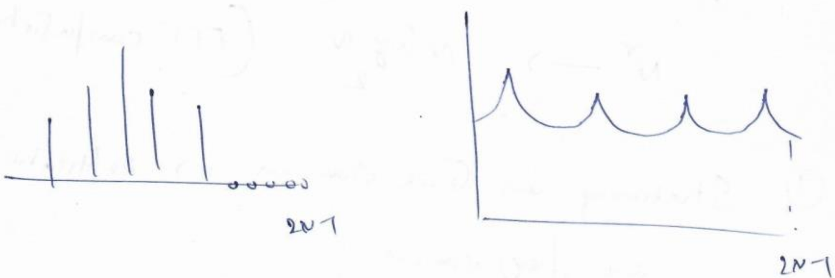
(1) Stretching in time domain => Replication in freq domain.



(2) Decimation => remove samples => aliasing in freq domain



(3) Appending zeros in time domain => interpolation in freq domain



2D Impulse and its properties

$$\delta(t,z) = \begin{cases} 1 & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$$

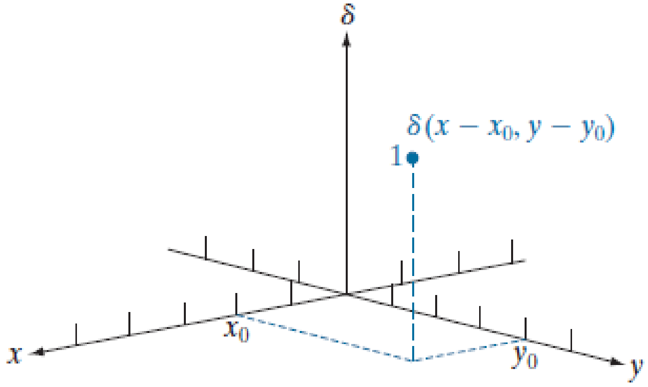
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

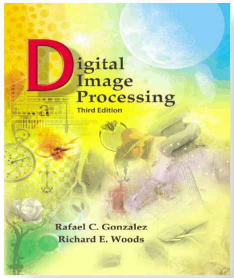
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0,z-z_0)\,dt\,dz = f(t_0,z_0)$$

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0,y-y_0) = f(x_0,y_0)$$





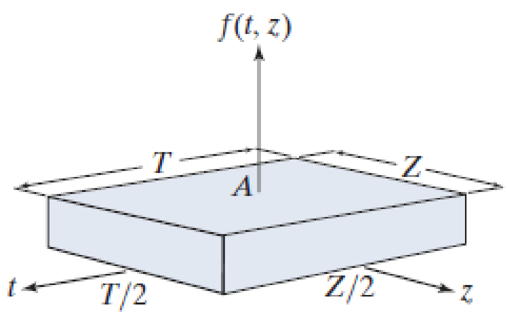
2D Continuous FT pair

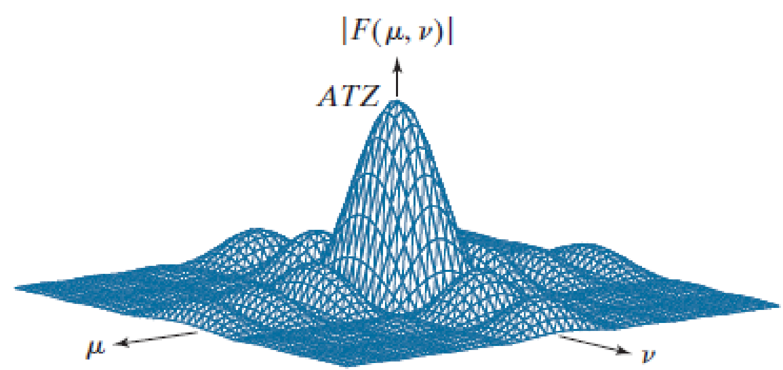
$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t + \nu z)}\,dt\,dz$$

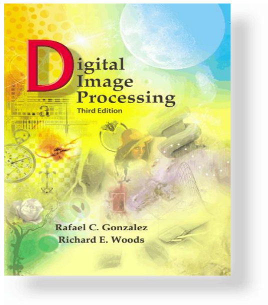
$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t + \nu z)}\,d\mu\,d\nu$$

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t + \nu z)}\,dt\,dz = \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A\,e^{-j2\pi(\mu t + \nu z)}\,dt\,dz$$

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$







2D Sampling and Sampling Theorem

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

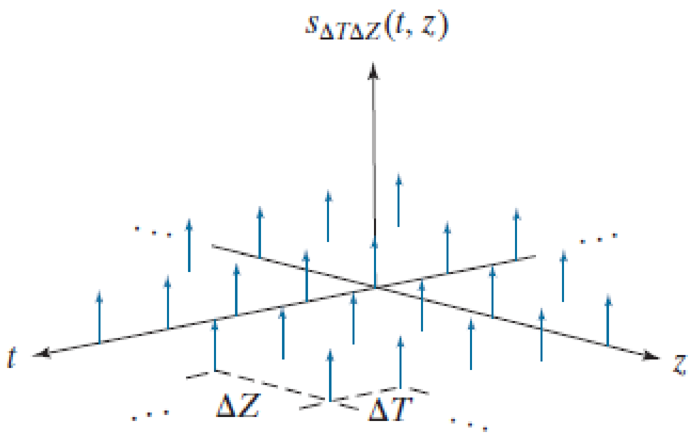
$$F(\mu,\nu) = 0 \quad \text{for} \quad |\mu| \geq \mu_{\max} \quad \text{and} \quad |\nu| \geq \nu_{\max}$$

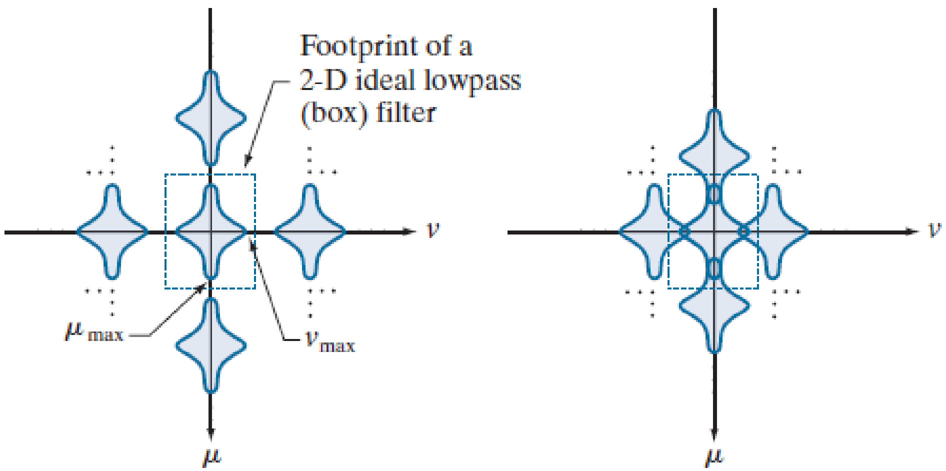
$$\Delta T < \frac{1}{2\mu_{\max}}$$

$$\Delta Z < \frac{1}{2\nu_{\max}}$$

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

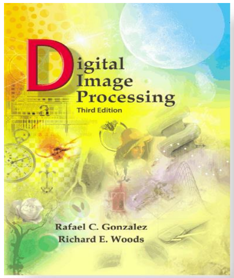
$$\frac{1}{\Delta Z} > 2\nu_{\max}$$



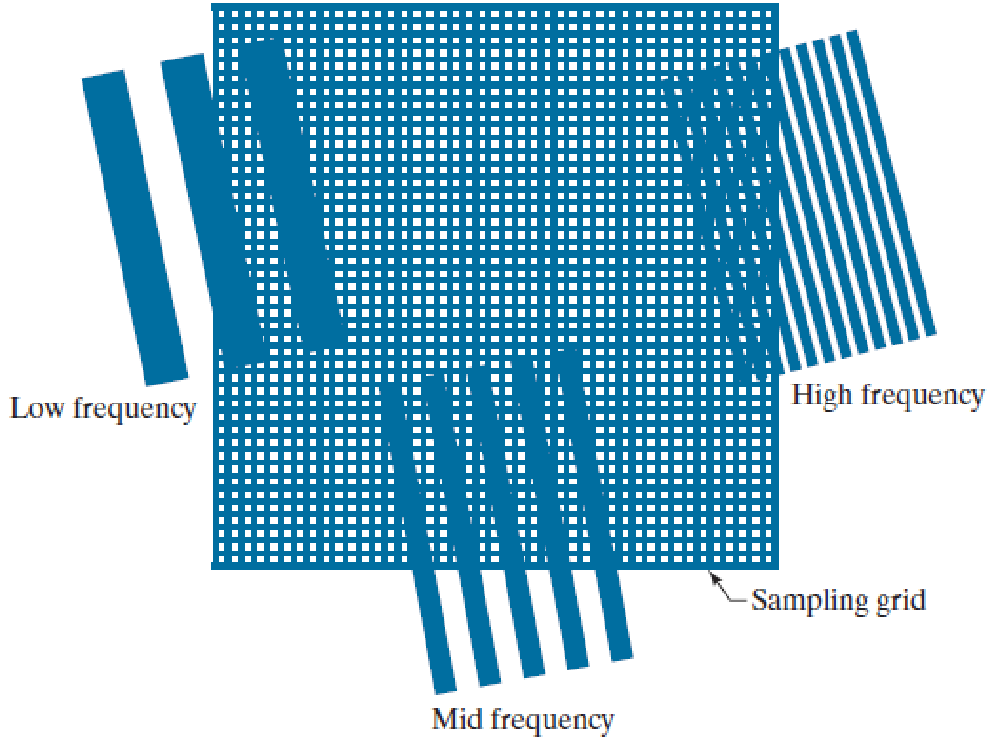


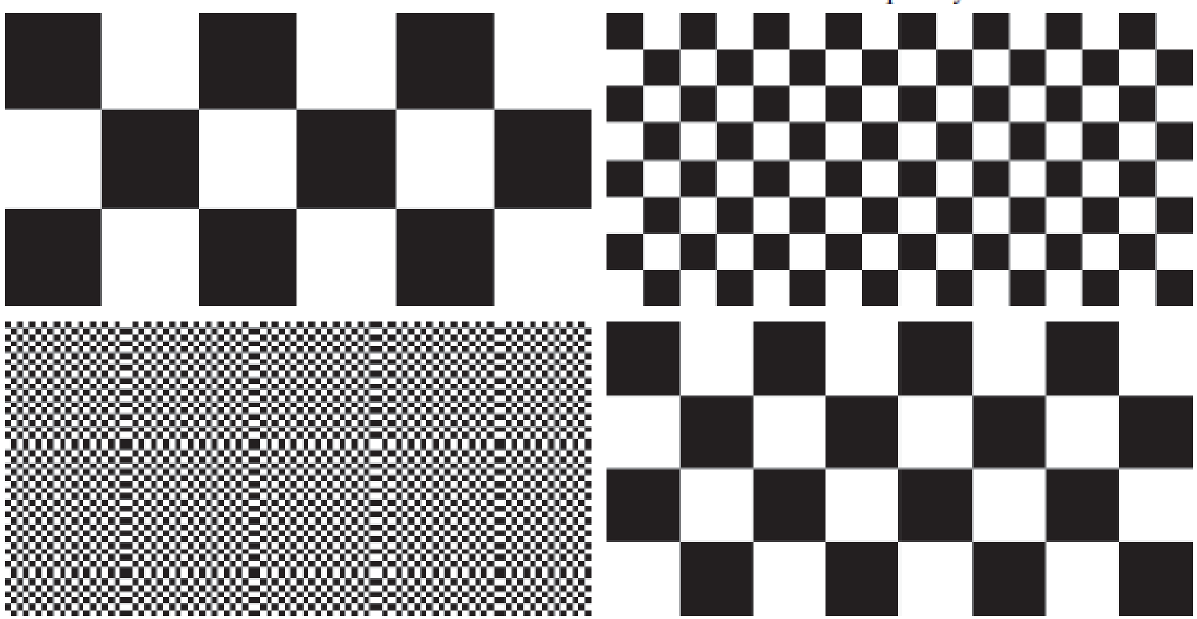
a b

FIGURE 4.16 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) an under-sampled, band-limited function.



Aliasing in Images





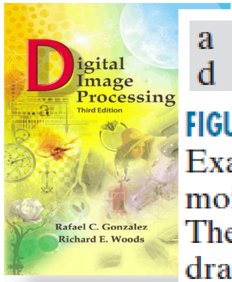
a b
c d

FIGURE 4.18 Aliasing. In (a) and (b) the squares are of sizes 16 and 6 pixels on the side. In (c) and (d) the squares are of sizes 0.95 and 0.48 pixels, respectively. Each small square in (c) is one pixel. Both (c) and (d) are aliased. Note how (d) masquerades as a "normal" image.

Illustration of the effect of Anti-aliasing filter (Image resampling & Interpolation)

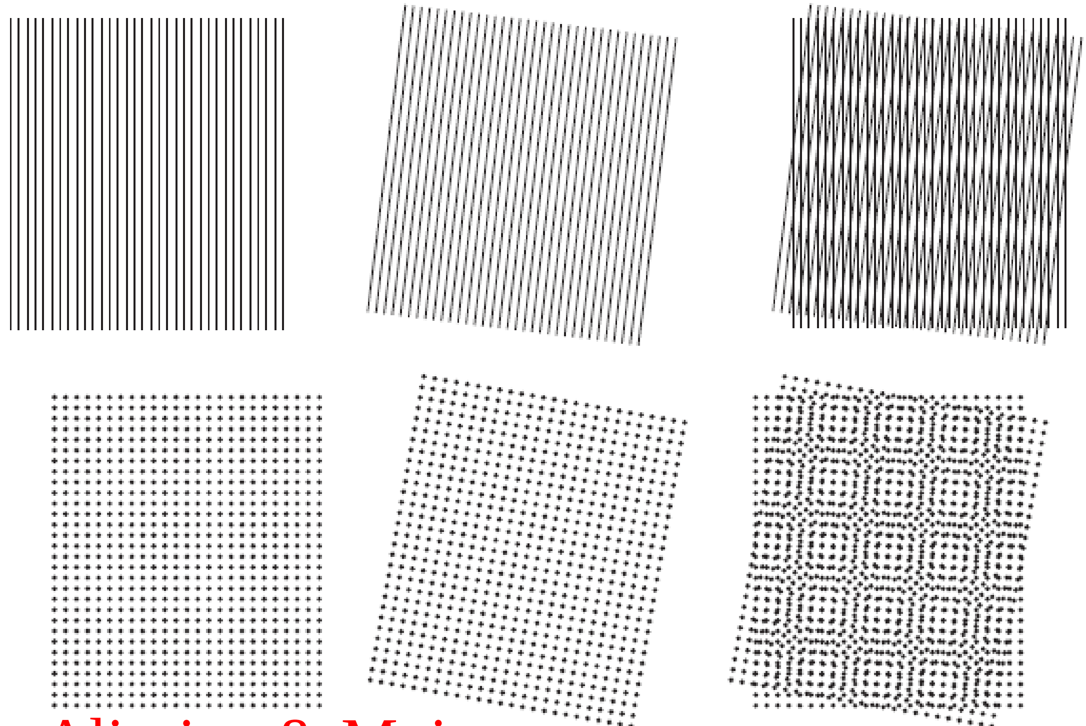
a b c

FIGURE 4.19 Illustration of aliasing on resampled natural images. (a) A digital image of size 772×548 pixels with visually negligible aliasing. (b) Result of resizing the image to 33% of its original size by pixel deletion and then restoring it to its original size by pixel replication. Aliasing is clearly visible. (c) Result of blurring the image in (a) with an averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)



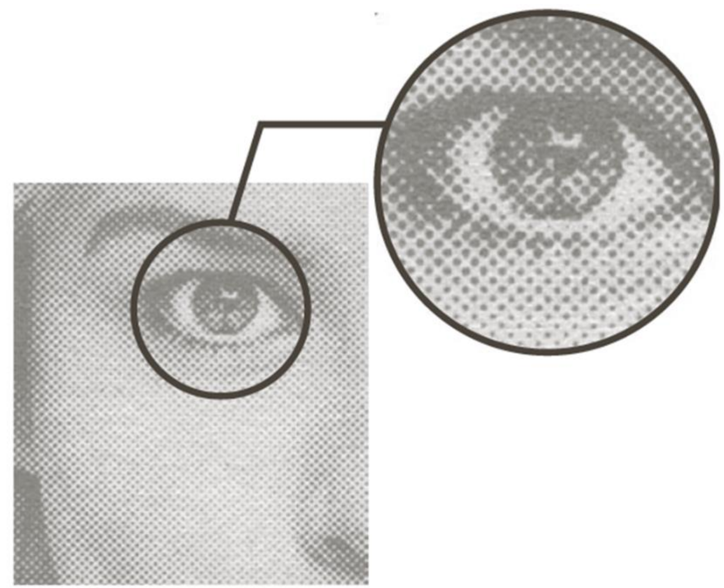
a b c
d e f

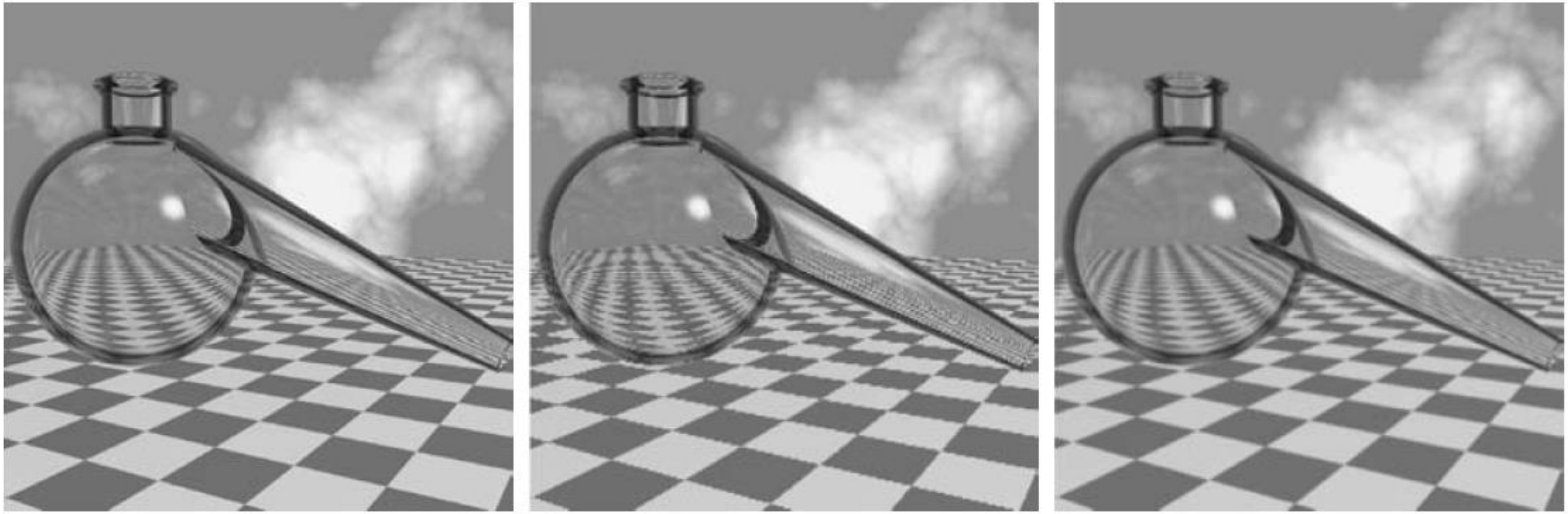
FIGURE 4.20 Examples of the moiré effect. These are vector drawings, not digitized patterns. Superimposing one pattern on the other is analogous to multiplying the patterns.



Aliasing & Moire patterns

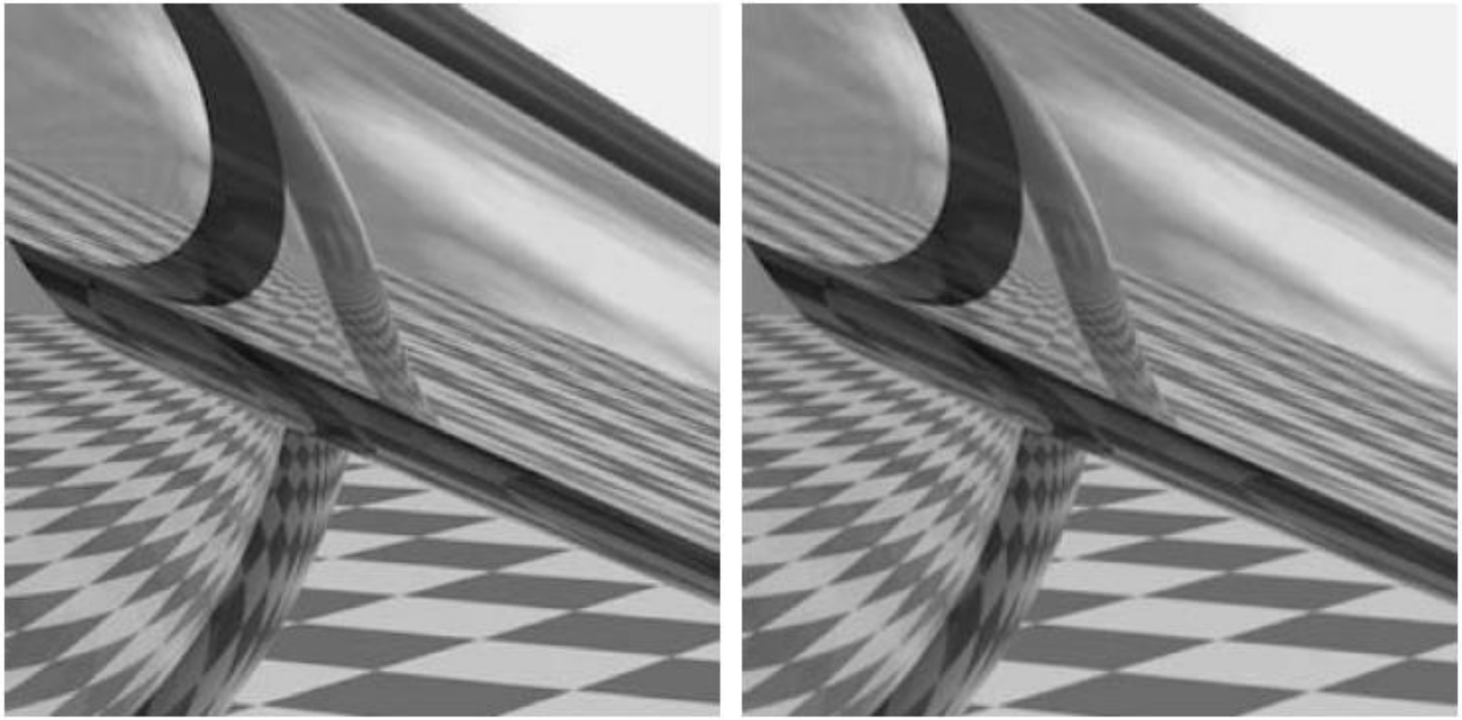
Aliasing due to
half-tone dots
and sampling grid





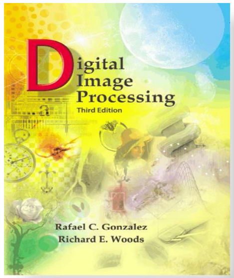
a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)



a b

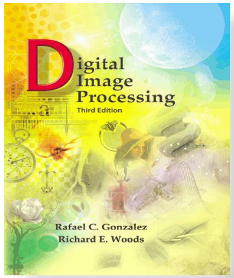
FIGURE 4.19 Image zooming. (a) A 1024×1024 digital image generated by pixel replication from a 256×256 image extracted from the middle of Fig. 4.18(a). (b) Image generated using bi-linear interpolation, showing a significant reduction in jaggies.



2-D Discrete FT and its Inverse

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M + vy/N)}$$

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M + vy/N)}$$



Properties of 2-D DFT and IDFT

Relationships between spatial and frequency intervals

$$\Delta u = \frac{1}{M\Delta T} \qquad \Delta v = \frac{1}{N\Delta Z}$$

Translation & Rotation

$$f(x,y)e^{j2\pi(u_0 x/M + v_0 y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad u = \omega\cos\varphi \qquad v = \omega\sin\varphi$$

$$f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$$

Properties of 2-D DFT and IDFT

Periodicity

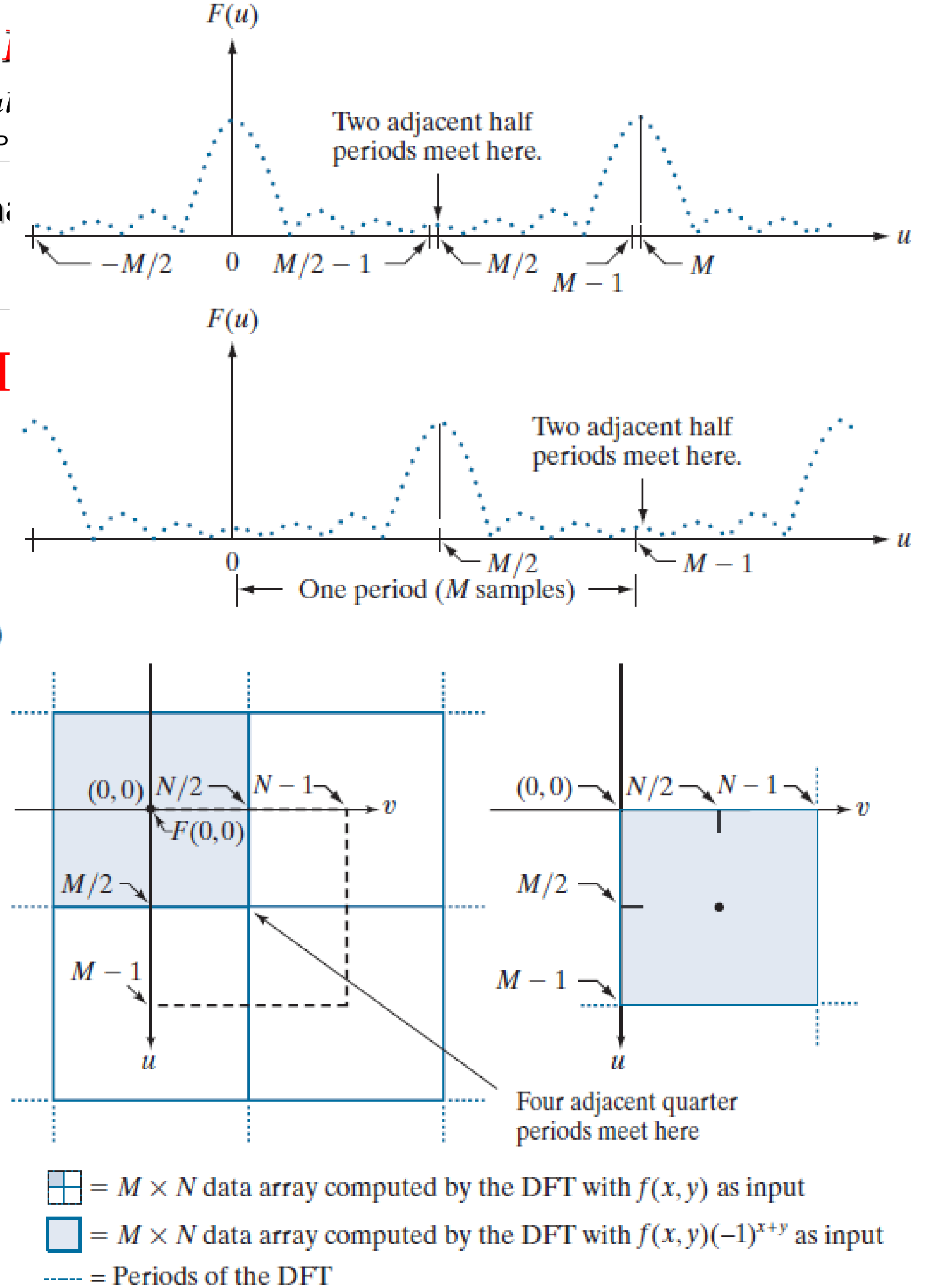
$$F(u,v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$$

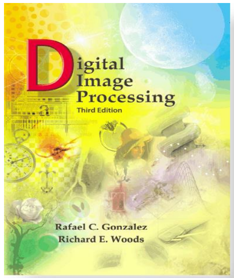
$$f(x,y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$$

$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u - u_0)$$

$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$





Chapter 4
Filtering in the Frequency Domain

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u,v)$ and $I(u,v)$ are the real and imaginary parts of $F(u,v)$, respectively. Use of the word *complex* indicates that a function has nonzero real and imaginary parts.

	Spatial Domain[†]		Frequency Domain[†]
1)	$f(x,y)$ real	$\Leftrightarrow$	$F^*(u,v) = F(-u,-v)$
2)	$f(x,y)$ imaginary	$\Leftrightarrow$	$F^*(-u,-v) = -F(u,v)$
3)	$f(x,y)$ real	$\Leftrightarrow$	$R(u,v)$ even; $I(u,v)$ odd
4)	$f(x,y)$ imaginary	$\Leftrightarrow$	$R(u,v)$ odd; $I(u,v)$ even
5)	$f(-x,-y)$ real	$\Leftrightarrow$	$F^*(u,v)$ complex
6)	$f(-x,-y)$ complex	$\Leftrightarrow$	$F(-u,-v)$ complex
7)	$f^*(x,y)$ complex	$\Leftrightarrow$	$F^*(-u,-v)$ complex
8)	$f(x,y)$ real and even	$\Leftrightarrow$	$F(u,v)$ real and even
9)	$f(x,y)$ real and odd	$\Leftrightarrow$	$F(u,v)$ imaginary and odd
10)	$f(x,y)$ imaginary and even	$\Leftrightarrow$	$F(u,v)$ imaginary and even
11)	$f(x,y)$ imaginary and odd	$\Leftrightarrow$	$F(u,v)$ real and odd
12)	$f(x,y)$ complex and even	$\Leftrightarrow$	$F(u,v)$ complex and even
13)	$f(x,y)$ complex and odd	$\Leftrightarrow$	$F(u,v)$ complex and odd

[†]Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M-1]$, and y and v in the range $[0, N-1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an *odd* complex function. As before, "$\Leftrightarrow$" indicates a Fourier transform pair.

Symmetry properties

Fourier Spectrum & Phase Angle

$$F(u,v) = R(u,v) + jI(u,v) \\ = |F(u,v)|e^{j\phi(u,v)}$$

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

$$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

$$P(u,v) = |F(u,v)|^2 \\ = R^2(u,v) + I^2(u,v)$$

$$|F(u,v)| = |F(-u,-v)|$$

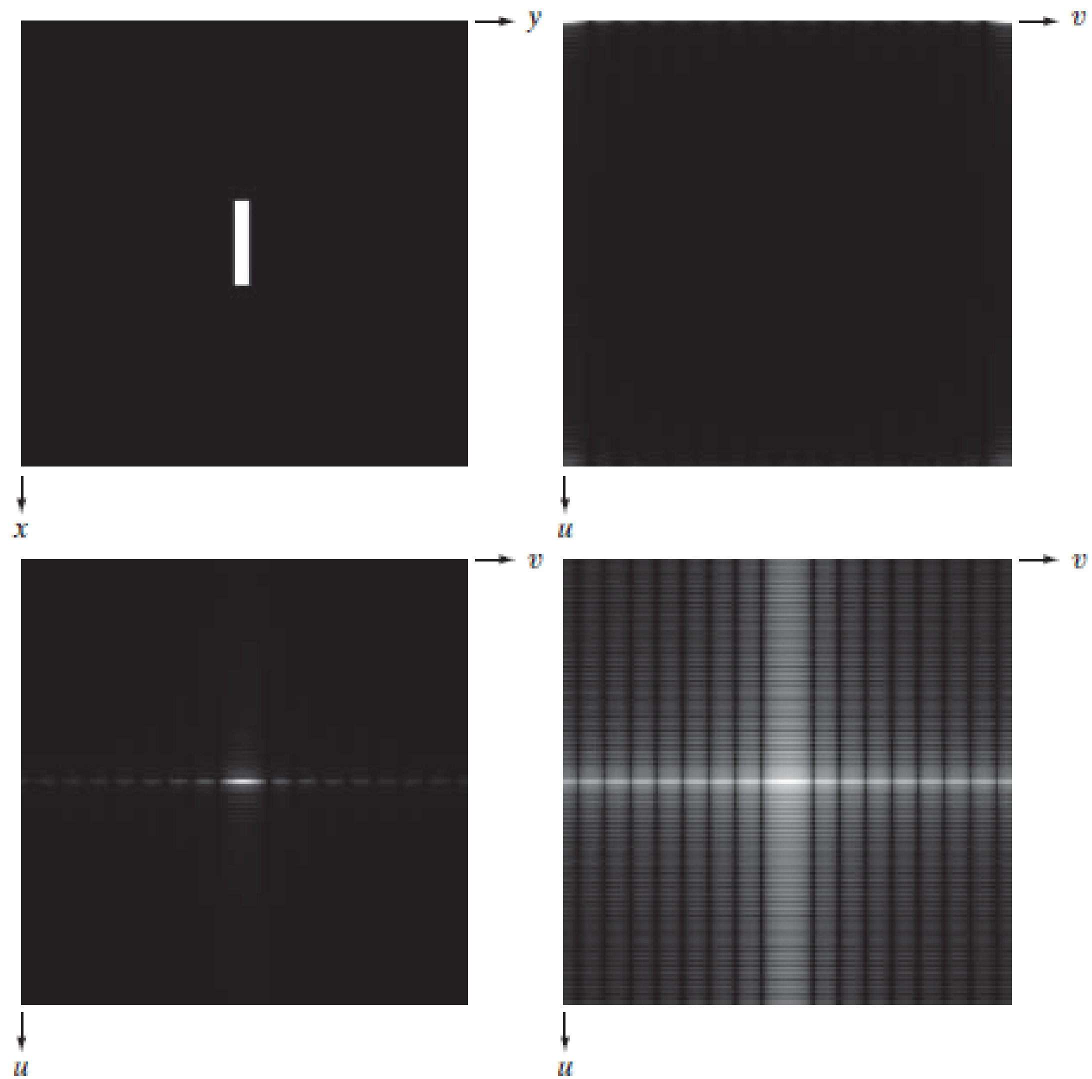
$$\phi(u,v) = -\phi(-u,-v)$$

$$F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \\ = MN\bar{f}$$

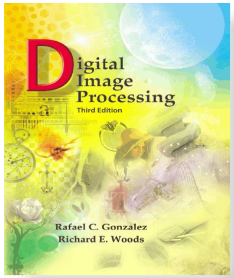
$$|F(0,0)| = MN|\bar{f}|$$

The Spectrum of a Rectangle

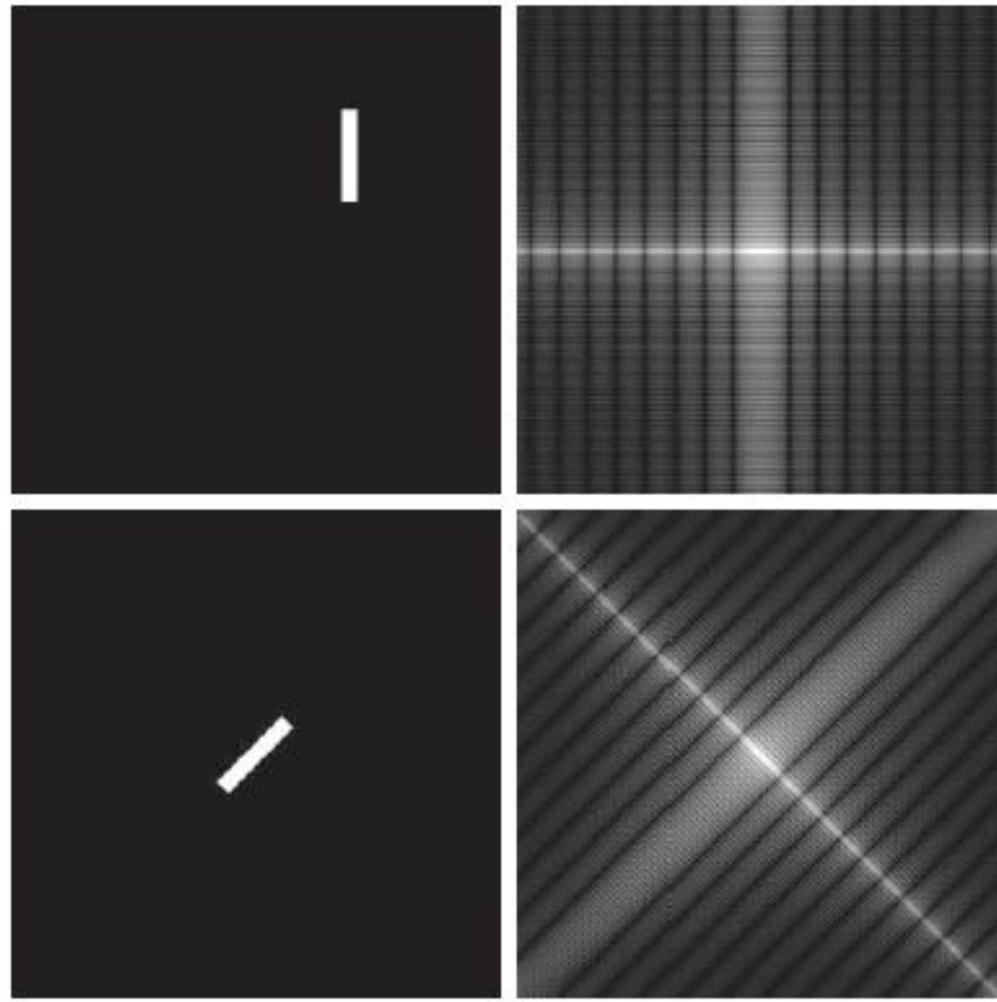


a b
c d

FIGURE 4.23 (a) Image. (b) Spectrum, showing small, bright areas in the four corners (you have to look carefully to see them). (c) Centered spectrum. (d) Result after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The right-handed coordinate convention used in the book places the origin of the spatial and frequency domains at the top left (see Fig. 2.19).

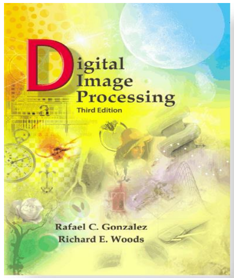


Spectrum of a Rectangle in Translation and Rotation



a b
c d

FIGURE 4.24 (a) The rectangle in Fig. 4.23(a) translated. (b) Corresponding spectrum. (c) Rotated rectangle. (d) Corresponding spectrum. The spectrum of the translated rectangle is identical to the spectrum of the original image in Fig. 4.23(a).



The phase spectrum of a Rectangle : Centered, Translated and Rotated Positions

a b c

FIGURE 4.25 Phase angle images of (a) centered, (b) translated, and (c) rotated rectangles.

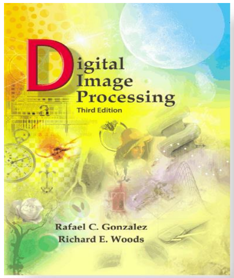
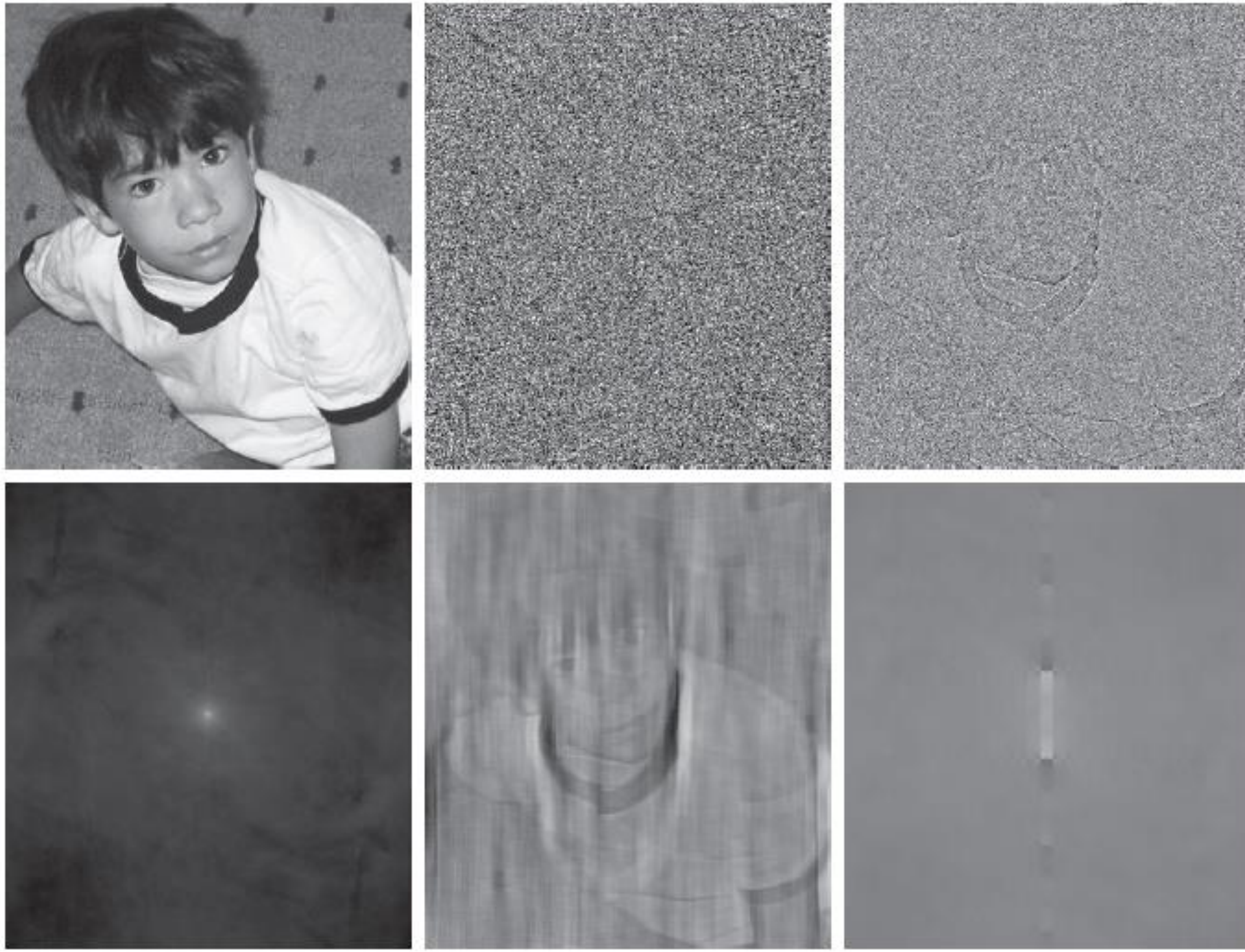


Image reconstruction using
(c) only phase,
(d) only mag,
(e) boy-phase + rect-mag and
(f) rectphase + boy-mag



a	b	c
d	e	f

FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape feature are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Bo image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum o the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's imag

2-D Discrete Convolution Theorem

$$(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

$$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$$

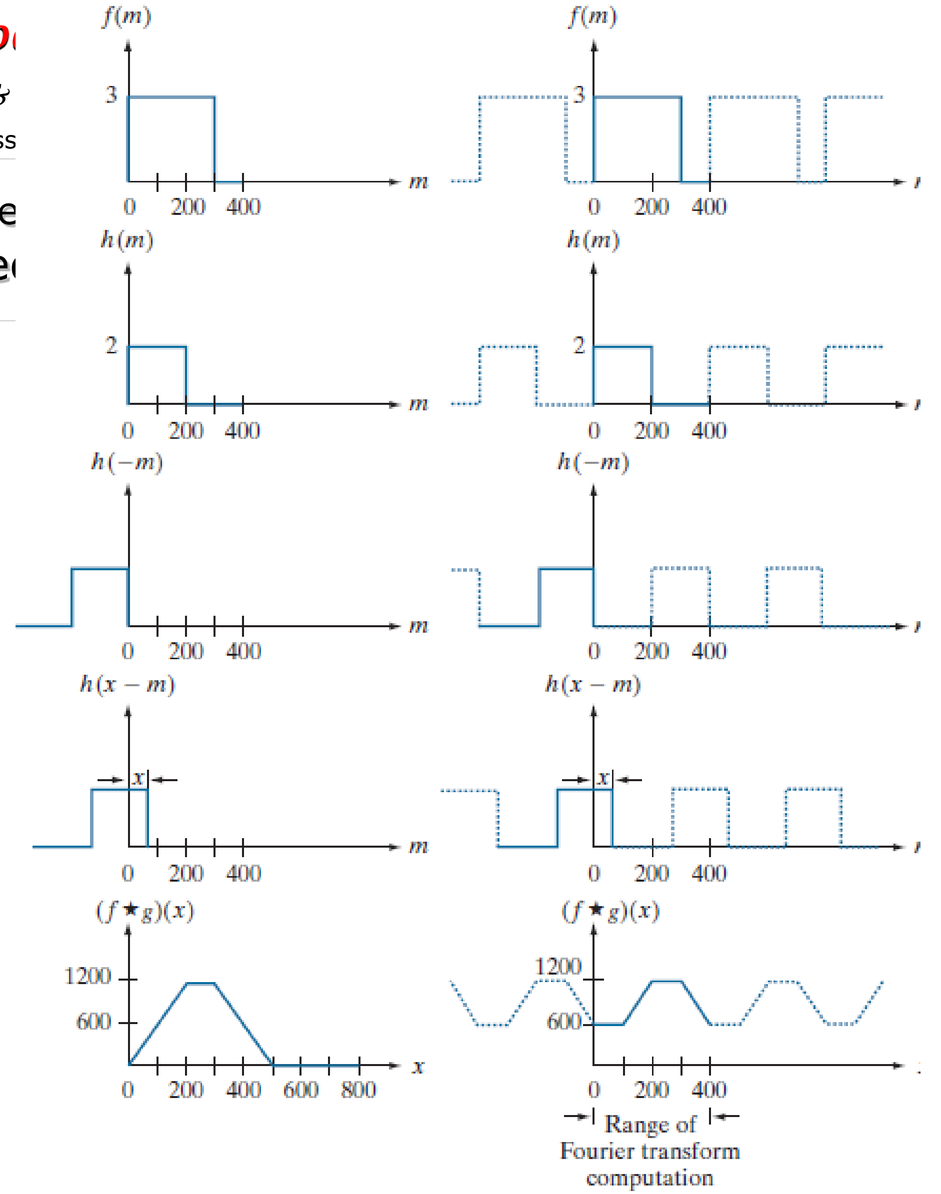
$$(f \bullet h)(x,y) \Leftrightarrow \frac{1}{MN}(F \star H)(u,v)$$

$$P \geq A + B - 1$$

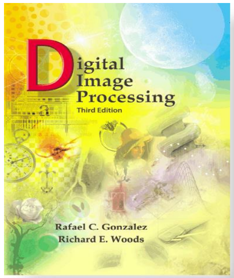
$$f_p(x,y) = \begin{cases} f(x,y) & 0 \leq x \leq A-1 \text{ and } 0 \leq y \leq B-1 \\ 0 & A \leq x \leq P \text{ or } B \leq y \leq Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y) & 0 \leq x \leq C-1 \text{ and } 0 \leq y \leq D-1 \\ 0 & C \leq x \leq P \text{ or } D \leq y \leq Q \end{cases}$$

$$P \geq A + C - 1 \qquad Q \geq B + D - 1$$

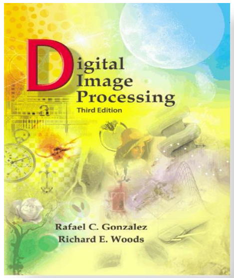


Frequency leakage due to padding



Summary of 2-D DFT definitions

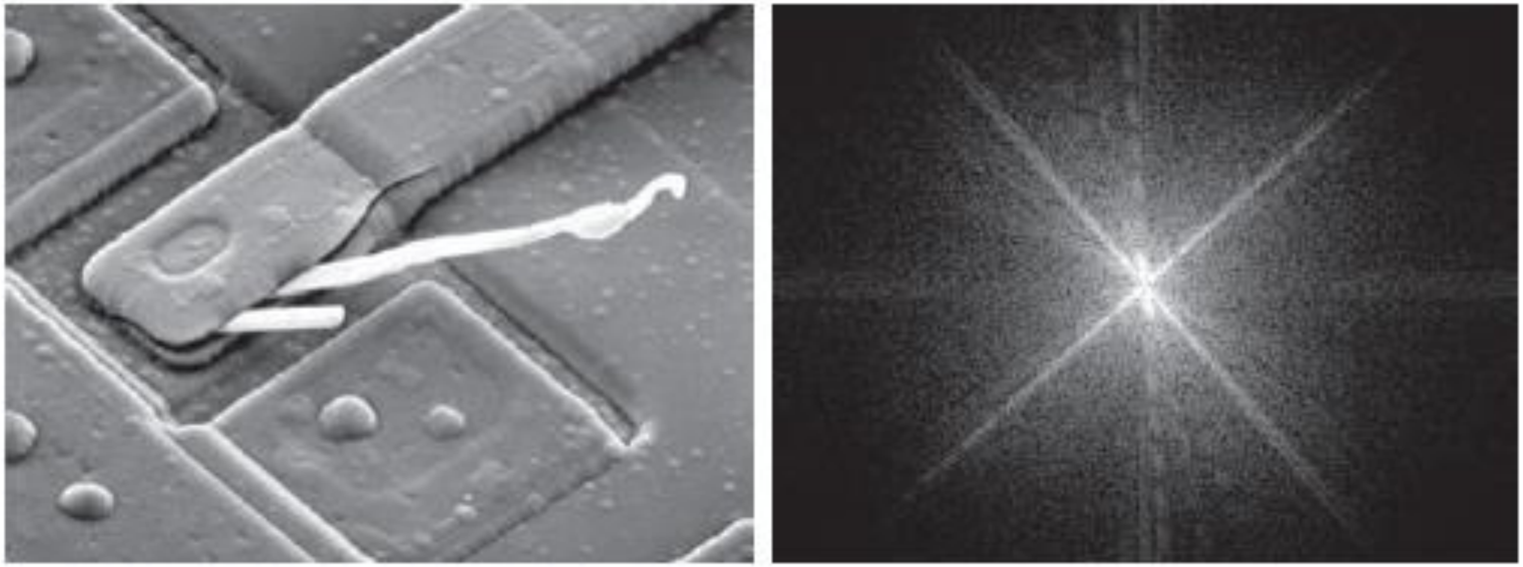
Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x,y)$	$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u,v)$	$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
3) Spectrum	$\lvert F(u,v)\rvert=\left[R^2(u,v)+I^2(u,v)\right]^{1/2}$ $R=\text{Real}(F);\ I=\text{Imag}(F)$
4) Phase angle	$\phi(u,v)=\tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
5) Polar representation	$F(u,v)=\lvert F(u,v)\rvert e^{j\phi(u,v)}$
6) Power spectrum	$P(u,v)=\lvert F(u,v)\rvert^2$
7) Average value	$\bar{f}=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)=\frac{1}{MN}F(0,0)$
8) Periodicity (k_1 and k_2 are integers)	$F(u,v)=F(u+k_1M,v)=F(u,v+k_2N)=F(u+k_1,v+k_2N)$ $f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)=f(x+k_1M,y+k_2N)$
9) Convolution	$(f\star h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$
10) Correlation	$(f\,☆\,h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m,y+n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.
12) Obtaining the IDFT using a DFT algorithm	$MNf^*(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.



Properties of 2-D DFT pairs

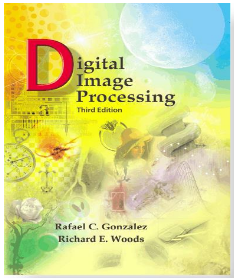
Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x,y) + bf_2(x,y) \Leftrightarrow aF_1(u,v) + bF_2(u,v)$
3) Translation (general)	$f(x,y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
5) Rotation	$f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$ $r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}(y/x) \quad \omega = \sqrt{u^2+v^2} \quad \varphi = \tan^{-1}(v/u)$
6) Convolution theorem†	$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$ $(f \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \star H)(u,v)]$
7) Correlation theorem†	$(f \,☆\, h)(x,y) \Leftrightarrow (F^* \bullet H)(u,v)$ $(f^* \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \,☆\, H)(u,v)]$
8) Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9) Rectangle	$\mathrm{rec}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{jMN}{2}[\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0)]$
11) Cosine	$\cos(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{1}{2}[\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0)]$
The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12) Differentiation (the expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu); \quad \dfrac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13) Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

Association of Spatial Features Frequency domain features



a b

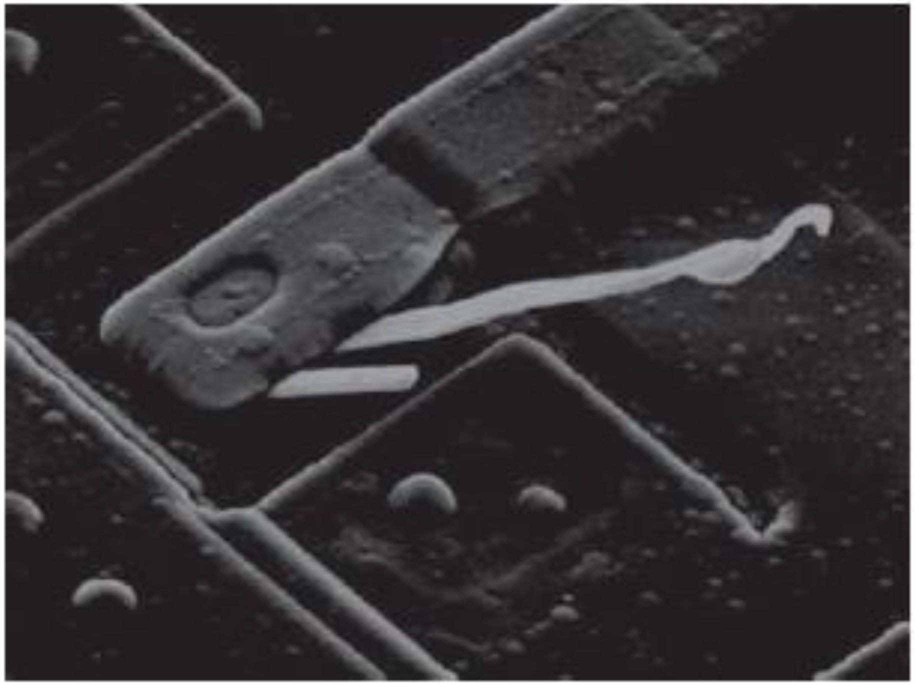
FIGURE 4.28 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

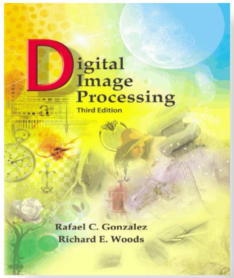


Filtering in Frequency Domain

$$g(x, y) = \text{Real}\left\{\Im^{-1}\left[H(u,v)F(u,v)\right]\right\}$$

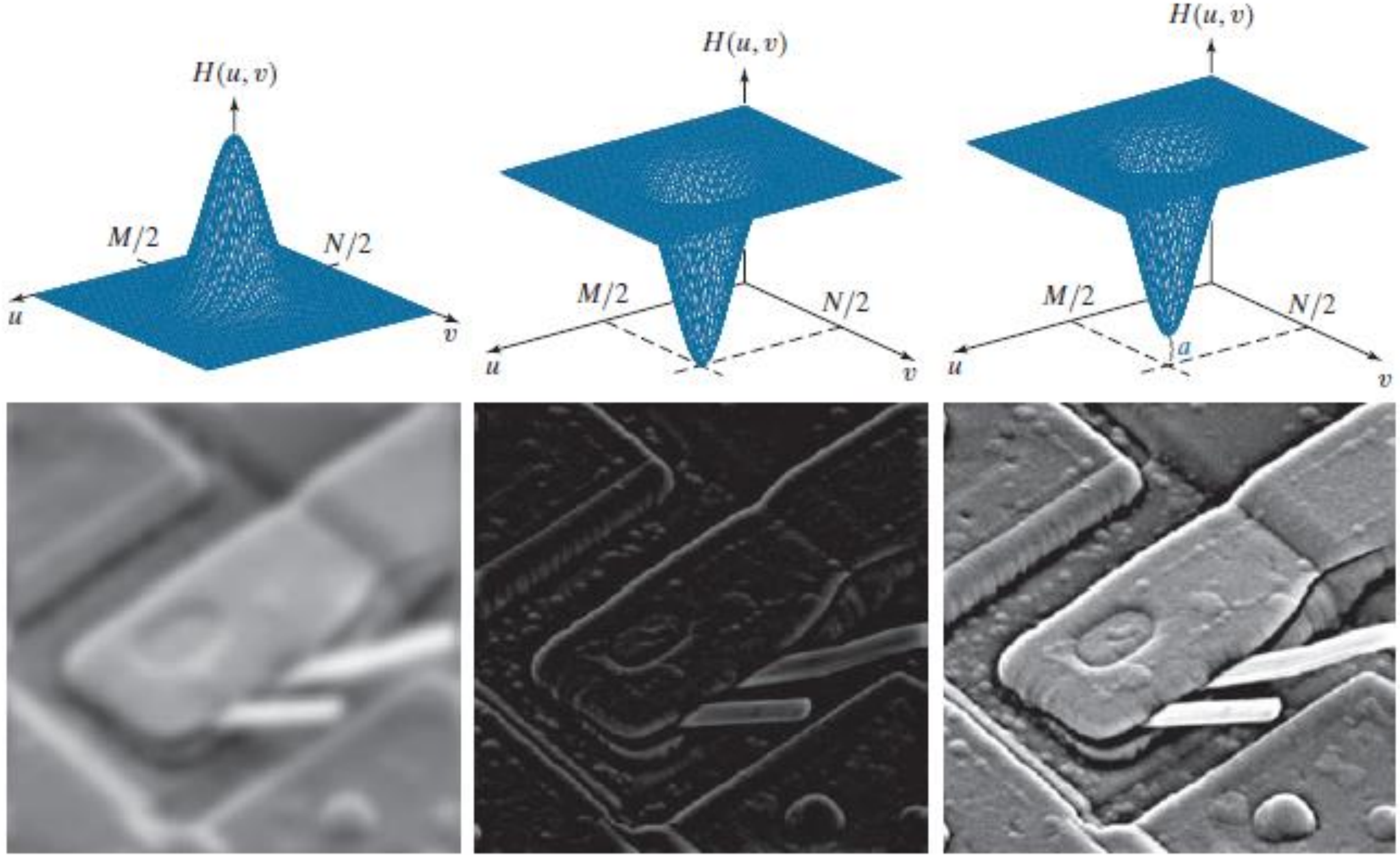
FIGURE 4.29 Result of filtering the image in Fig. 4.28(a) with a filter transfer function that sets to 0 the dc term, $F(P/2, Q/2)$, in the centered Fourier transform, while leaving all other transform terms unchanged.





Chapter 4
Filtering in the Frequency Domain

Filtering in Frequency Domain



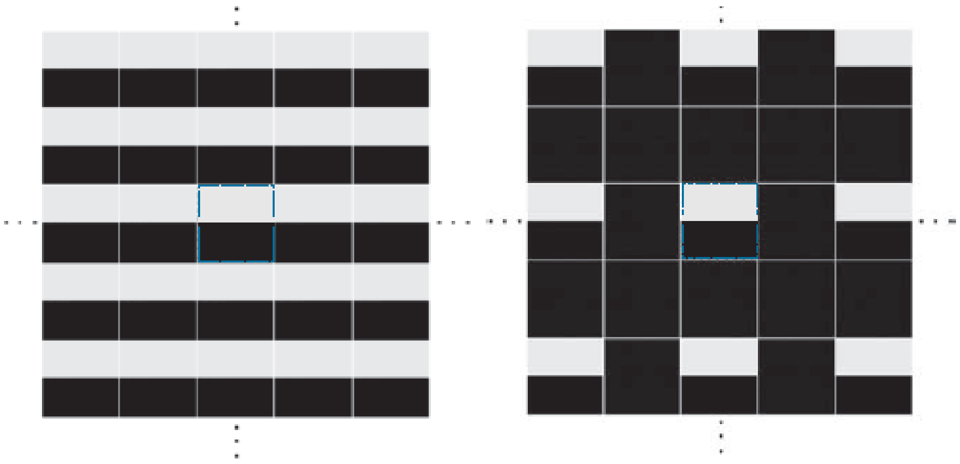
a b c
d e f

FIGURE 4.30 Top row: Frequency domain filter transfer functions of (a) a lowpass filter, (b) a highpass filter, and (c) an offset highpass filter. Bottom row: Corresponding filtered images obtained using Eq. (4-104). The offset in (c) is $a = 0.85$, and the height of $H(u, v)$ is 1. Compare (f) with Fig. 4.28(a).

Illustration of Padding in Frequency Domain Filtering to avoid wrap-around error

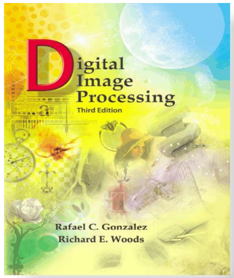
a b c

FIGURE 4.31 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with zero padding. Compare the vertical edges in (b) and (c).



a b

FIGURE 4.32 (a) Image periodicity without image padding. (b) Periodicity after padding with 0's (black). The dashed areas in the center correspond to the image in Fig. 4.31(a). Periodicity is inherent when using the DFT. (The thin white lines in both images are superimposed for clarity; they are not part of the data.)



Artifacts of padding the filter in spatial domain

a c
b d

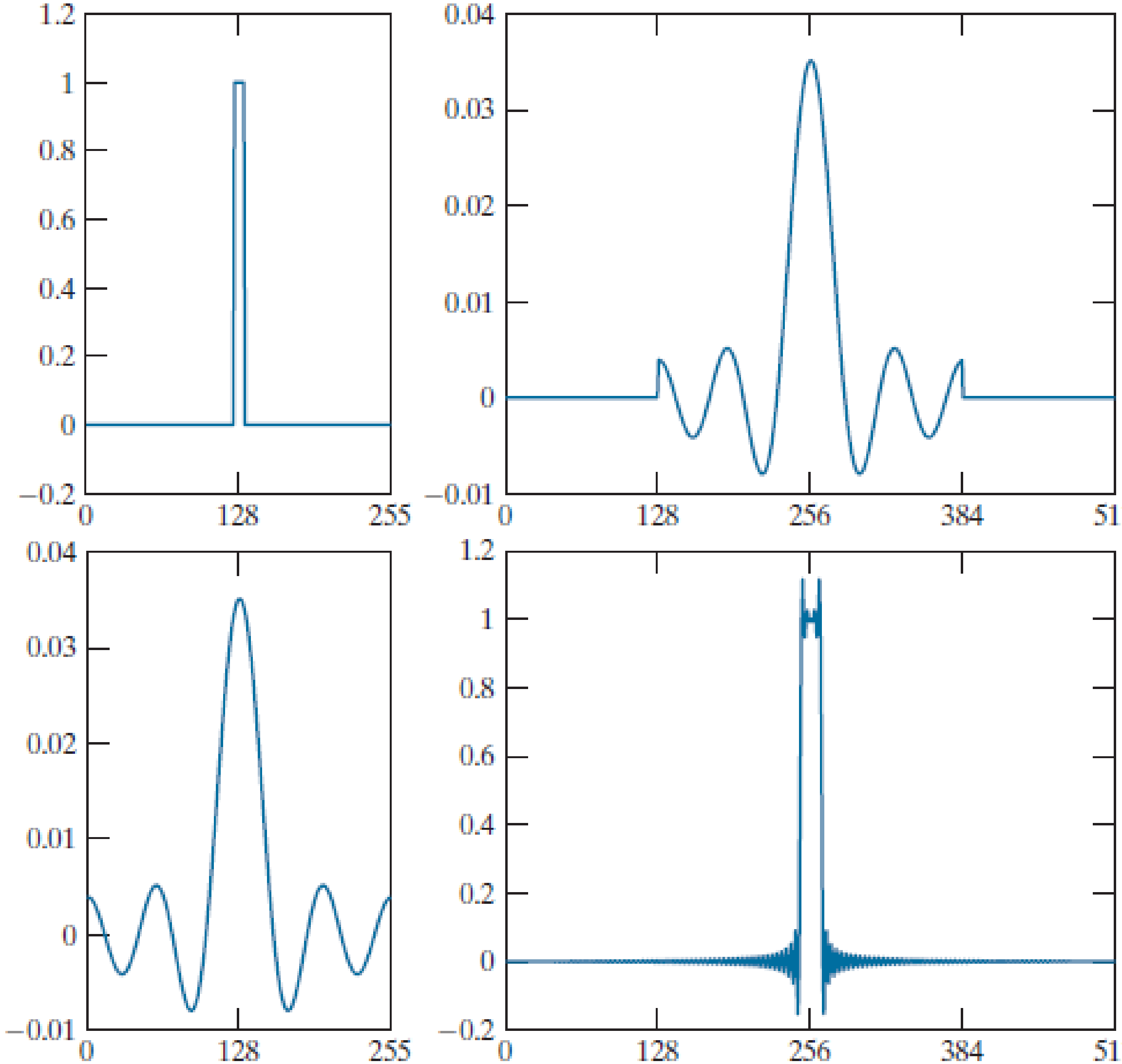
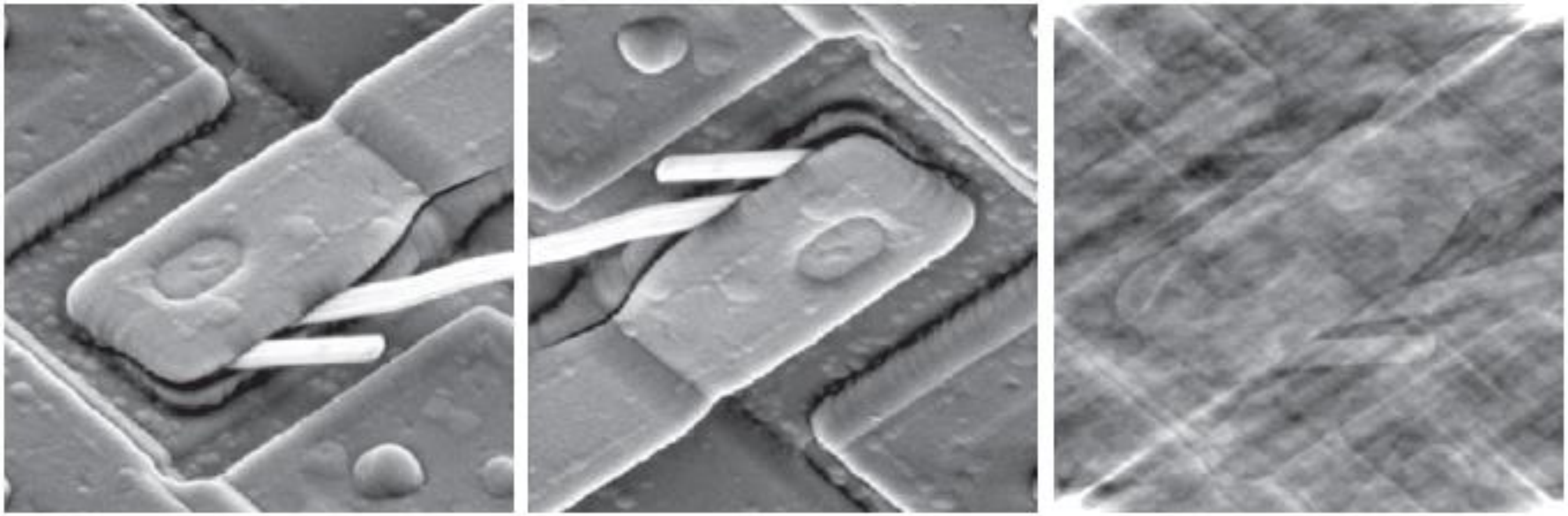


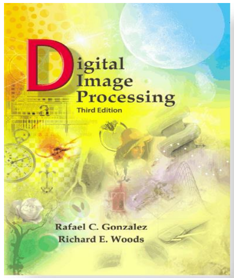
FIGURE 4.33 (a) Filter transfer function specified in the (centered) frequency domain. (b) Spatial representation (filter kernel) obtained by computing the IDFT of (a). (c) Result of padding (b) to twice its length (note the discontinuities). (d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). Part (b) of the figure is below (a), and (d) is below (c).

Significance of Phase in Image reconstruction



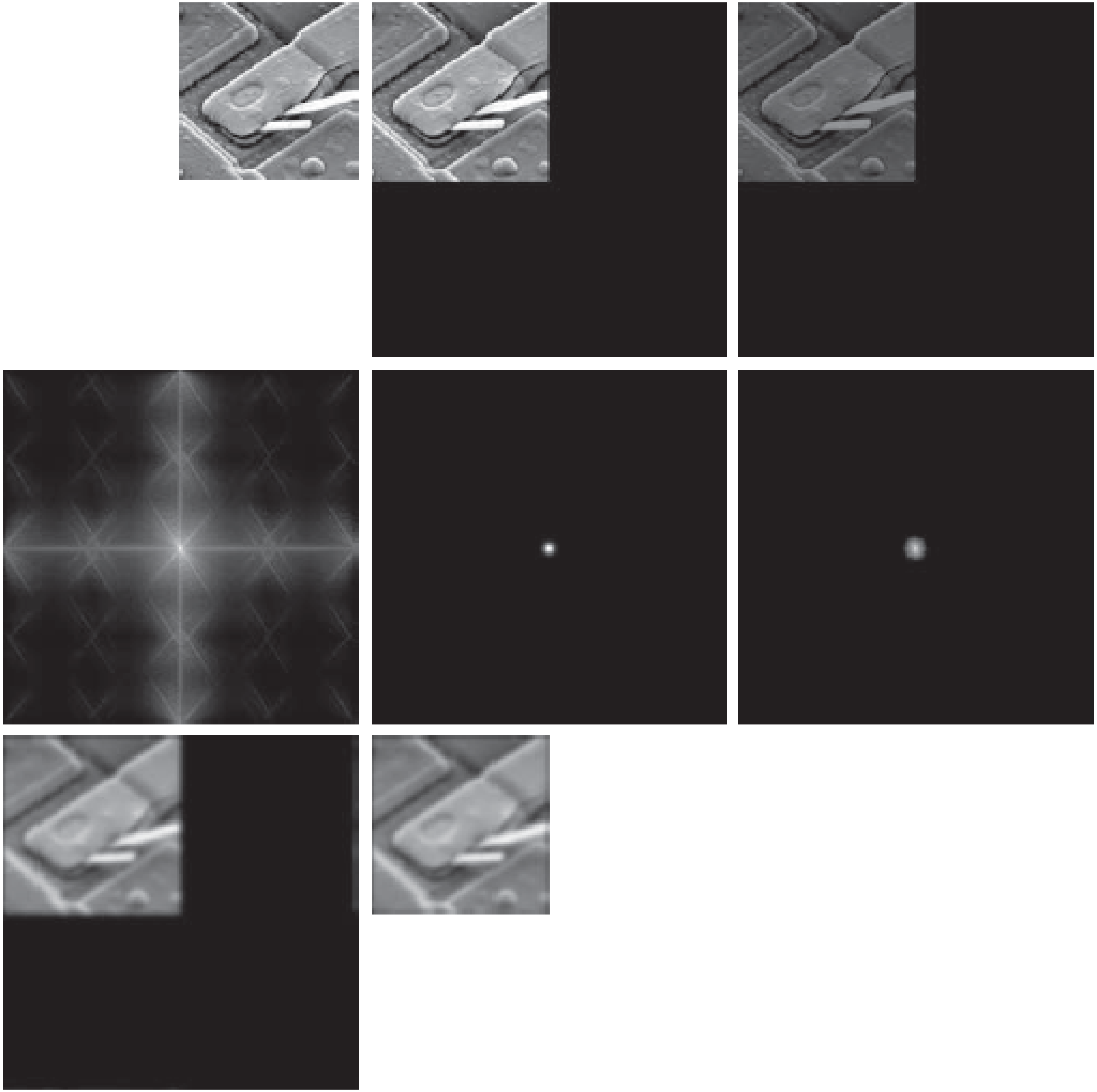
a b c

FIGURE 4.34 (a) Original image. (b) Image obtained by multiplying the phase angle array by –1 in Eq. (4-86) and computing the IDFT. (c) Result of multiplying the phase angle by 0.25 and computing the IDFT. The magnitude of the transform, $|F(u,v)|$, used in (b) and (c) was the same.



Steps in Filter Design in Frequency Domain

1. f(x,y) is MxN. Pad to PxQ. Typically, P=2M, Q=2N
2. Form fp(x,y) of size PxQ by adding necessary zeros to f(x,y)
3. Multiply fp(x,y) by (–1)^(x+y) to center transform
4. Compute F(u,v) by DFT of fp(x,y)
5. Use real symmetric filter H(u,v), of size PxQ & center at (P/2,Q/2). Form G(u,v)=H(u,v)F(u,v)
6. Compute gp(x,y) by product of real part of IDFT of G(u,v) and (–1)^(x+y).
7. Extract g(x,y) taking MxN at left top corner of gp(x,y)



a b c
d e f
g h

FIGURE 4.35
(a) An $M \times N$ image, f.
(b) Padded image, f_p of size $P \times Q$.
(c) Result of multiplying f_p by $(-1)^{x+y}$.
(d) Spectrum of F. (e) Centered Gaussian lowpass filter transfer function, H, of size $P \times Q$.
(f) Spectrum of the product HF.
(g) Image g_p, the real part of the IDFT of HF, multiplied by $(-1)^{x+y}$.
(h) Final result, g, obtained by extracting the first M rows and N columns of g_p.

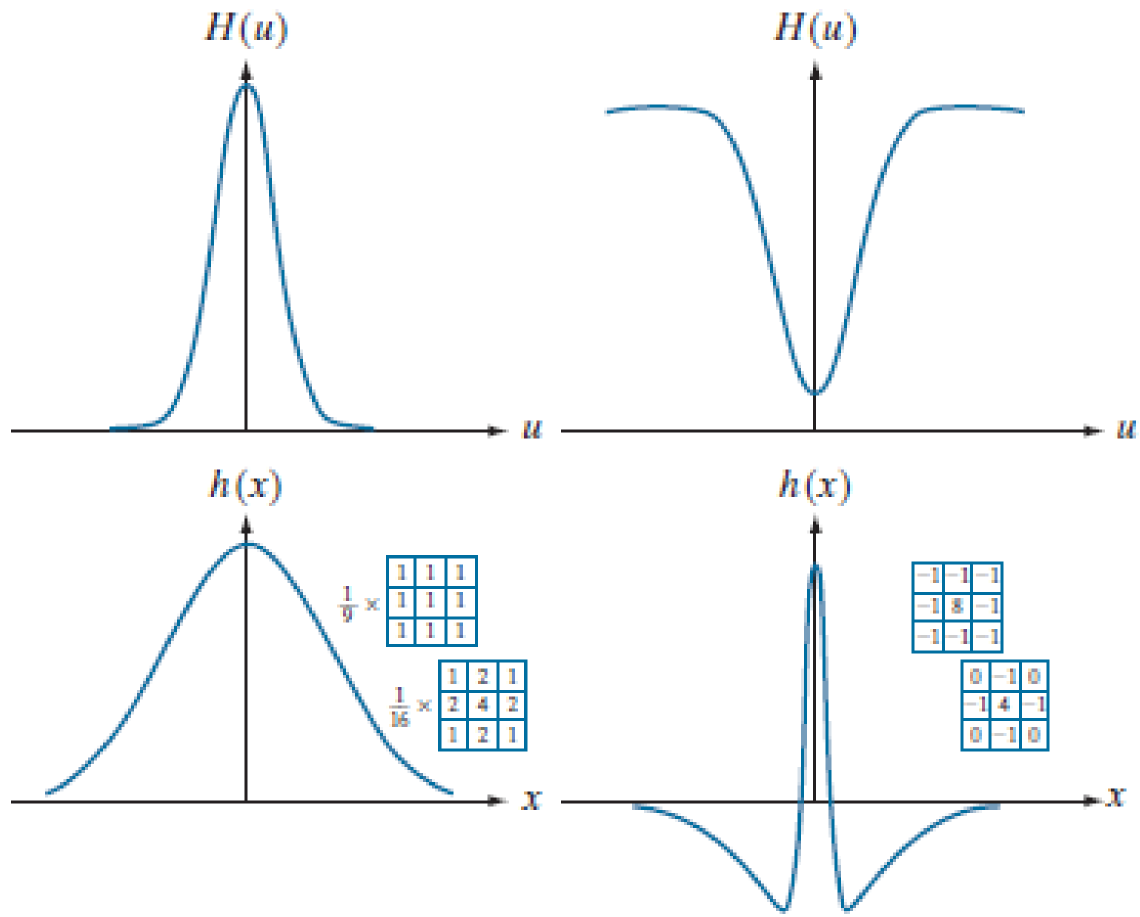
Correspondence between filtering in spatial and frequency domains

$$H(u) = Ae^{-u^2/2\sigma^2}$$

$$h(x) = \sqrt{2\pi}\sigma A\, e^{-2\pi^2\sigma^2 x^2}$$

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

$$h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2 x^2}$$



Chapter 4
Filtering in the Frequency Domain

a b

FIGURE 4.37 (a) Image of a building, and (b) its Fourier spectrum.

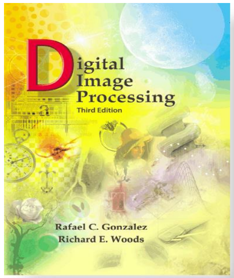


Illustration of Sobel operator (Filter) implementation in spatial and frequency domains

a b
c d

FIGURE 4.38
(a) A spatial kernel and perspective plot of its corresponding frequency domain filter transfer function.
(b) Transfer function shown as an image.
(c) Result of filtering Fig. 4.37(a) in the frequency domain with the transfer function in (b).
(d) Result of filtering the same image in the spatial domain with the kernel in (a). The results are identical.

−1	0	1
−2	0	2
−1	0	1