Fourier Series

Lecture 14: 27-Aug-12

Dr. P P Das

Source

- This presentation is lifted from:

http://www.ele.uri.edu/courses/ele436/FourierSeries.ppt

Content

- Periodic Functions
- Fourier Series
- Complex Form of the Fourier Series
- Impulse Train
- Analysis of Periodic Waveforms
- Half-Range Expansion
- Least Mean-Square Error Approximation

Fourier Series

Periodic Functions

The Mathematic Formulation

- Any function that satisfies

$$f(t) = f(t + T)$$

where T is a constant and is called the *period* of the function.

Example:

$$f(t) = \cos\frac{t}{3} + \cos\frac{t}{4}$$ Find its period.

$$f(t) = f(t+T) \Rightarrow \cos\frac{t}{3} + \cos\frac{t}{4} = \cos\frac{1}{3}(t+T) + \cos\frac{1}{4}(t+T)$$

Fact: $\cos\theta = \cos(\theta + 2m\pi)$

$$\frac{T}{3} = 2m\pi \qquad \frac{T}{4} = 2n\pi \Rightarrow T = 6m\pi \qquad T = 8n\pi \Rightarrow T = 24\pi$$

smallest T

Example:

$$f(t) = \cos\omega_1 t + \cos\omega_2 t$$ Find its period.

$$f(t) = f(t+T) \Rightarrow \cos\omega_1 t + \cos\omega_2 t = \cos\omega_1(t+T) + \cos\omega_2(t+T)$$

$$\omega_1 T = 2m\pi$$

$$\omega_2 T = 2n\pi$$

$$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{m}{n} \Rightarrow \frac{\omega_1}{\omega_2}$$ must be a rational number

Example:

$$f(t) = \cos 10t + \cos(10+\pi)t$$

Is this function a periodic one?

$$\frac{\omega_1}{\omega_2} = \frac{10}{10+\pi}$$

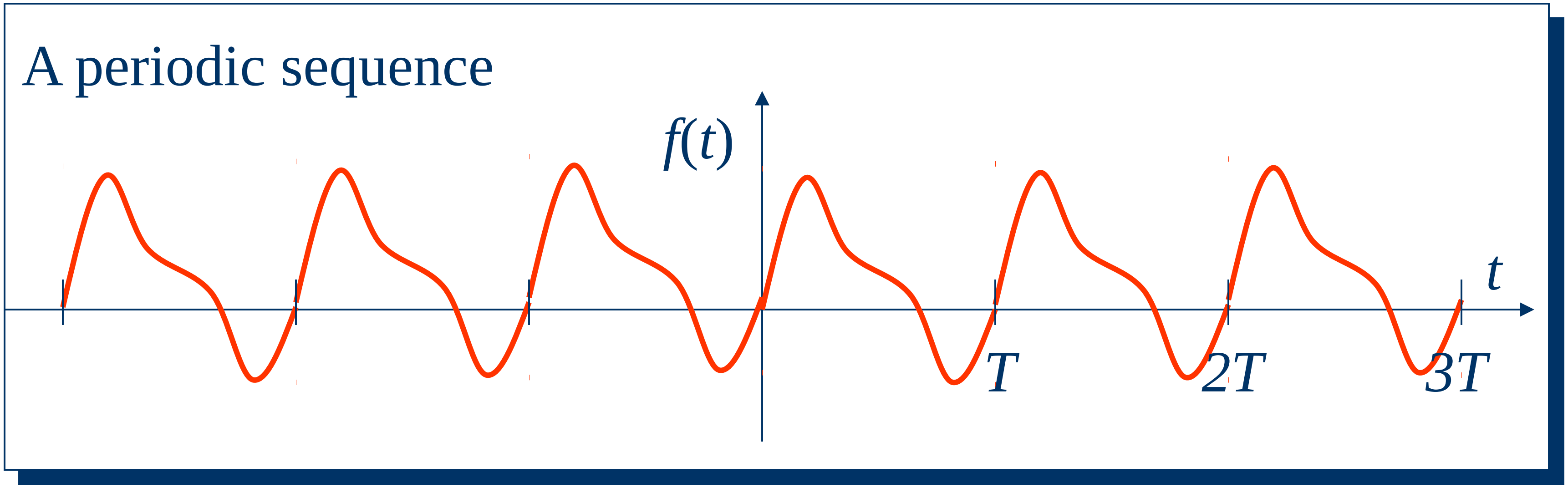
not a rational number

Fourier Series

Fourier Series

Introduction

- Decompose a periodic input signal into *primitive periodic components*.



Synthesis

$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}}_{\text{Odd Part}}$$

T is a period of all the above signals

Let $\omega_0 = 2\pi/T$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Orthogonal Functions

- Call a set of functions $\{\phi_k\}$ *orthogonal* on an interval $a < t < b$ if it satisfies

$$\int_a^b \phi_m(t)\phi_n(t)\,dt = \begin{cases} 0 & m \neq n \\ r_n & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi / T$.

$$\int_{T/2}^{T/2} \cos(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{T/2}^{T/2} \sin(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

We now prove this one

$$\int_{T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0, \qquad \text{for all } m \text{ and } n$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

Proof

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt \qquad m \neq n$$

$$=\frac{1}{2}\int_{-T/2}^{T/2}\cos[(m+n)\omega_0 t]dt+\frac{1}{2}\int_{-T/2}^{T/2}\cos[(m-n)\omega_0 t]dt$$

$$=\frac{1}{2}\frac{1}{(m+n)\omega_0}\sin[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2}+\frac{1}{2}\frac{1}{(m-n)\omega_0}\sin[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}$$

$$=\frac{1}{2}\frac{1}{(m+n)\omega_0}\underbrace{2\sin[(m+n)\pi]}_{0}+\frac{1}{2}\frac{1}{(m-n)\omega_0}\underbrace{2\sin[(m-n)\pi]}_{0}$$

$$=0$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

Proof

$$\cos^2\alpha = \frac{1}{2}[1+\cos 2\alpha]$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt$$

$m = n$

$$= \int_{-T/2}^{T/2}\cos^2(m\omega_0 t)dt \quad = \frac{1}{2}\int_{-T/2}^{T/2}[1+\cos 2m\omega_0 t]dt$$

$$= \frac{1}{2}t\Big|_{-T/2}^{T/2} + \underbrace{\frac{1}{4m\omega_0}\sin 2m\omega_0 t]\Big|_{-T/2}^{T/2}}_{0}$$

$$= \frac{T}{2}$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi/T$.

$$\int_{-T/2}^{T/2} \mathrm{c}\ldots \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \mathrm{c}\ldots$$

$$\left\{ \begin{array}{l} 1, \\ \cos\omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \cdots \\ \sin\omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \cdots \end{array} \right\}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \ldots$$

an orthogonal set.

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0, \quad \text{for all } m \text{ and } n$$

Decomposition

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt \qquad n = 1, 2, \cdots$$

Proof

Use the following facts:

$$\int_{T/2}^{T/2} \cos(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

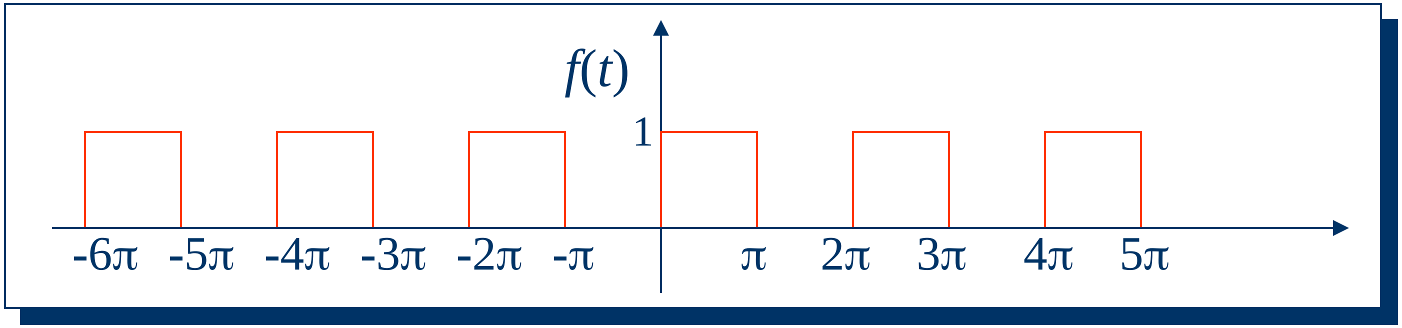
$$\int_{T/2}^{T/2} \sin(m\omega_0 t)\,dt = 0, \qquad m \neq 0$$

$$\int_{T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0, \qquad \text{for all } m \text{ and } n$$

Example (Square Wave)



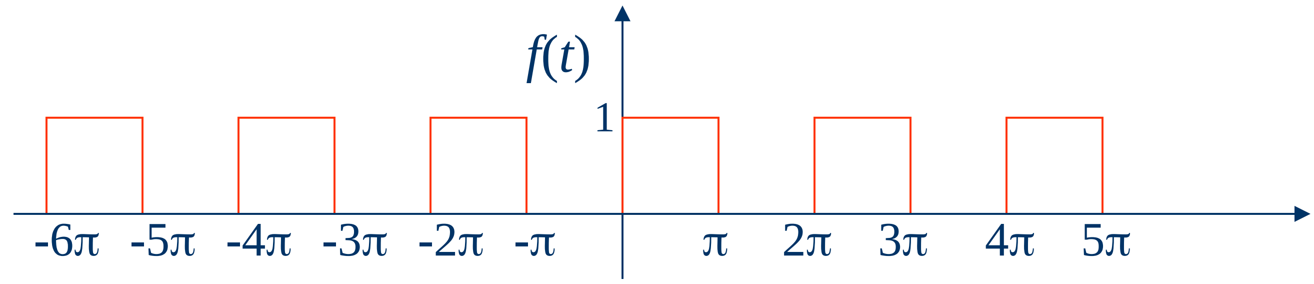
$$a_0 = \frac{2}{2\pi}\int_0^{\pi} 1 dt = 1$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} \cos nt dt = \frac{1}{n\pi}\sin nt\Big|_0^{\pi} = 0 \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{2\pi}\int_0^{\pi} \sin nt dt = -\frac{1}{n\pi}\cos nt\Big|_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1, 3, 5, \cdots \\ 0 & n = 2, 4, 6, \cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right]$$

Example (Square Wave)



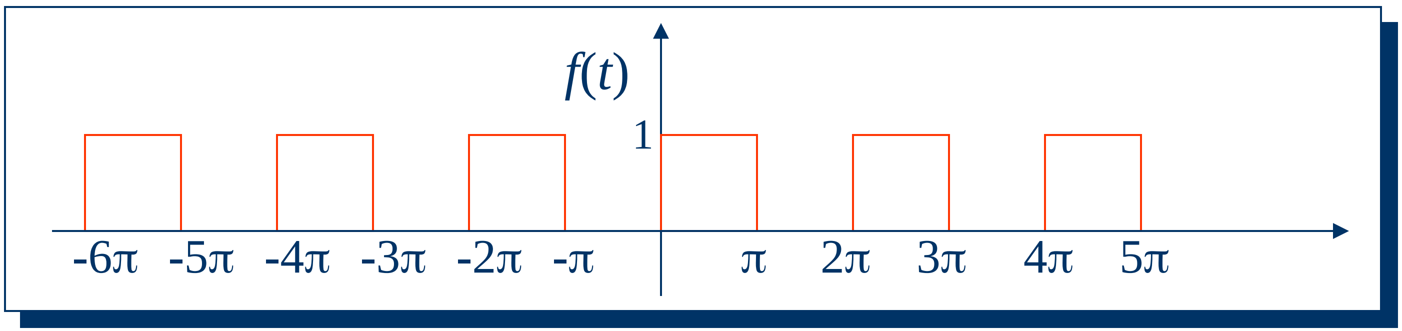
$$a_0 = \frac{2}{2\pi}\int_0^{\pi} 1\,dt = 1$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} \cos nt\,dt = \frac{1}{n\pi}\sin nt\Big|_0^{\pi} = 0 \qquad n = 1, 2, \cdots$$

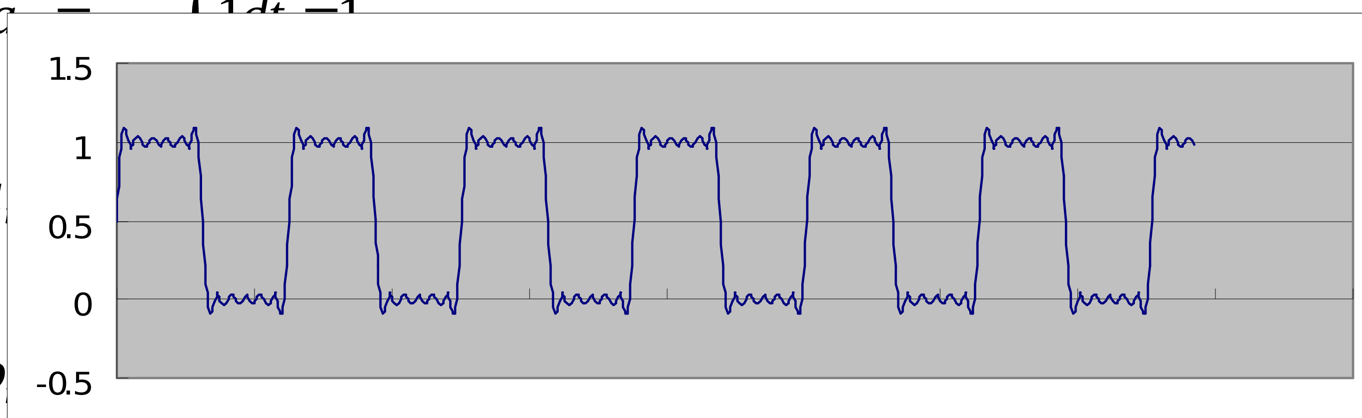
$$b_n = \frac{1}{2\pi}\int_0^{\pi} \sin nt\,dt = -\frac{1}{n\pi}\cos nt\Big|_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1, 3, 5, \cdots \\ 0 & n = 2, 4, 6, \cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right]$$

Example (Square Wave)



$a \; = \; \frac{2}{} \int^{\pi} 1dt = 1$



a

b

$n = 1,3,5,\cdots$

$n = 2,4,6,\cdots$

Harmonics

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{2\pi nt}{T}+\sum_{n=1}^{\infty}b_n\sin\frac{2\pi nt}{T}$$

$$f(t)=\underbrace{\underbrace{\frac{a_0}{2}}_{\text{DC Part}}+\underbrace{\sum_{n=1}^{\infty}a_n\cos(n\omega_0 t)}_{\text{Even Part}}+\underbrace{\sum_{n=1}^{\infty}b_n\sin(n\omega_0 t)}_{\text{Odd Part}}}_{T\text{ is a period of all the above signals}}$$

Harmonics

Define $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$, called the *fundamental angular frequency*.

Define $\omega_n = n\omega_0$, called the *n*-th *harmonic* of the periodic function.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\Rightarrow$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \omega_n t \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\cos \theta_n \cos \omega_n t + \sin \theta_n \sin \omega_n t \right)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

Amplitudes and Phase Angles

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

harmonic amplitude

phase angle

$$C_0 = \frac{a_0}{2}$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Fourier Series

Complex Form of the Fourier Series

Complex Exponentials

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j\sin n\omega_0 t$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j\sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

Complex Form of the Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - \frac{j}{2}\sum_{n=1}^{\infty} b_n \left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty} \left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right]$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt - j\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}\,dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

If $f(t)$ is real,

$$c_{-n} = c_n^*$$

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$n = \pm 1, \pm 2, \pm 3, \cdots$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$$c_0 = \frac{1}{2} a_0$$

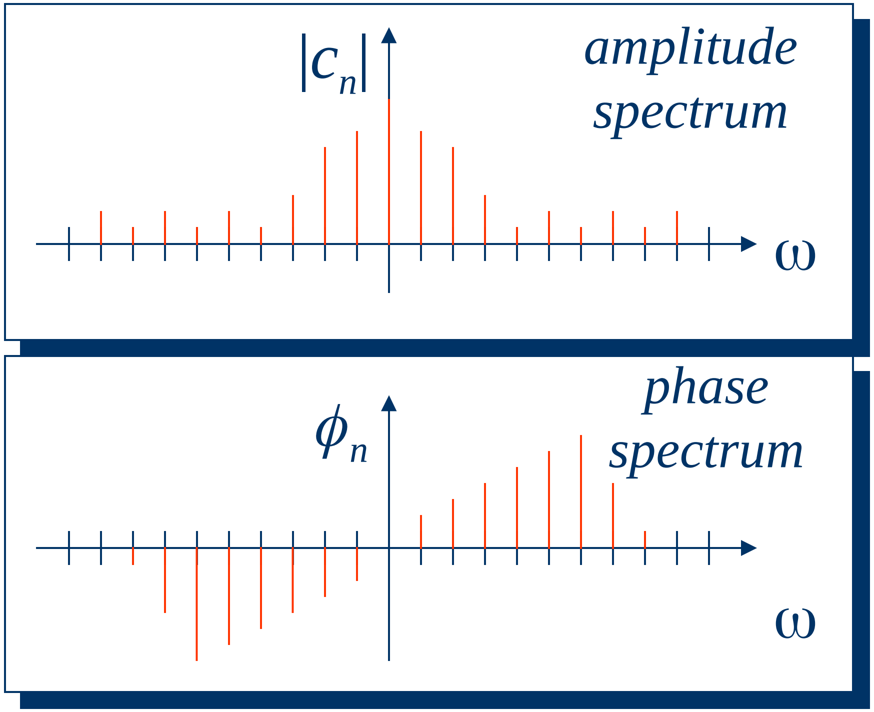
Complex Frequency Spectra

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

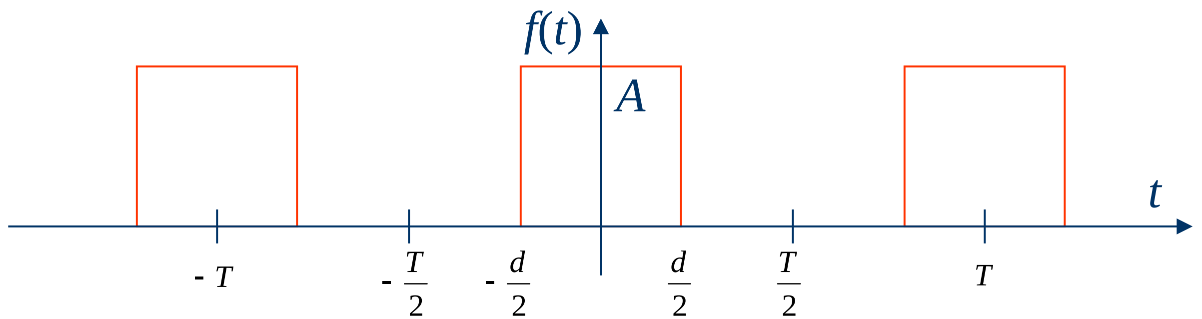
$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) \qquad n = \pm 1, \pm 2, \pm 3, \cdots$$

$$c_0 = \frac{1}{2} a_0$$



Example



$$c_n = \frac{A}{T}\int_{-d/2}^{d/2} e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\,\frac{1}{-jn\omega_0}\,e^{-jn\omega_0 t}\Big|_{-d/2}^{d/2}$$

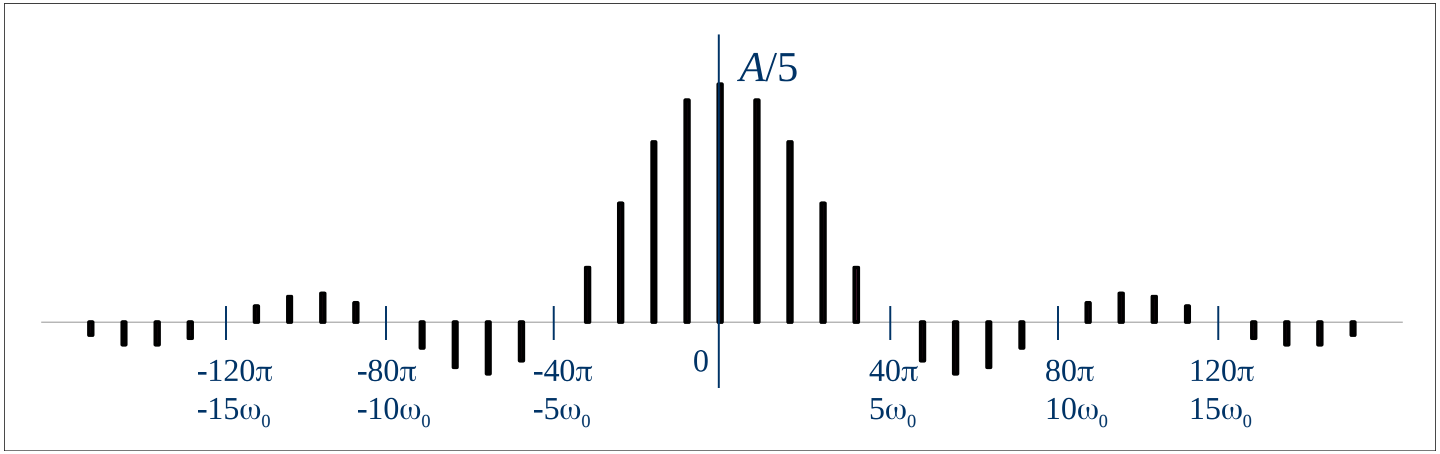
$$= \frac{A}{T}\left[\frac{1}{-jn\omega_0}e^{-jn\omega_0 d/2} - \frac{1}{-jn\omega_0}e^{jn\omega_0 d/2}\right]$$

$$= \frac{A}{T}\,\frac{1}{-jn\omega_0}\,(-2j\sin n\omega_0 d/2)$$

$$= \frac{A}{T}\,\frac{1}{\frac{1}{2}n\omega_0}\sin n\omega_0 d/2$$

$$= \frac{Ad}{T}\,\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

Example

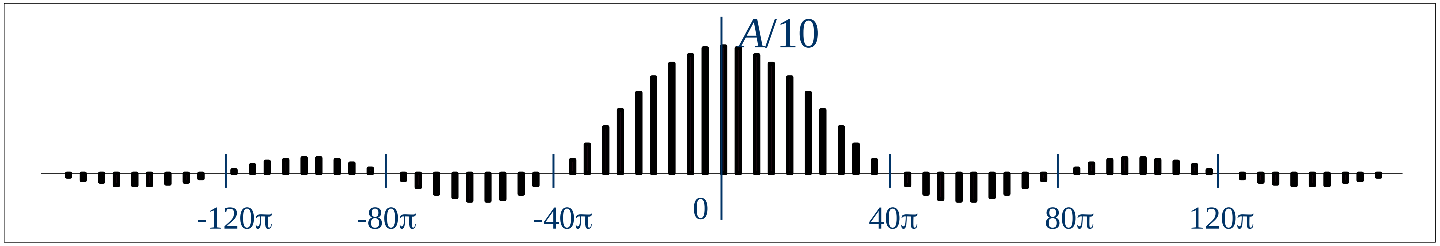


$$c_n = \frac{Ad}{T}\,\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20},\quad \boxed{T = \frac{1}{4},}\quad \frac{d}{\mathrm{T}} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 8\pi$$

Example



$$c_n = \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20},\ \boxed{T = \frac{1}{2},}\ \frac{d}{\mathrm{T}} = \frac{1}{10}$$

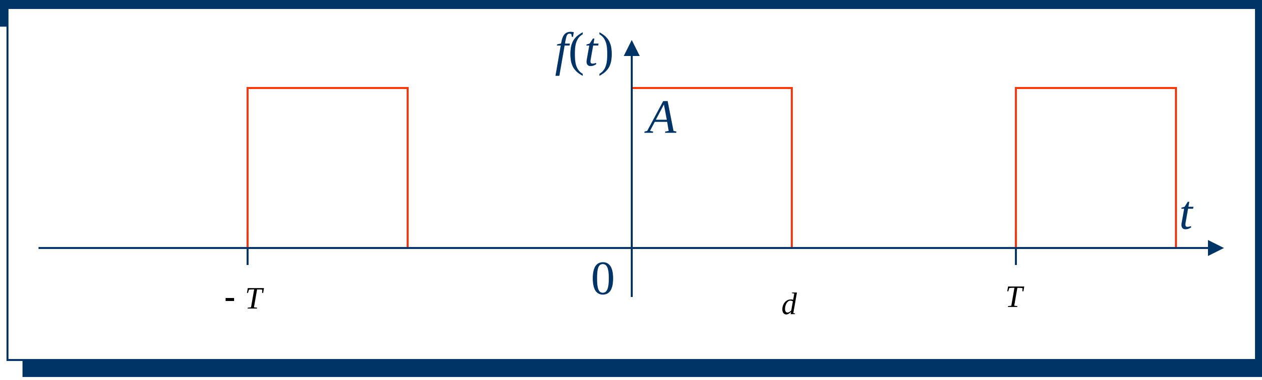
$$\omega_0 = \frac{2\pi}{T} = 4\pi$$

Fourier Series & Fourier Transform

Lecture 15-16: 28-Aug-12

Dr. P P Das

Example



$$c_n = \frac{A}{T}\int_0^d e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\frac{1}{-jn\omega_0} e^{-jn\omega_0 t}\Big|_0^d$$

$$= \frac{A}{T}\left[\frac{1}{-jn\omega_0} e^{-jn\omega_0 d} - \frac{1}{-jn\omega_0}\right]$$

$$= \frac{A}{T}\frac{1}{jn\omega_0}\left(1 - e^{-jn\omega_0 d}\right)$$

$$= \frac{A}{T}\frac{1}{jn\omega_0} e^{-jn\omega_0 d/2}\left(e^{jn\omega_0 d/2} - e^{-jn\omega_0 d/2}\right)$$

$$= \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)} e^{-jn\omega_0 d/2}$$

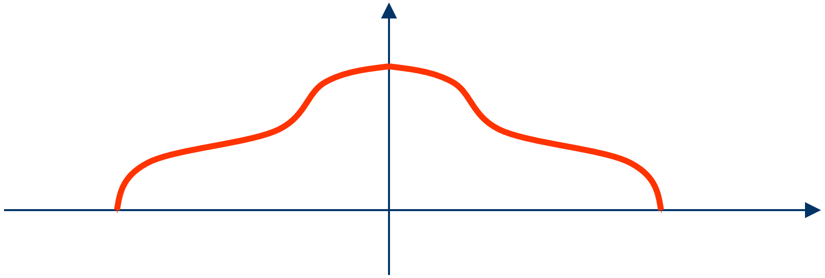
Fourier Series

Analysis of Periodic Waveforms

Waveform Symmetry

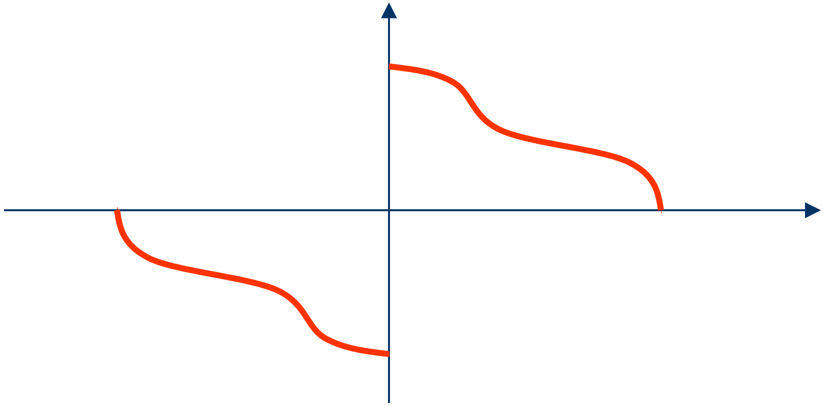
- Even Functions

$$f(t) = f(-t)$$



- Odd Functions

$$f(t) = -f(-t)$$



Decomposition

- Any function $f(t)$ can be expressed as the sum of an even function $f_e(t)$ and an odd function $f_o(t)$.

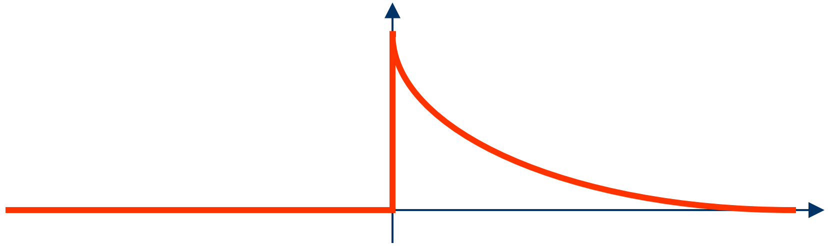
$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \tfrac{1}{2}[f(t) + f(-t)]$$ Even Part

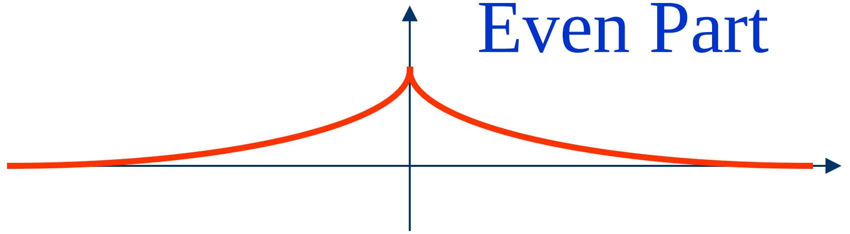
$$f_o(t) = \tfrac{1}{2}[f(t) - f(-t)]$$ Odd Part

Example

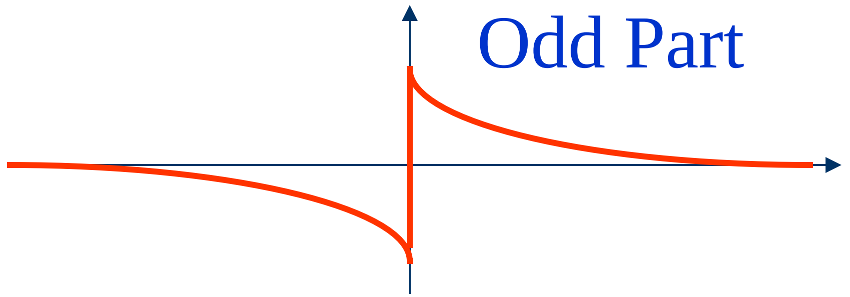
$$f(t)=\begin{cases} e^{-t} & t>0 \\ 0 & t<0 \end{cases}$$



$$f_e(t)=\begin{cases} \frac{1}{2}e^{-t} & t>0 \\ \frac{1}{2}e^{t} & t<0 \end{cases}$$

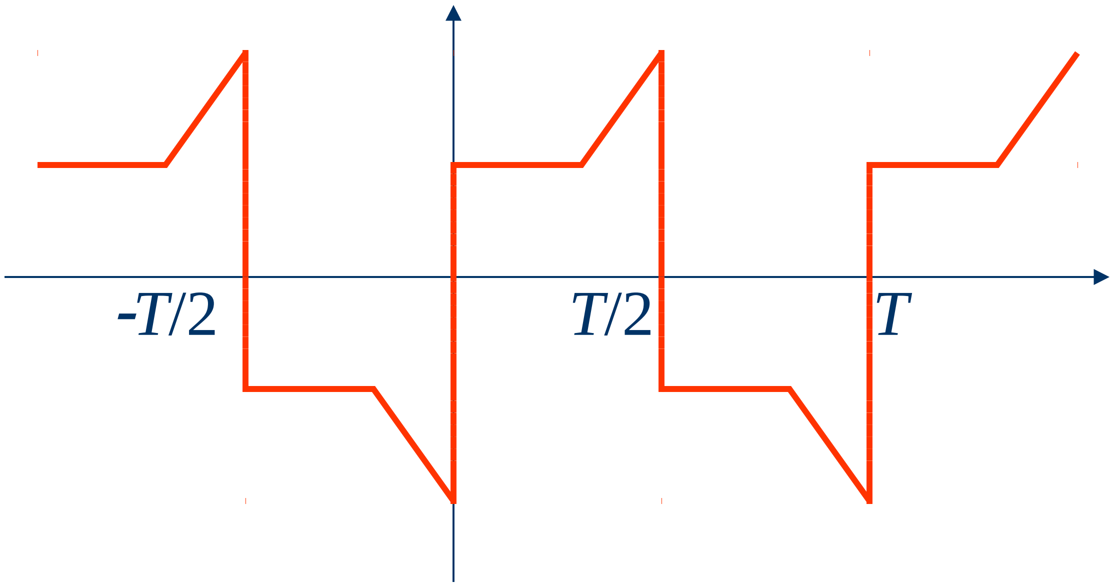


$$f_o(t)=\begin{cases} \frac{1}{2}e^{-t} & t>0 \\ -\frac{1}{2}e^{t} & t<0 \end{cases}$$



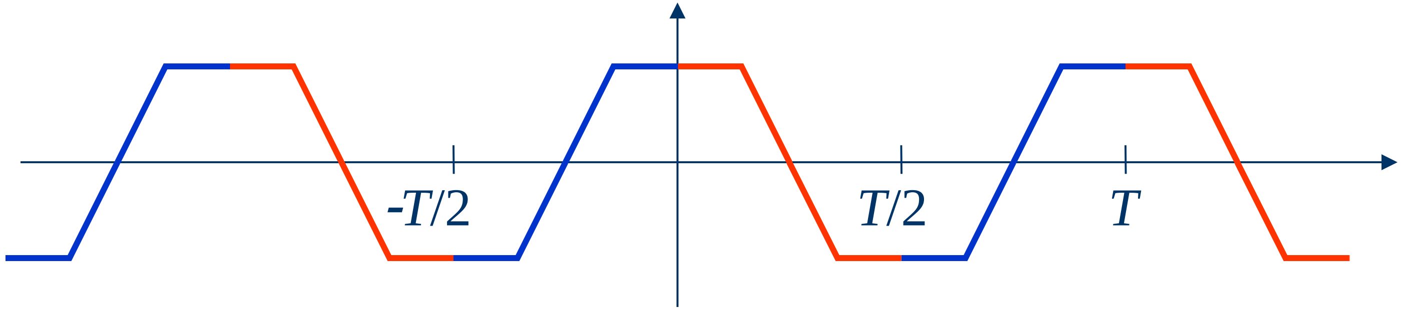
Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t+T/2\right)$$

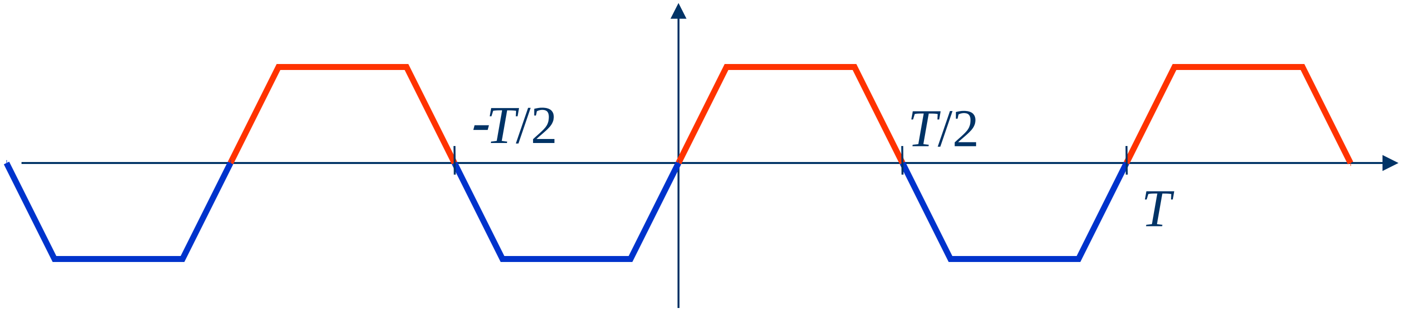


Quarter-Wave Symmetry

Even Quarter-Wave Symmetry

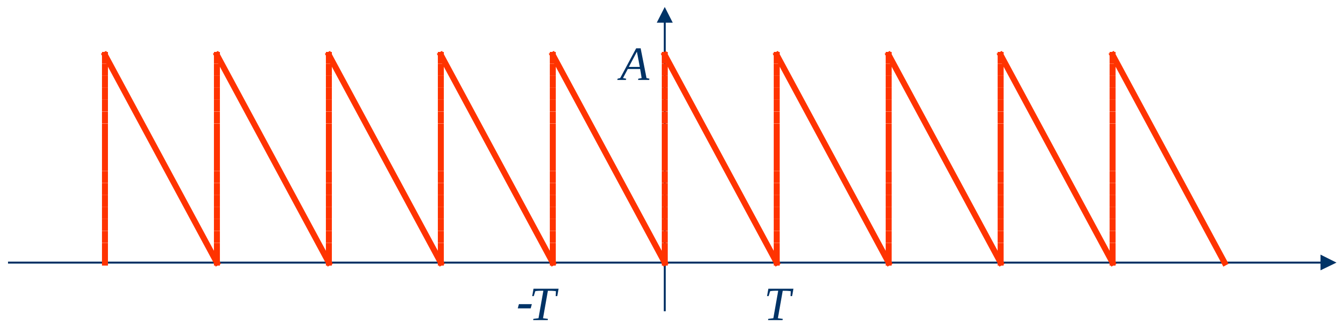


Odd Quarter-Wave Symmetry

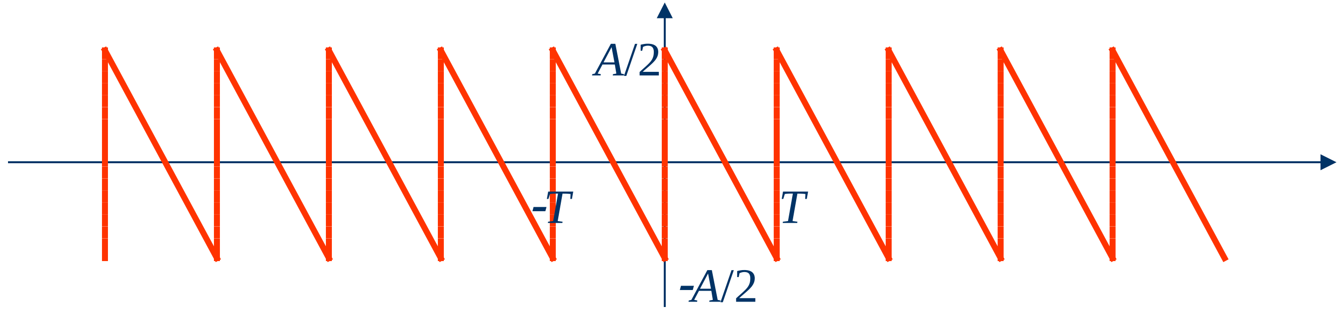


Hidden Symmetry

- The following is a asymmetry periodic function:



- Adding a constant to get symmetry property.

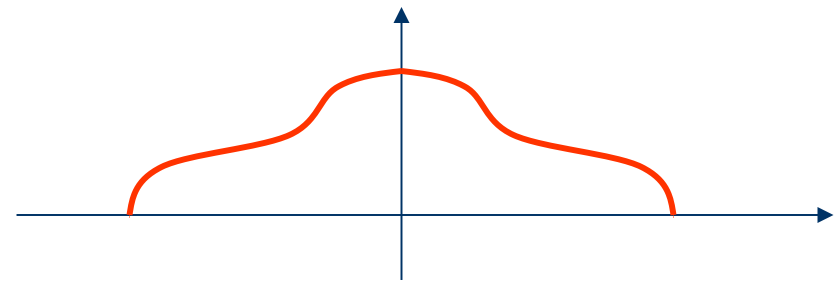


Fourier Coefficients of Symmetrical Waveforms

- The use of symmetry properties simplifies the calculation of Fourier coefficients.
  - Even Functions
  - Odd Functions
  - Half-Wave
  - Even Quarter-Wave
  - Odd Quarter-Wave
  - Hidden

Fourier Coefficients of Even Functions

$$f(t) = f(-t)$$

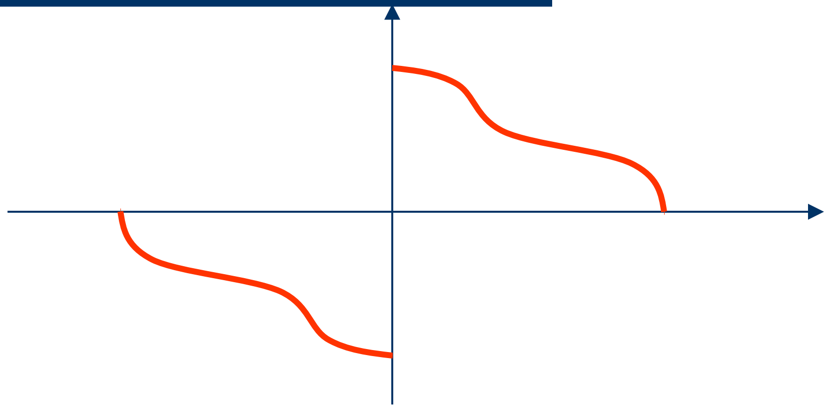


$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_n = \boxed{\frac{4}{T}\int_0^{T/2}} f(t)\cos(n\omega_0 t)\,dt$$

Fourier Coefficients of Odd Functions

$$f(t) = -f(-t)$$

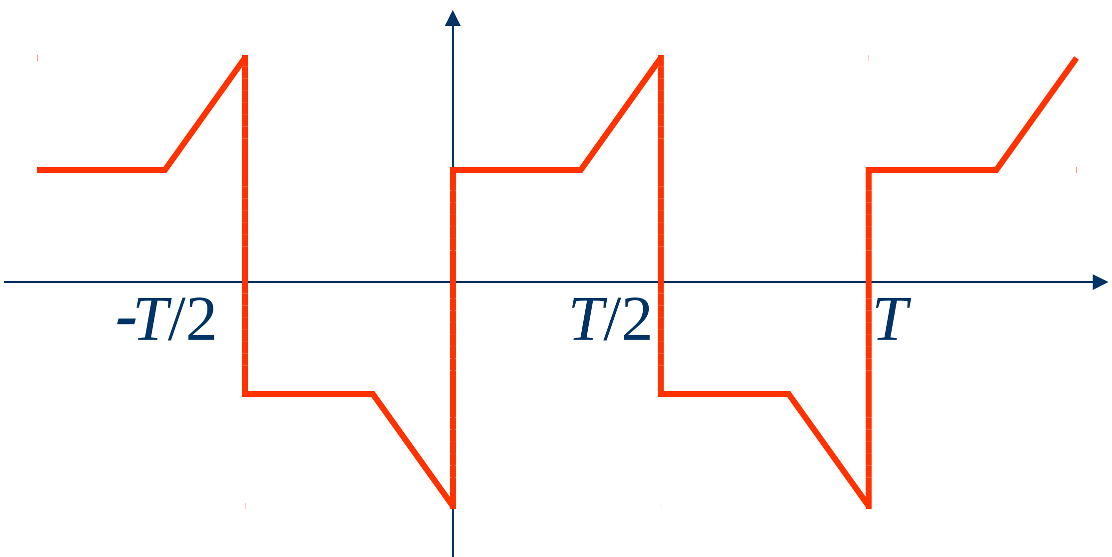


$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$b_n = \boxed{\frac{4}{T} \int_0^{T/2}} f(t) \sin(n\omega_0 t) dt$$

Fourier Coefficients for Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t + T/2\right)$$



The Fourier series contains only odd harmonics.

Fourier Coefficients for Half-Wave Symmetry

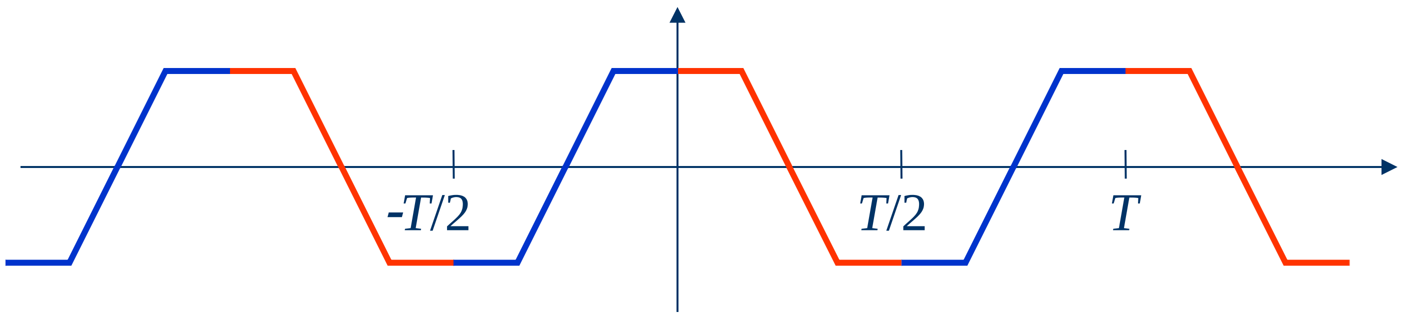
$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t + T/2\right)$$

$$\Rightarrow \quad f(t) = \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt & \text{for } n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt & \text{for } n \text{ odd} \end{cases}$$

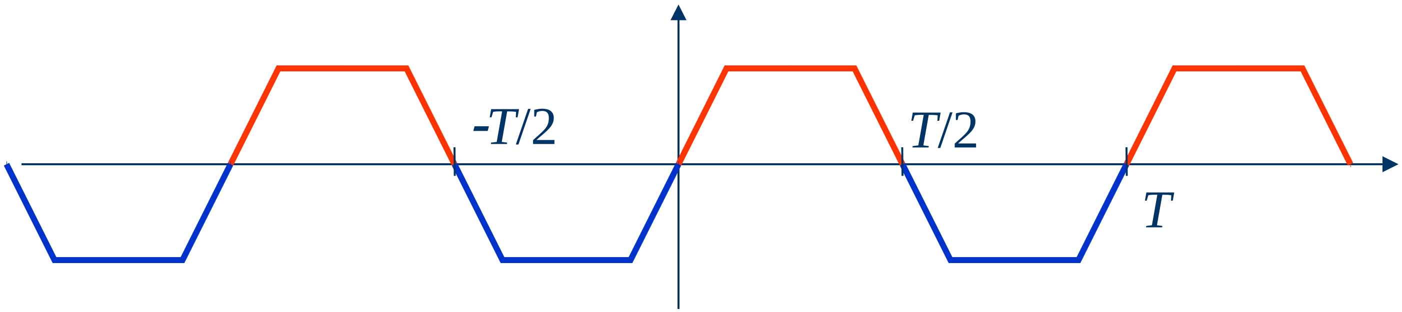
Fourier Coefficients for Even Quarter-Wave Symmetry



$$f(t) = \sum_{n=1}^{\infty} a_{2n-1} \cos[(2n-1)\omega_0 t]$$

$$a_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \cos[(2n-1)\omega_0 t] dt$$

Fourier Coefficients for Odd Quarter-Wave Symmetry

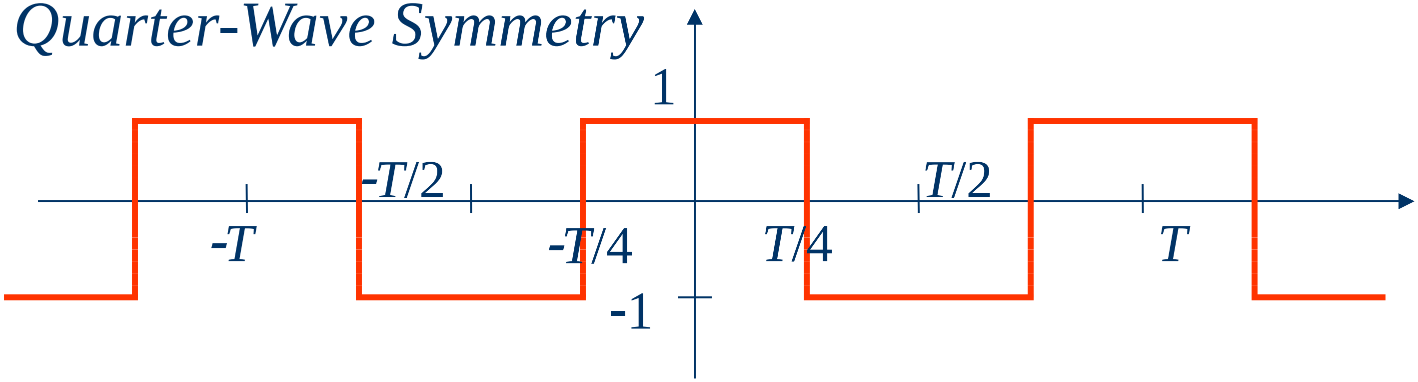


$$f(t) = \sum_{n=1}^{\infty} b_{2n-1} \sin[(2n-1)\omega_0 t]$$

$$b_{2n-1} = \frac{8}{T} \int_0^{T/4} f(t) \sin[(2n-1)\omega_0 t] dt$$

Example

Even Quarter-Wave Symmetry



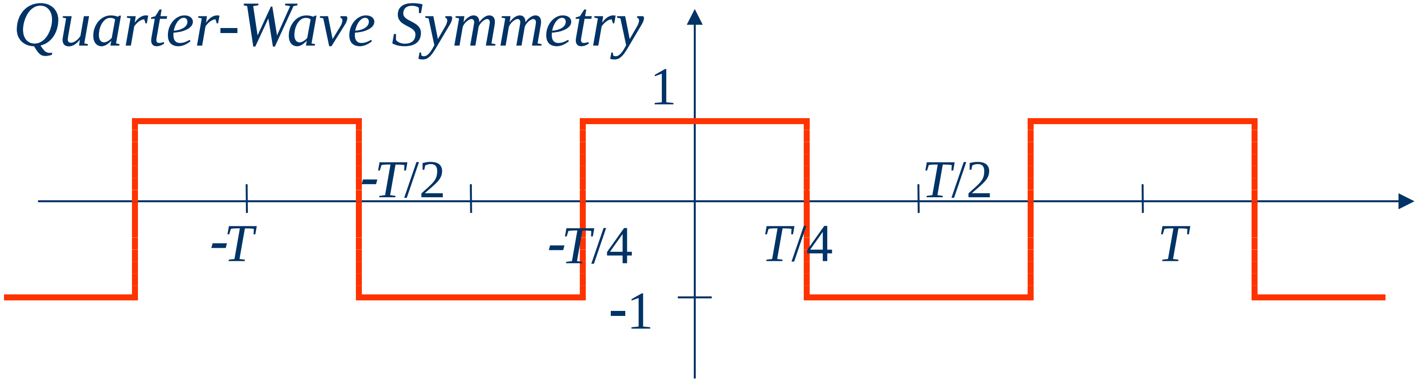
$$a_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\cos[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4}\cos[(2n-1)\omega_0 t]dt$$

$$= \frac{8}{(2n-1)\omega_0 T}\sin[(2n-1)\omega_0 t]\Big|_0^{T/4} = (-1)^{n-1}\frac{4}{(2n-1)\pi}$$

$$f(t) = \frac{4}{\pi}\left[\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t + \cdots\right]$$

Example

Even Quarter-Wave Symmetry

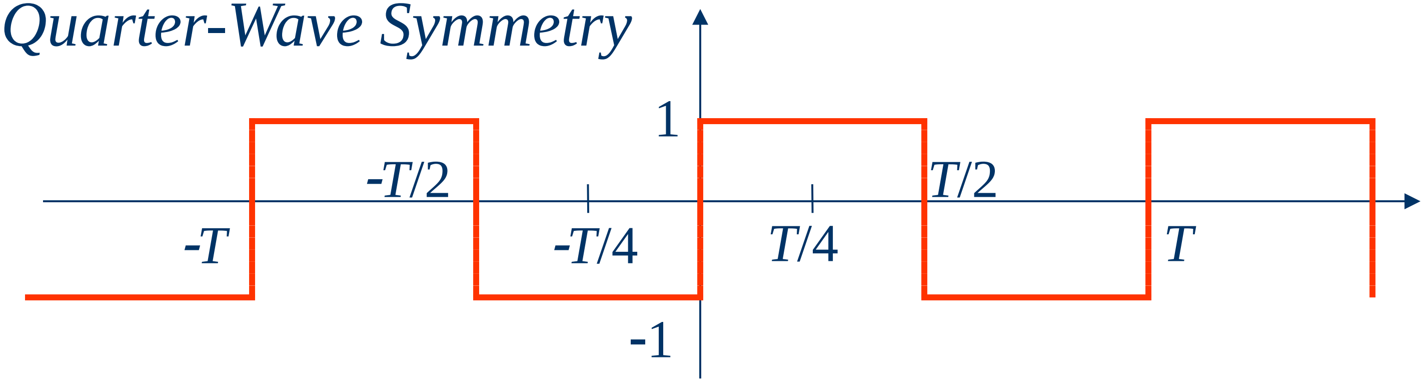


$$a_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\cos[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4}\cos[(2n-1)\omega_0 t]dt$$

$$= \frac{8}{(2n-1)\omega_0 T}\sin[(2n-1)\omega_0 t]\Big|_0^{T/4} = (-1)^{n-1}\frac{4}{(2n-1)\pi}$$

Example

Odd Quarter-Wave Symmetry



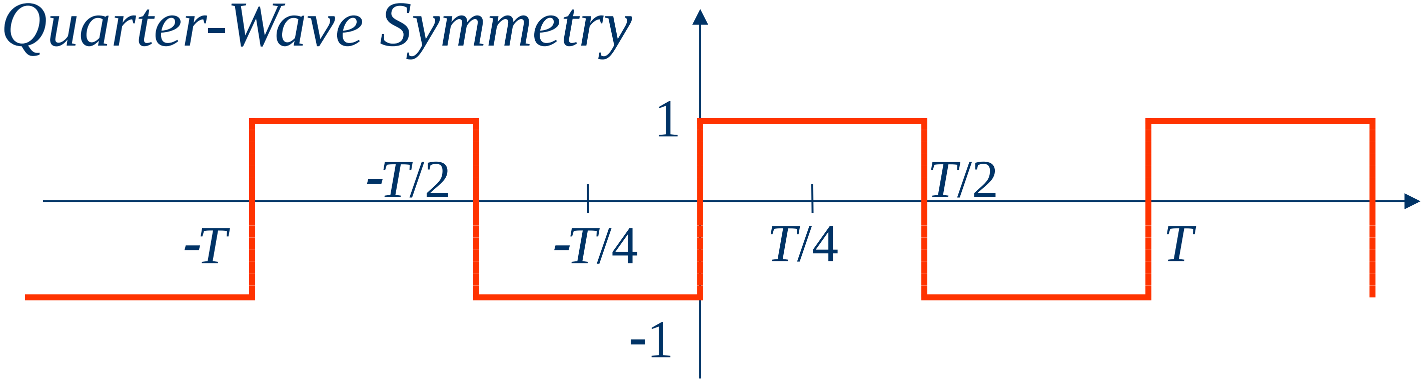
$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4} \sin[(2n-1)\omega_0 t]dt$$

$$= \frac{-8}{(2n-1)\omega_0 T}\cos[(2n-1)\omega_0 t]\Big|_0^{T/4} = \frac{4}{(2n-1)\pi}$$

$$f(t) = \frac{4}{\pi}\left[\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right]$$

Example

Odd Quarter-Wave Symmetry



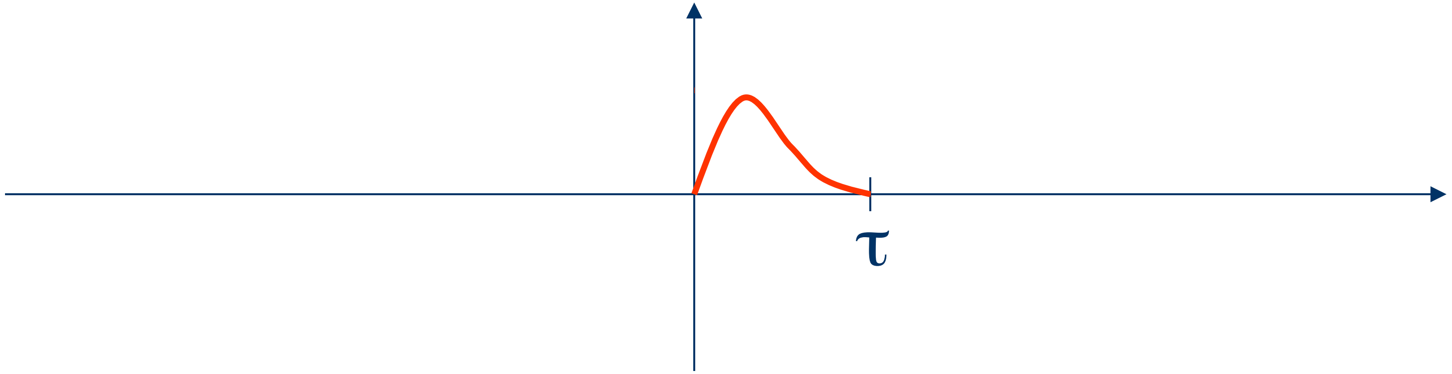
$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4} \sin[(2n-1)\omega_0 t]dt$$

$$= \frac{-8}{(2n-1)\omega_0 T}\cos[(2n-1)\omega_0 t]\Big|_0^{T/4} = \frac{4}{(2n-1)\pi}$$

Fourier Series

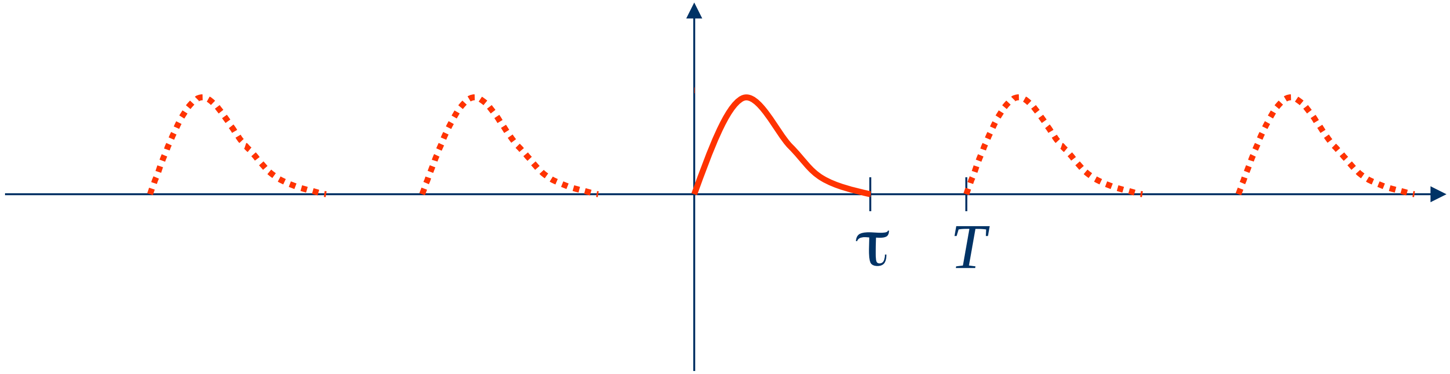
Half-Range Expansions

Non-Periodic Function Representation



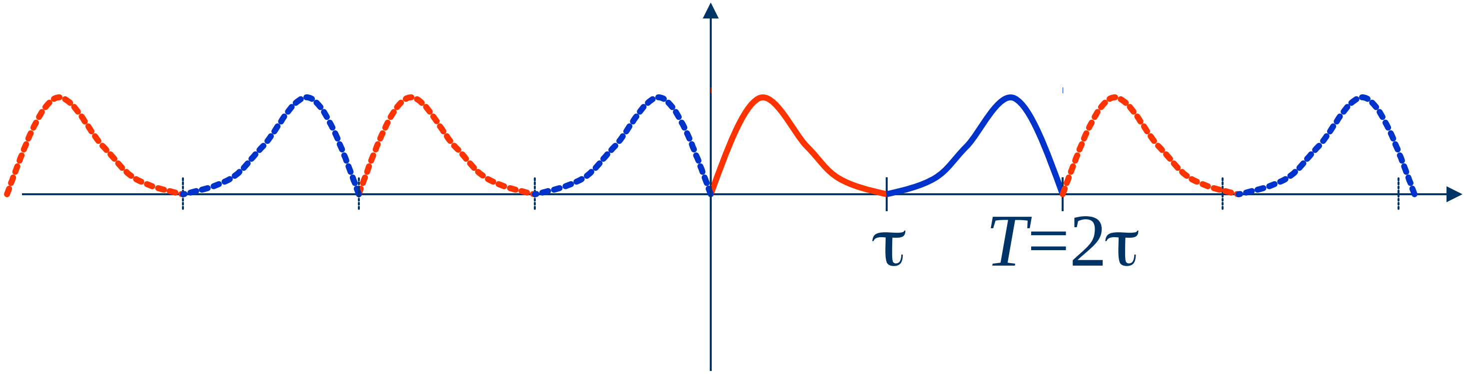
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Without Considering Symmetry



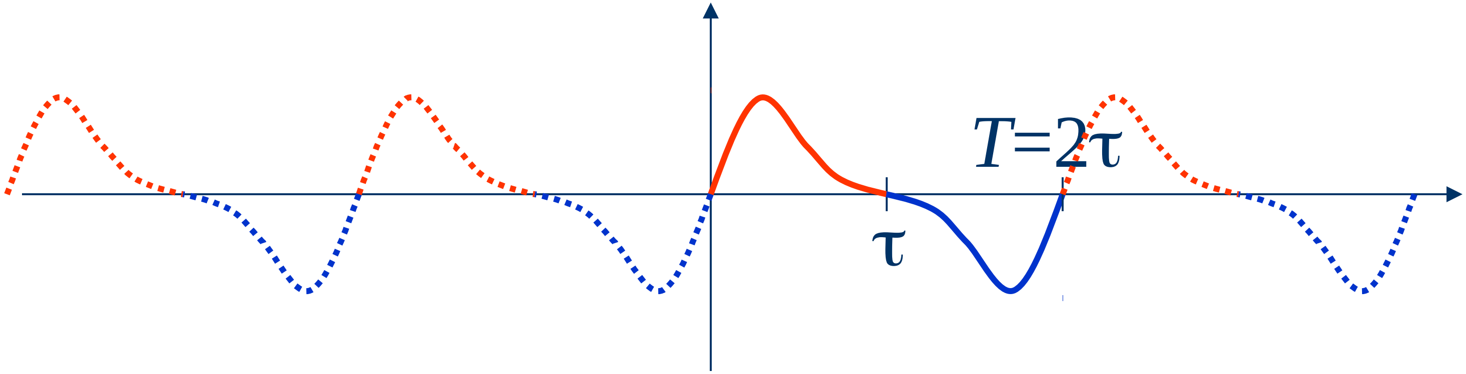
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Symmetry



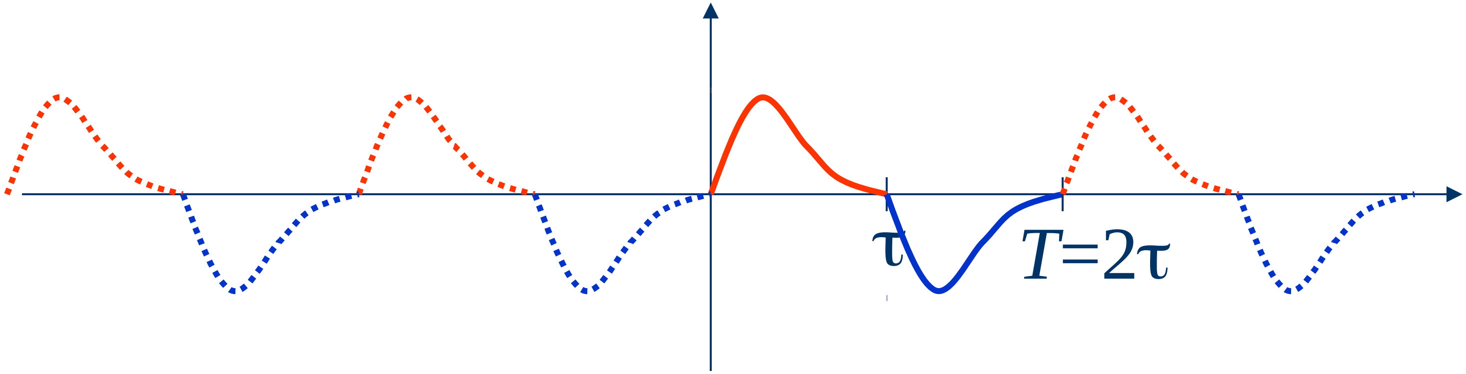
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Odd Symmetry



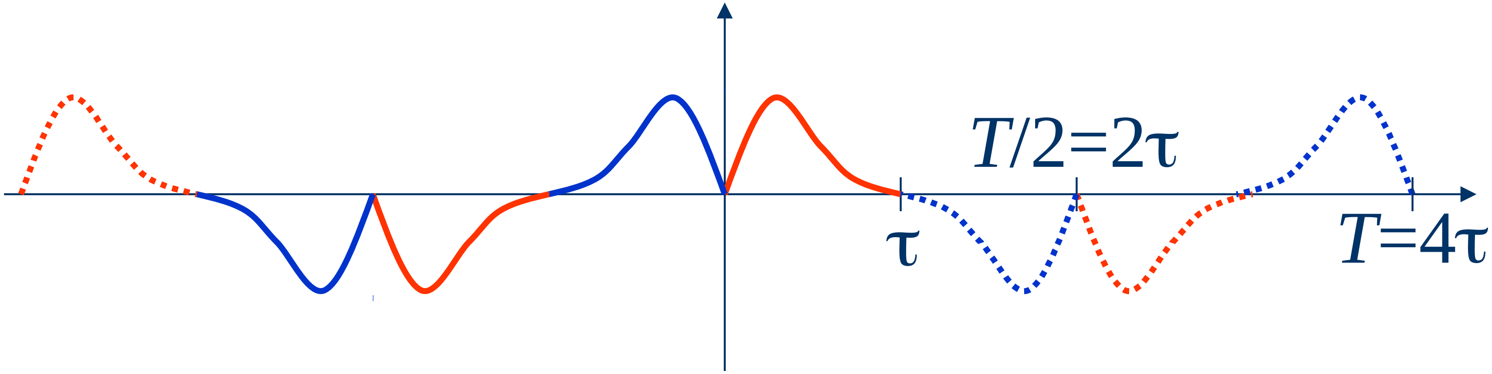
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Half-Wave Symmetry



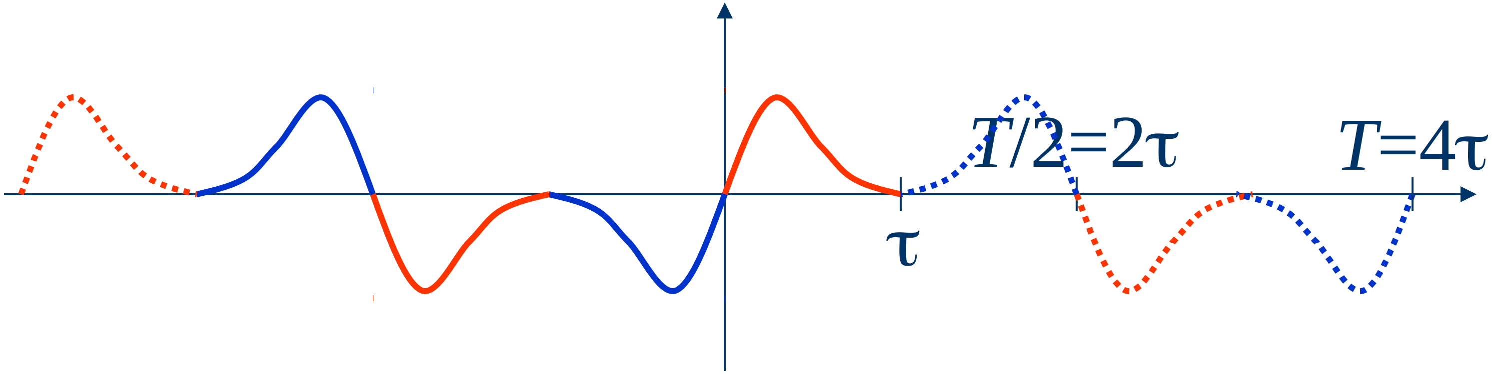
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Quarter-Wave Symmetry



- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Odd Quarter-Wave Symmetry



- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Fourier Series

Least Mean-Square Error Approximation

Approximation a function

Use $$S_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right)$$

to represent $f(t)$ on interval $-T/2 < t < T/2$.

Define $\varepsilon_k(t) = f(t) - S_k(t)$

$$E_k = \frac{1}{T} \int_{-T/2}^{T/2} [\varepsilon_k(t)]^2 \, dt$$

Mean-Square Error

Approximation a function

Show that using $S_k(t)$ to represent $f(t)$ has least mean-square property.

$$E_k = \frac{1}{T}\int_{-T/2}^{T/2} [\varepsilon_k(t)]^2\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) \right]^2 dt$$

Proven by setting $\partial E_k/\partial a_i = 0$ and $\partial E_k/\partial b_i = 0$.

Approximation a function

$$E_k = \frac{1}{T}\int_{-T/2}^{T/2} [\varepsilon_k(t)]^2 dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \right]^2 dt$$

$$\frac{\partial E_k}{\partial a_0} = \frac{a_0}{2} - \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = 0$$

$$\frac{\partial E_k}{\partial a_n} = a_n - \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt = 0$$

$$\frac{\partial E_k}{\partial b_n} = b_n - \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt = 0$$

Mean-Square Error

$$E_k = \frac{1}{T}\int_{-T/2}^{T/2} [\varepsilon_k(t)]^2 \, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right) \right]^2 dt$$

$$\boxed{E_k = \frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2 \, dt - \frac{a_0^2}{4} - \frac{1}{2}\sum_{n=1}^{k} (a_n^2 + b_n^2)}$$

Mean-Square Error

$$E_k = \frac{1}{T}\int_{T/2}^{T/2}[\varepsilon_k(t)]^2\,dt$$

$$= \frac{1}{T}\int_{T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)\right]^2 dt$$

$$\frac{1}{T}\int_{T/2}^{T/2}[f(t)]^2\,dt \geq \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{k}(a_n^2 + b_n^2)$$

Mean-Square Error

$$E_k = \frac{1}{T}\int_{T/2}^{T/2}[\varepsilon_k(t)]^2\,dt$$

$$= \frac{1}{T}\int_{T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{k}\left(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)\right]^2 dt$$

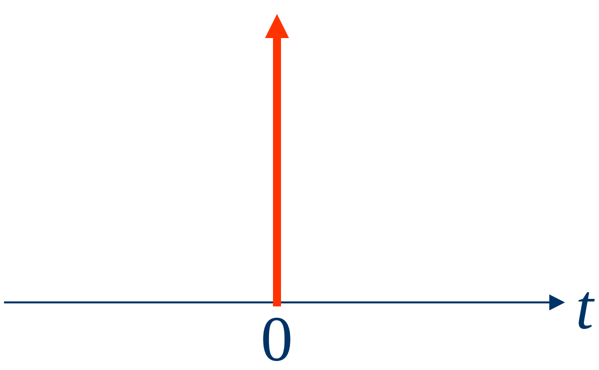
$$\boxed{\frac{1}{T}\int_{T/2}^{T/2}[f(t)]^2\,dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

Fourier Series

Impulse Train

Dirac Delta Function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



Also called *unit impulse function*, but not actually a function in true sense.

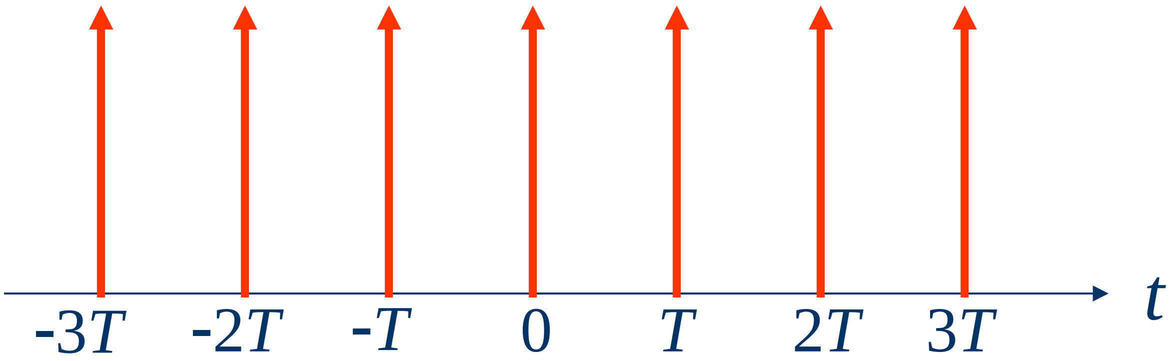
Sifting Property

$$\boxed{\int_{\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)}$$

ϕ(t): Test Function

$$\int_{\infty}^{\infty} \delta(t)\phi(t)dt = \int_{\infty}^{\infty} \delta(t)\phi(0)dt = \phi(0)\int_{\infty}^{\infty} \delta(t)dt = \phi(0)$$

Impulse Train



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\int_{\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)$$

Fourier Series of the Impulse Train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$a_0 = \frac{2}{T}\int_{T/2}^{T/2} \delta_T(t)dt = \frac{2}{T}$$

$$a_n = \frac{2}{T}\int_{T/2}^{T/2} \delta_T(t)\cos(n\omega_0 t)dt = \frac{2}{T}$$

$$b_n = \frac{2}{T}\int_{T/2}^{T/2} \delta_T(t)\sin(n\omega_0 t)dt = 0$$

$$\delta_T(t) = \frac{1}{T} + \frac{2}{T}\sum_{n=-\infty}^{\infty} \cos n\omega_0 t$$

Complex Form
Fourier Series of the Impulse Train

$$\int_{\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{T/2}^{T/2} \delta_T(t)dt = \frac{1}{T}$$

$$c_n = \frac{1}{T}\int_{T/2}^{T/2} \delta_T(t)e^{-jn\omega_0 t}dt = \frac{1}{T}$$

$$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Fourier Transform

Impulse Train

Fourier Series / Transform

- Fourier Series deals with discrete variable (harmonics of $\omega_0 = 2\pi f_0$)
- Fourier Transform deals with continuous frequency

Fourier Series / Transform

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn2\pi f_0 t}dt$$

$$\Im\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn2\pi f_0 t}$$

$$f(t) = \Im^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

$$\omega_0 = 2\pi f_0$$

Example

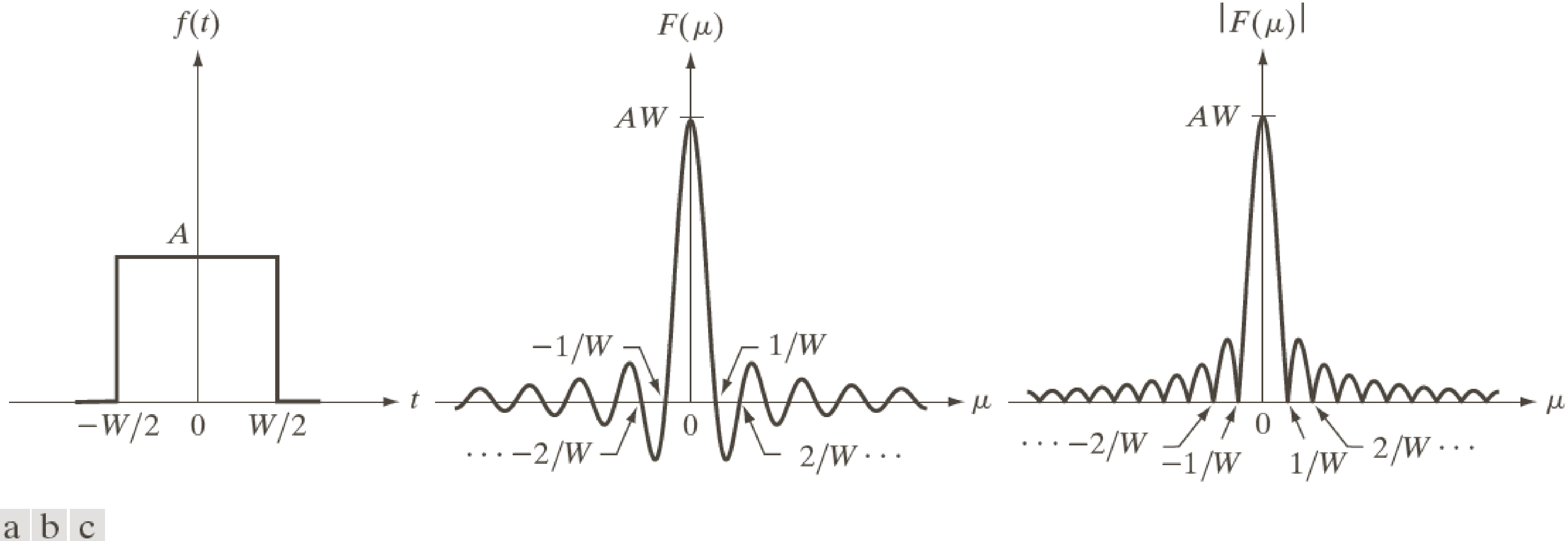


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Example

$$F(\mu) = \int_{\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

$$= A \frac{1}{-j2\pi\mu} e^{-j2\pi\mu t} \Big|_{-W/2}^{W/2}$$

$$= \frac{A}{j2\pi\mu} \left(e^{j\pi\mu W} - e^{-j\pi\mu W} \right)$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

Example

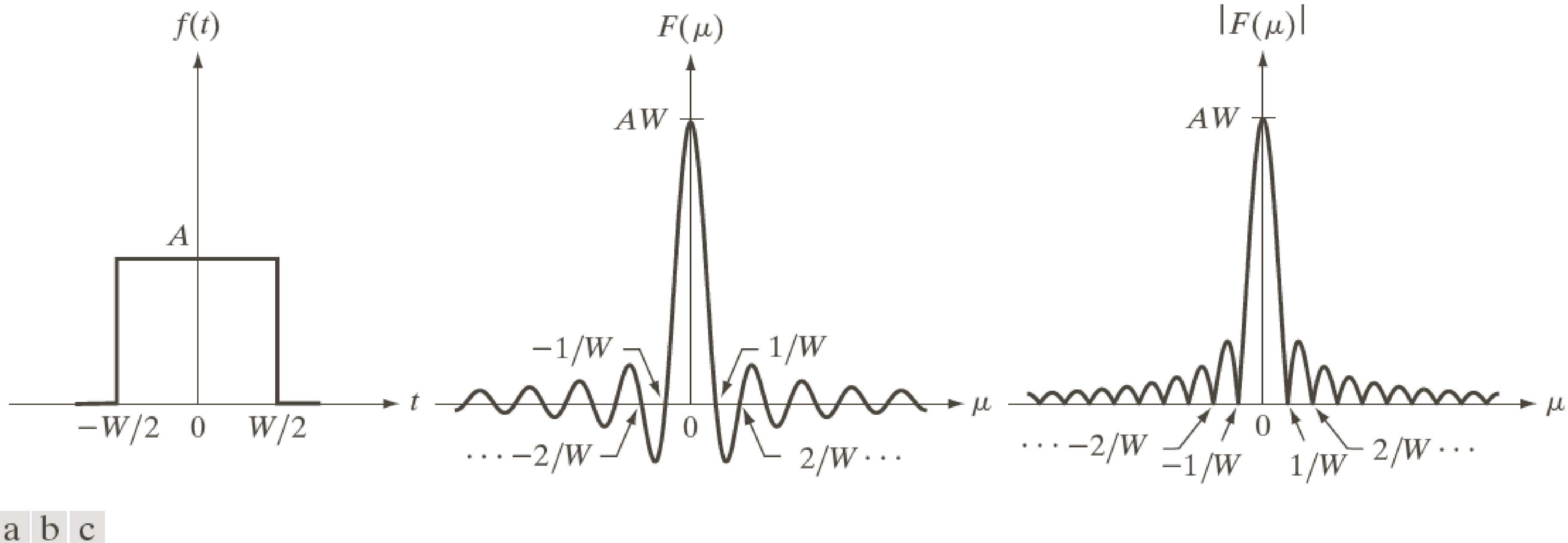


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Example: Unit Impulse

$$F(\mu) = \int_{\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt$$

$$= e^{-j2\pi\mu 0} = e^{0} = 1$$

$$F(\mu) = \int_{\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt$$

$$= e^{-j2\pi\mu t_0}$$

Fourier Series of the Impulse Train

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T}$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

Fourier Transform of the Impulse Train

$$\Im\{e^{j\frac{2\pi n}{\Delta T}t}\} = \delta(\mu - \frac{n}{\Delta T})$$

$$S(\mu) = \Im\{s_{\Delta T}(t)\} = \Im\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\}$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

Fourier Transform

Convolution

Convolution

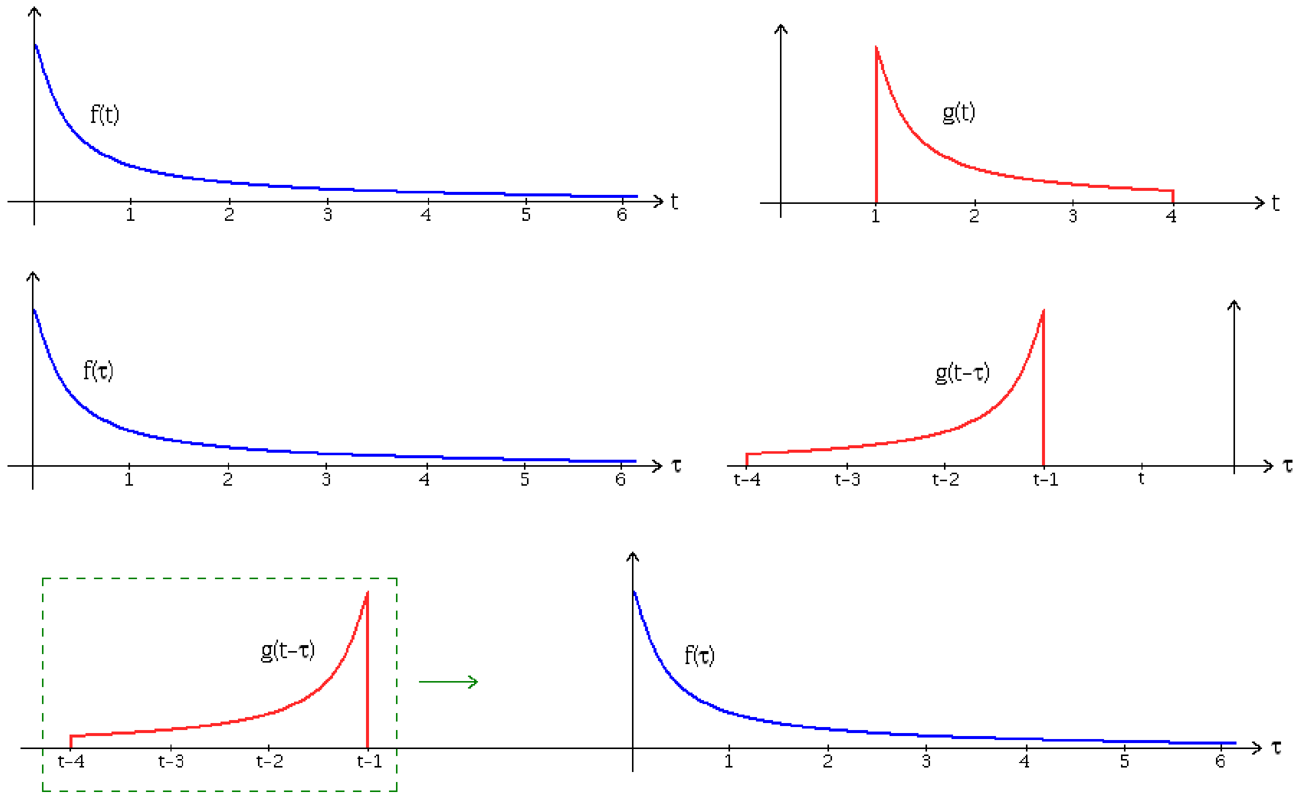
- A mathematical operator which computes the "amount of overlap" between two functions. Can be thought of as a general moving average
- Discrete domain: $(f * g)(m) = \sum_n f(n) g(m - n)$
- Continuous domain: $(f * g)(t) = \int f(t - \tau) g(\tau) \, d\tau$

Discrete domain

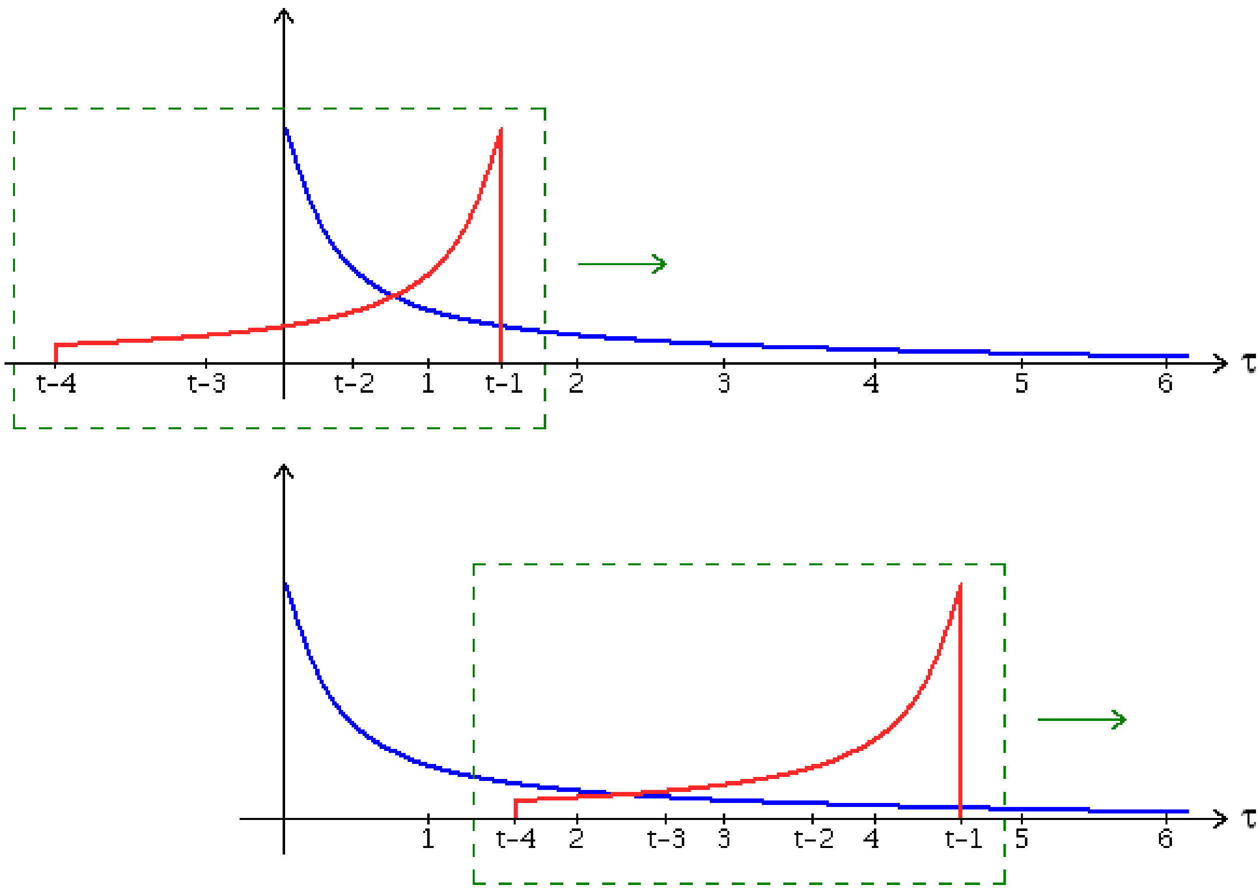
- Basic steps
  1. Flip (reverse) one of the digital functions.
  2. Shift it along the time axis by one sample.
  3. Multiply the corresponding values of the two digital functions.
  4. Summate the products from step 3 to get one point of the digital convolution.
  5. Repeat steps 1-4 to obtain the digital convolution at all times that the functions overlap.

- Example

Continuous domain example



Continuous domain example



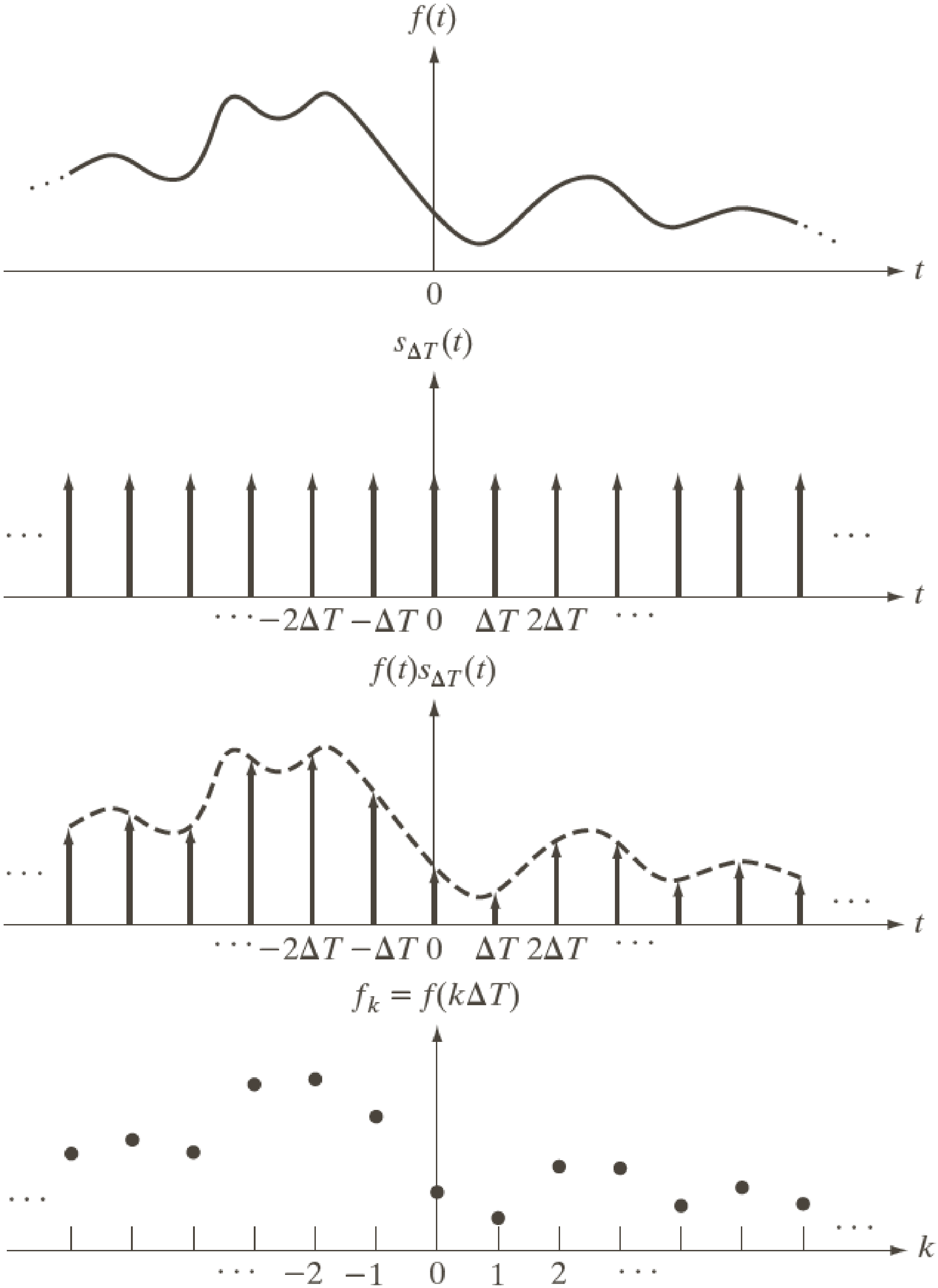
Convolution Theorem

$$\Im\{ f(t) \bullet g(t)\} = F(\mu)G(\mu)$$

$$\Im\{F(\mu) \bullet G(\mu)\} = f(t)g(t)$$

Fourier Transform

Sampling



a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

Sampling

$$\tilde{f}(t)$$

$$= f(t) s_{\Delta T}(t)$$

$$= \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

Sampling & FT

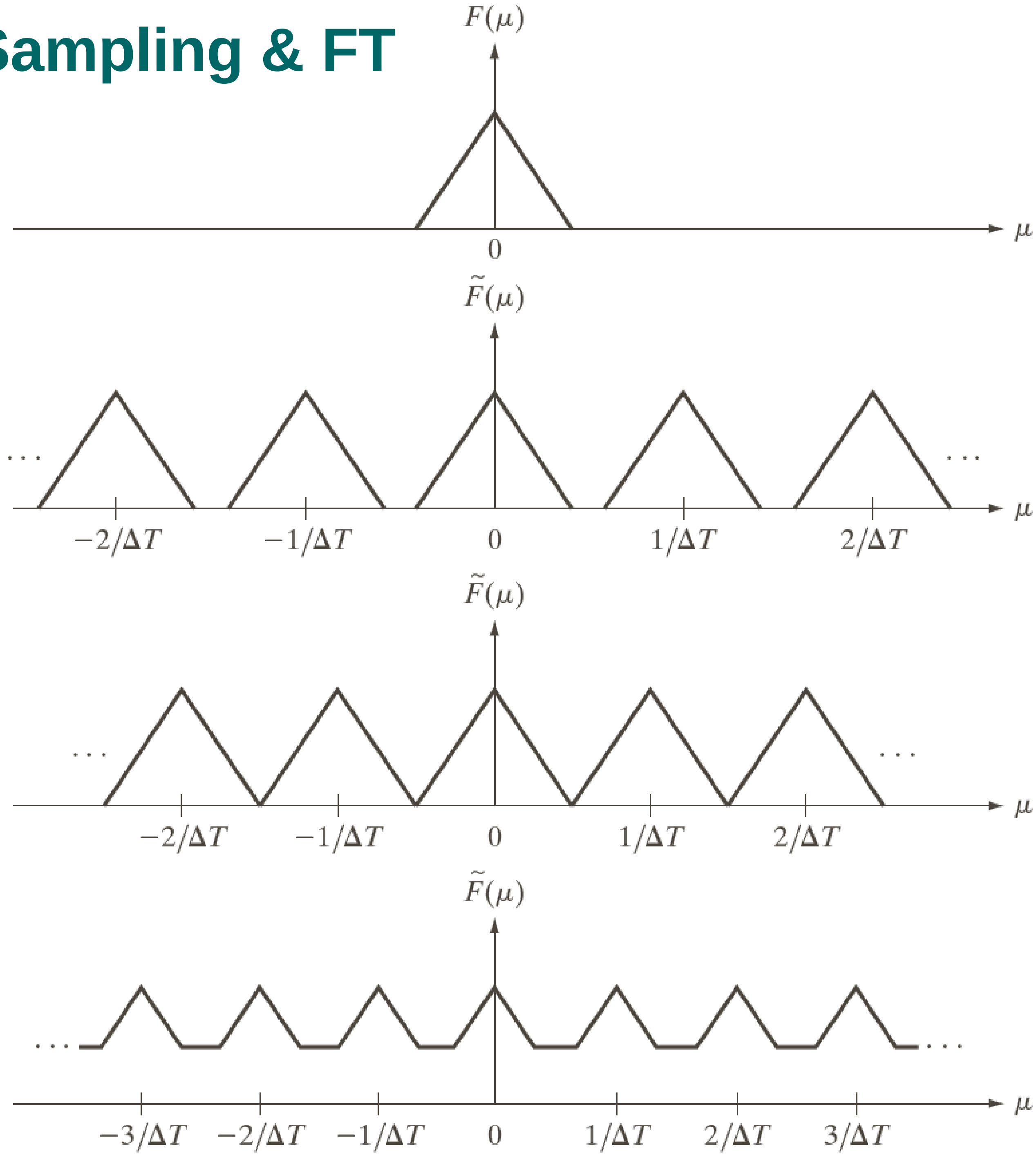
$$\tilde{F}(\mu)$$

$$= \Im\{\tilde{f}(t)\} = \Im\{f(t) s_{\Delta T}(t)\}$$

$$= F(\mu) \bullet S(\mu)$$

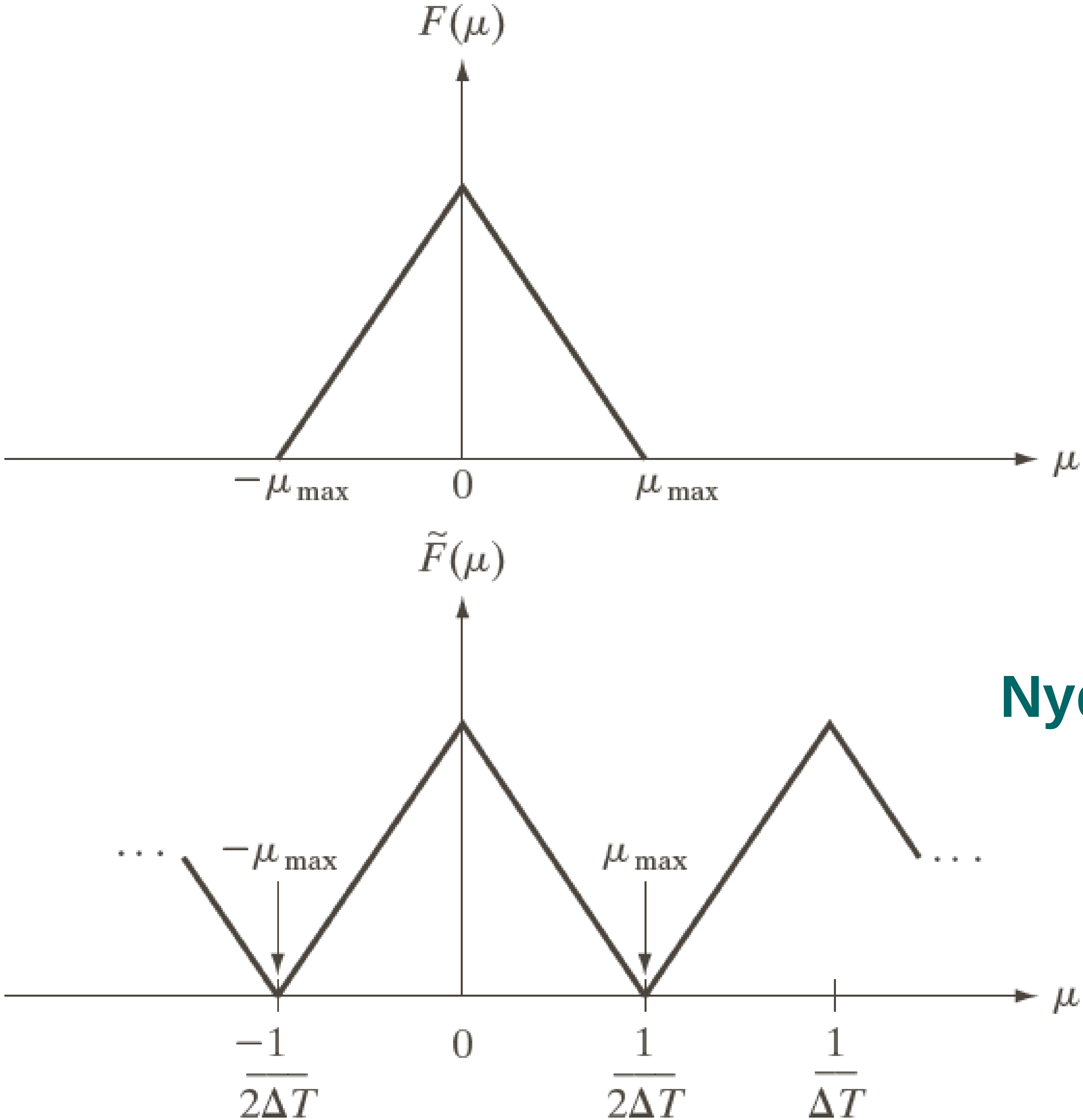
$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$$

Sampling & FT



a
b
c
d

FIGURE 4.6
(a) Fourier transform of a band-limited function.
(b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

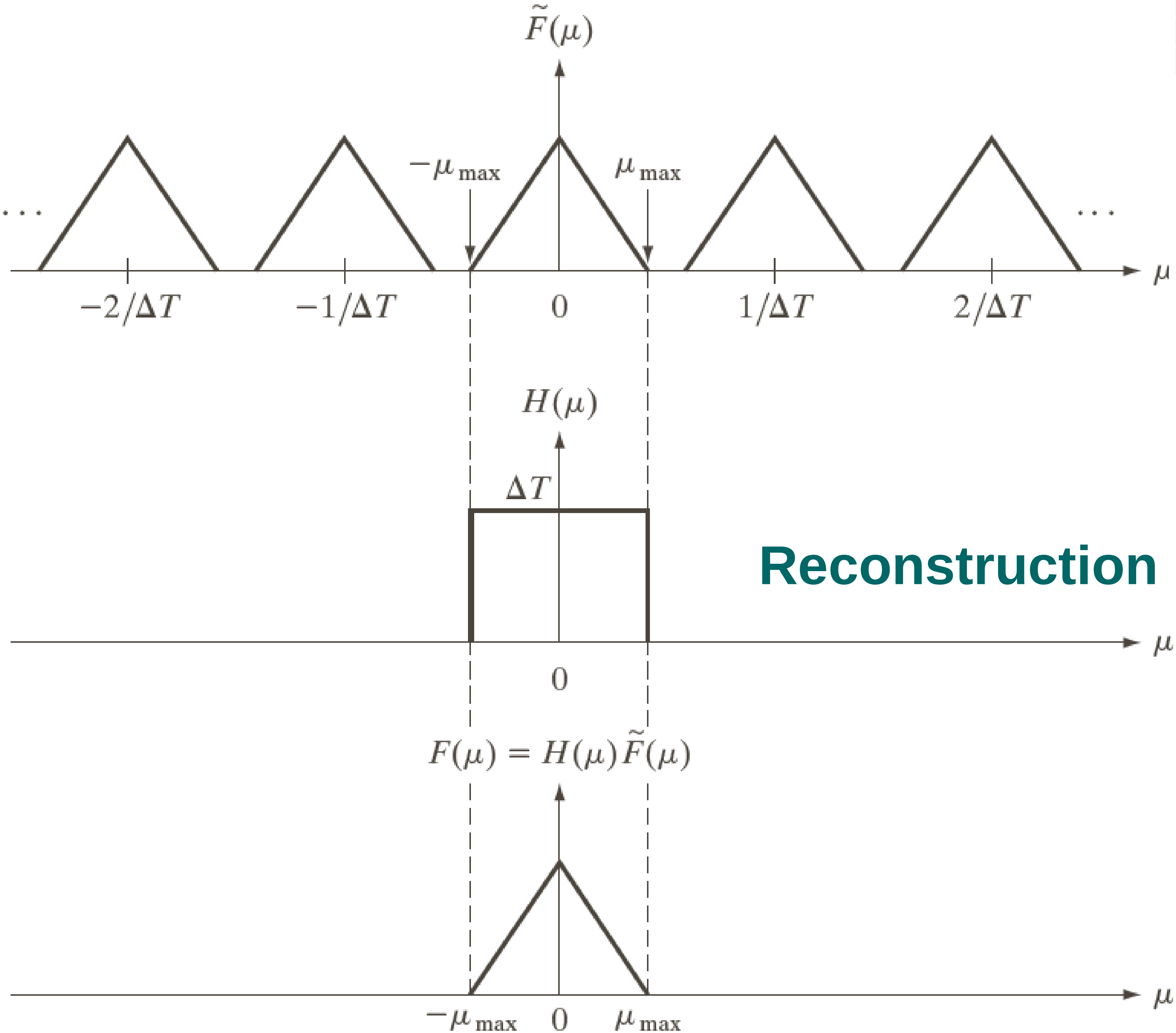


a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

Nyquist Rate

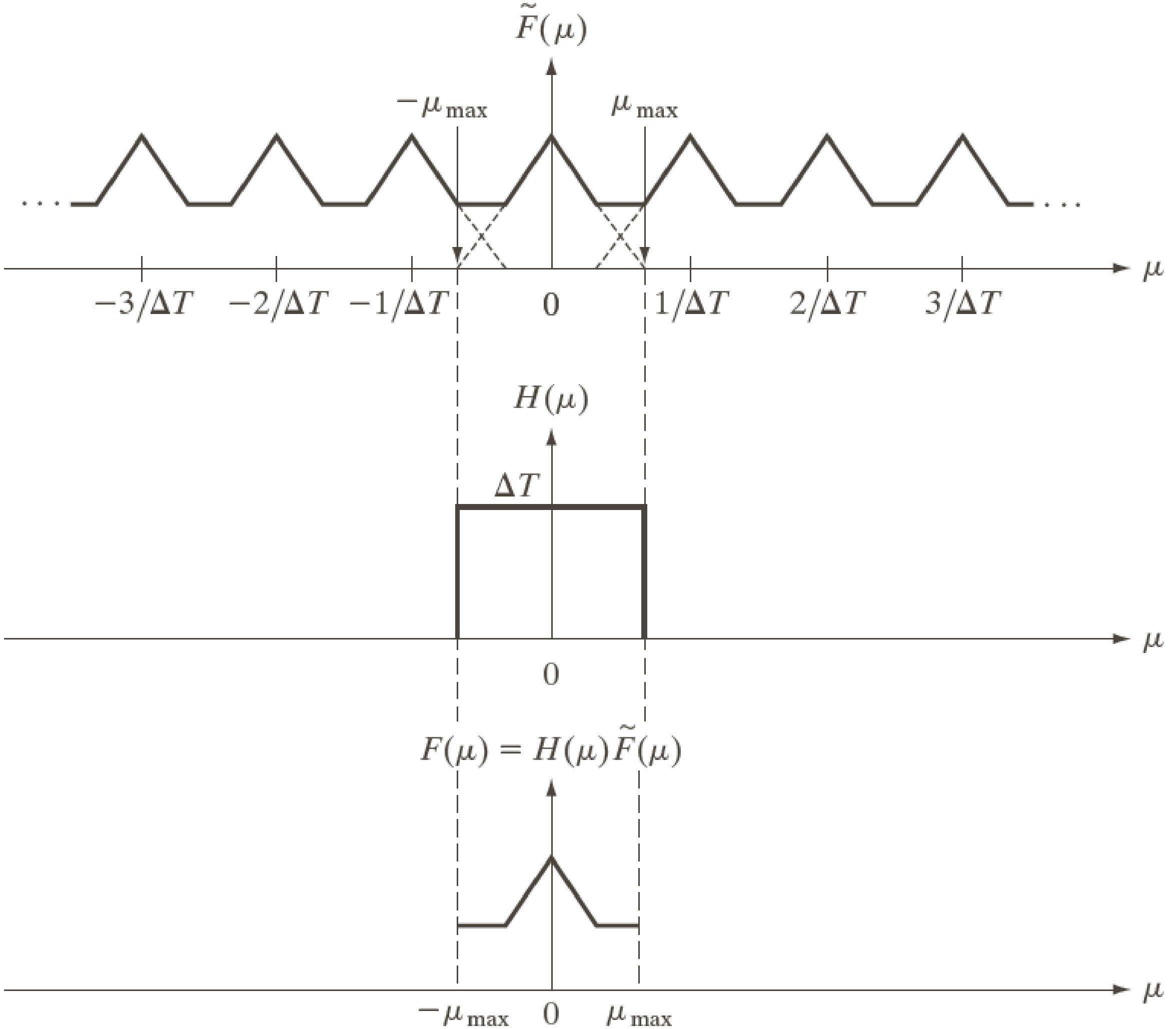
Reconstruction Filter



a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

Aliasing



a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Discrete & Fast Fourier Transform

Lecture 17: 03-Sep-12

Dr. P P Das

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$$

Sampling & FT

$$\tilde{F}(\mu)$$

$$= \Im\{\tilde{f}(t)\} = \Im\{f(t)s_{\Delta T}(t)\}$$

$$= F(\mu) \bullet S(\mu)$$

$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$$

Sampling & FT

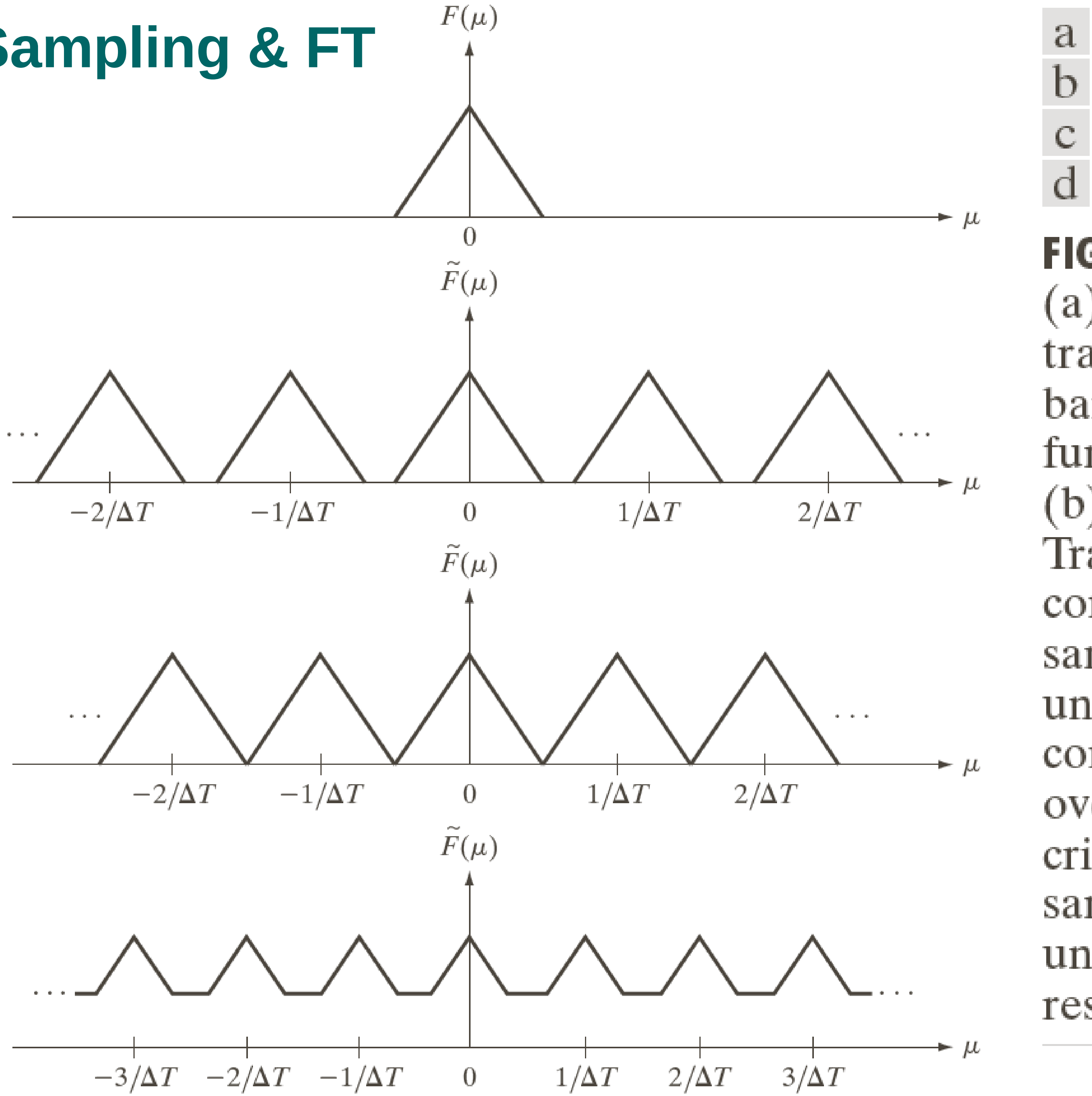


FIGURE 4.6
(a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

DFT: How to compute FT?

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) \right] e^{-j2\pi\mu t} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

$$f_n = \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) dt$$

$$= f(n\Delta T)$$

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

$$f_n = f(n\Delta T)$$

Discrete Fourier Transform

- f_n is a discrete function
- $\tilde{F}(\mu)$ is continuous and infinitely periodic with period $\dfrac{1}{\Delta T}$
- $\tilde{F}(\mu)$ need to be characterized over one period
- Sampling one period is the basis for the DFT
- Obtain M equally spaced samples of $\tilde{F}(\mu)$ over the

period $\mu = 0$ to $\mu = 1/\Delta T$

- $\mu = \dfrac{m}{M\Delta T}, m = 0,1,2,\cdots M - 1$

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

$$\mu = \frac{m}{M\Delta T}, m = 0,1,2,\cdots M-1$$

Discrete Fourier Transform

$$f_n = f(n\Delta T)$$

DFT

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}, m = 0,1,2,\cdots M-1$$

IDFT

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}, n = 0,1,2,\cdots M-1$$

DFT: Notations

t : Continuous (time) variable in 1 - D

μ : Continuous (frequency) variable in 1 - D

t, z : Continuous function variables in 2 - D

μ, v : Continuous transformvariables in 2 - D

x, y : Discrete function variables in 2 - D

u, v : Discrete transformvariables in 2 - D

DFT Pair

DFT

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}, u = 0,1,2,\cdots M-1$$

IDFT

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}, x = 0,1,2,\cdots M-1$$

Discrete Fourier Transform Pair

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}$$

DFT pair :

- Independent of Sampling Interval ΔT
- Independent of Frequency Interval $\mu = 1/\Delta T$
- Is applicable to any finite set of

discrete samples taken uniformly

DFT: Periodicity

DFT

$$F(u) = F(u + kM), k = 0,1,2,\cdots$$

IDFT

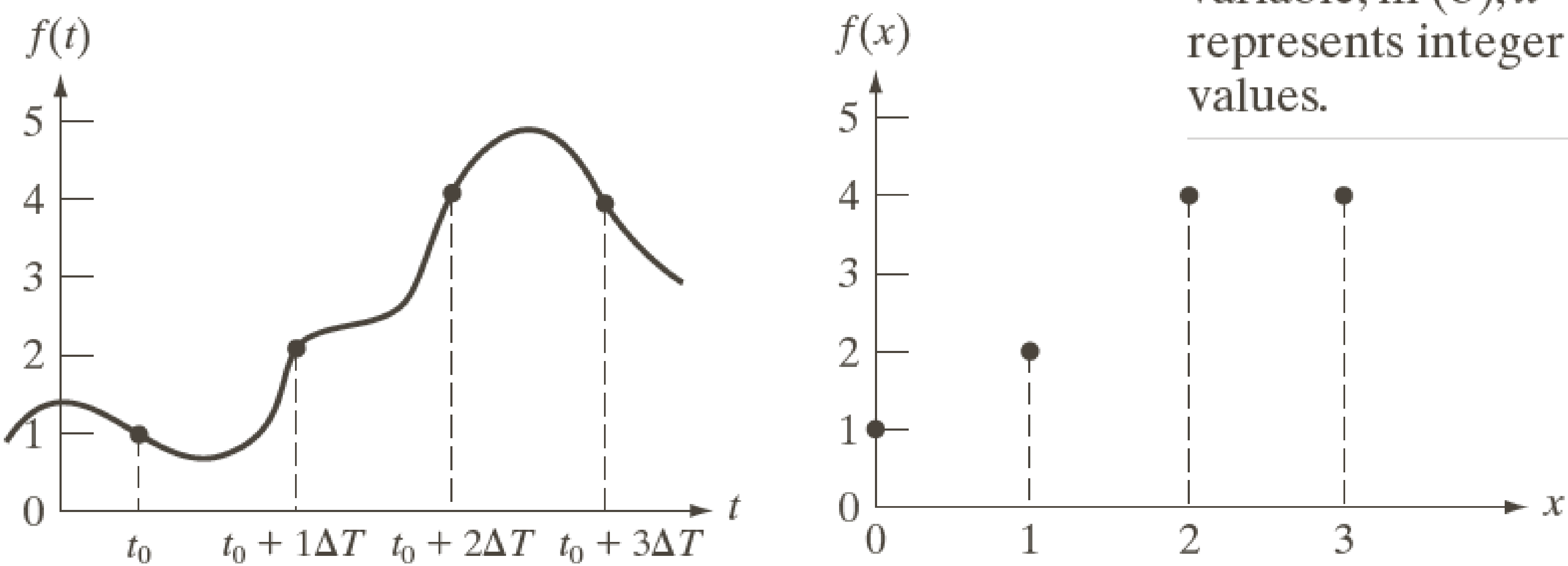
$$f(x) = f(x + kM), k = 0,1,2,\cdots$$

Sampling & Frequency Intervals

$$T = M\Delta T; \quad \Delta u = \frac{1}{M\Delta T} = \frac{1}{T}; \quad \Omega = M\Delta u = \frac{1}{\Delta T}$$

- Resolution in frequency Δu depends on the duration T over which $f(t)$ is sampled.
- Range of frequencies spanned by the DFT depends on the sampling interval ΔT

DFT: Example

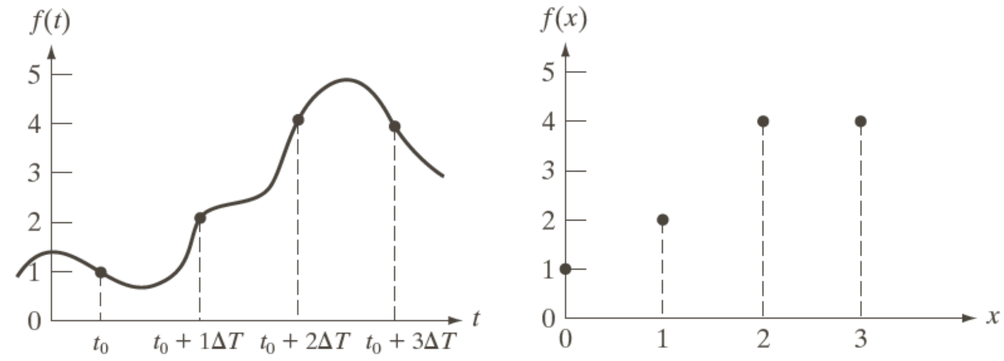


a b

FIGURE 4.11
(a) A function, and (b) samples in the x-domain. In (a), t is a continuous variable; in (b), x represents integer values.

$$f(0) = 1; \quad f(1) = 2; \quad f(2) = 4; \quad f(3) = 4$$

DFT: Example



$$F(0) = \sum_{x=0}^{3} f(x) = \left[f(0) + f(1) + f(2) + f(3) \right]$$

$$= 1 + 2 + 4 + 4 = 11$$

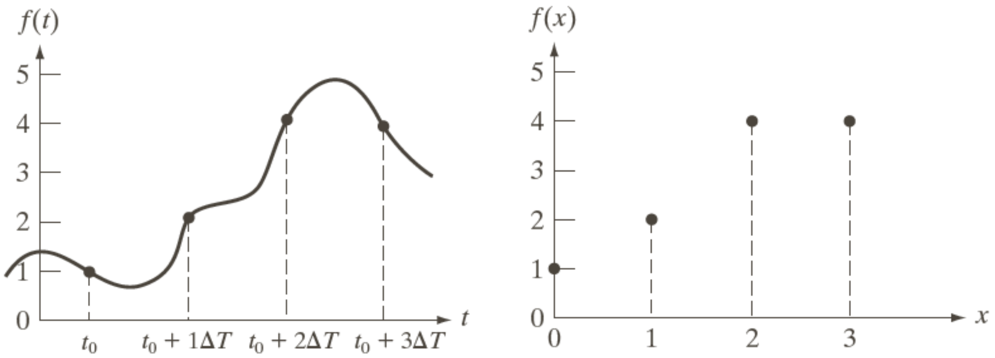
$$F(1) = \sum_{x=0}^{3} f(x) e^{-j2\pi(1)x/4}$$

$$= 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}$$

$$= 1 + (0 - 2j) + (-4 + 0j) + (0 + 4j) = -3 + 2j$$

$$F(2) = -(1 + 0j); \qquad F(3) = -(3 + 2j)$$

DFT: Example



$$f(0) = \frac{1}{4}\sum_{u=0}^{3} F(u)e^{j2\pi u(0)}$$

$$= \frac{1}{4}\sum_{u=0}^{3} F(u)$$

$$= \frac{1}{4}\left[11 - 3 + 2j - 1 - 3 - 2j\right] = \frac{1}{4}\left[4\right] = 1$$

1-D FFT in Matrix/Vector

$$\begin{cases} Z(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} z(n) \cdot W_N^{nk} \\ z(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} Z(k) \cdot W_N^{-nk} \end{cases}$$

- { z(n) } ⇔ { Z(k) }

  n, k = 0, 1, …, N-1, $W_N = \exp\{-j2\pi / N\}$

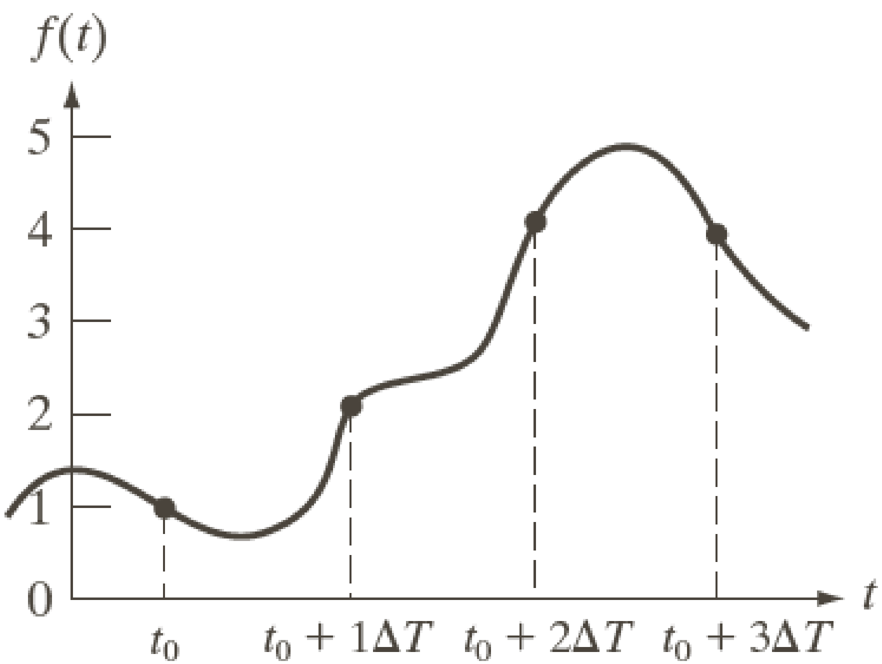
  ~ complex conjugate of primitive N^th root of unity

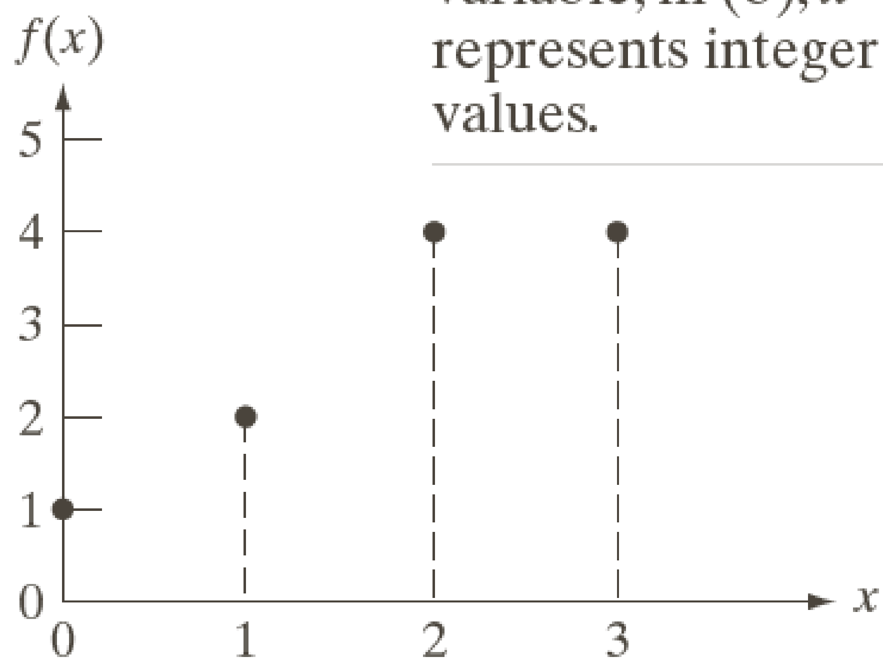
$$\begin{bmatrix} Z(0) \\ Z(1) \\ Z(2) \\ \vdots \\ Z(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & e^{-j2\pi \cdot 2/N} & \cdots & e^{-j2\pi(N-1)/N} \\ 1 & e^{-j2\pi \cdot 2/N} & e^{-j2\pi \cdot 4/N} & \cdots & e^{-j2\pi \cdot 2(N-1)/N} \\ \vdots & \vdots & & & \vdots \\ 1 & e^{-j2\pi(N-1)/N} & e^{-j2\pi \cdot 2(N-1)/N} & \cdots & e^{-j2\pi(N-1)^2/N} \end{bmatrix} \cdot \begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ \vdots \\ z(N-1) \end{bmatrix} = \begin{bmatrix} \underline{a}_0^{*T} \\ \underline{a}_1^{*T} \\ \vdots \\ \underline{a}_{N-1}^{*T} \end{bmatrix} \cdot \underline{z}$$

$$\begin{bmatrix} z(0) \\ z(1) \\ z(2) \\ \vdots \\ z(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j2\pi/N} & e^{j2\pi \cdot 2/N} & \cdots & e^{j2\pi(N-1)/N} \\ 1 & e^{j2\pi \cdot 2/N} & e^{j2\pi \cdot 4/N} & \cdots & e^{j2\pi \cdot 2(N-1)/N} \\ \vdots & \vdots & & & \vdots \\ 1 & e^{j2\pi(N-1)/N} & e^{j2\pi \cdot 2(N-1)/N} & \cdots & e^{j2\pi(N-1)^2/N} \end{bmatrix} \cdot \begin{bmatrix} Z(0) \\ Z(1) \\ Z(2) \\ \vdots \\ Z(N-1) \end{bmatrix} = \begin{bmatrix} \underline{a}_0 & \underline{a}_1 & \cdots & \underline{a}_{N-1} \end{bmatrix} \cdot \begin{bmatrix} Z(0) \\ Z(1) \\ \vdots \\ Z(N-1) \end{bmatrix}$$

DFT: Example

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

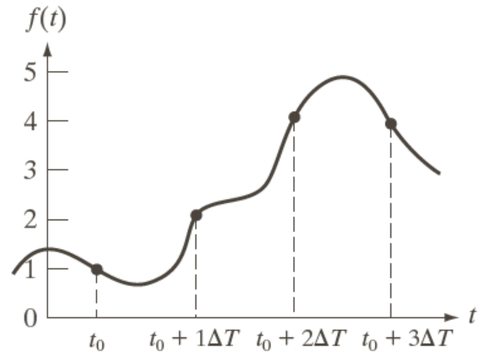


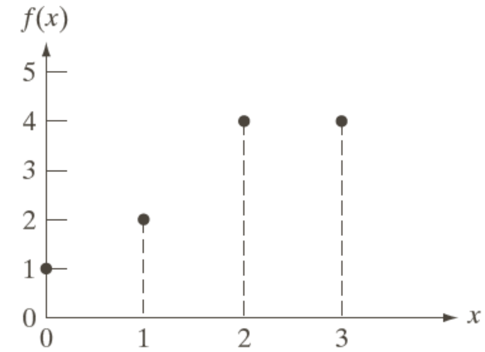


a b

FIGURE 4.11 (a) A function, and (b) samples in the x-domain. In (a), t is a continuous variable; in (b), x represents integer values.

$$f(0) = 1; \quad f(1) = 2; \quad f(2) = 4; \quad f(3) = 4$$

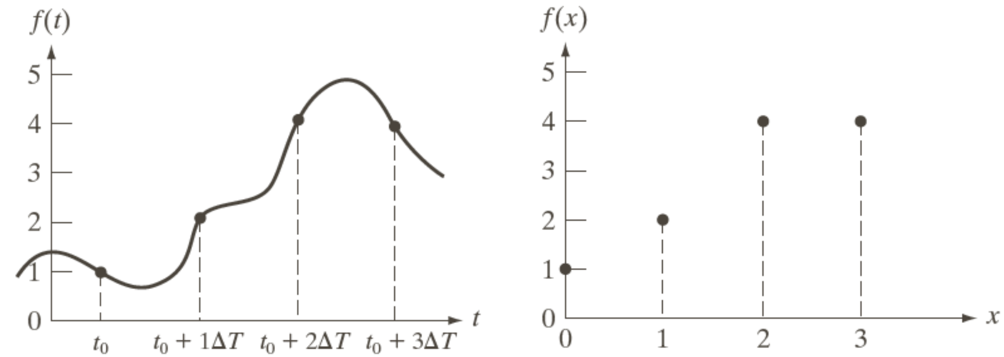




DFT: By Matrix

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/2} & e^{-j\pi} & e^{-j3\pi/2} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} \\ 1 & e^{-j3\pi/2} & e^{-j3\pi} & e^{-j9\pi/2} \end{bmatrix} . \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0-j & -1+0j & 0+j \\ 1 & -1+0j & 1+0j & -1+0j \\ 1 & 0+j & -1+0j & 0-j \end{bmatrix} . \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$



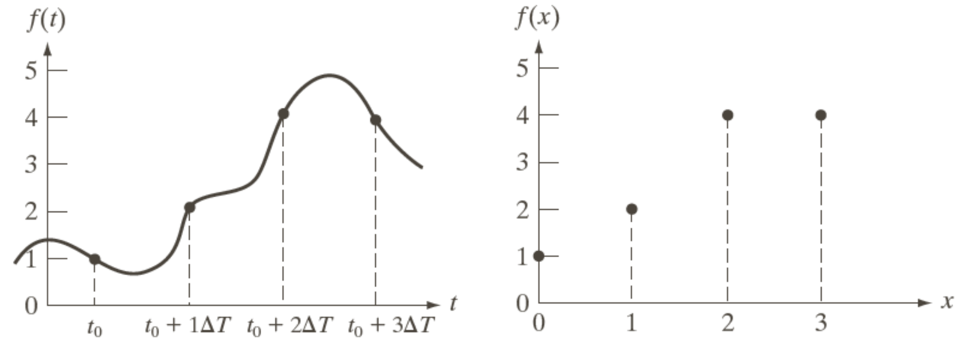
DFT: By Matrix

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0-j & -1+0j & 0+j \\ 1 & -1+0j & 1+0j & -1+0j \\ 1 & 0+j & -1+0j & 0-j \end{bmatrix} . \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

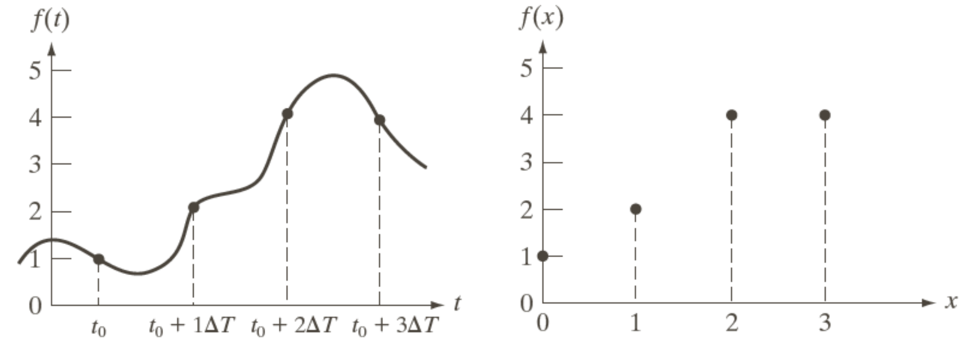
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+2+4+4 \\ 1-2j-4+4j \\ 1-2+4-4 \\ 1+2j-4-4j \end{bmatrix} = \begin{bmatrix} 11 \\ -3+2j \\ -1 \\ -3-2j \end{bmatrix}$$

IDFT: By Matrix



$$4\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\pi/2} & e^{j\pi} & e^{j3\pi/2} \\ 1 & e^{j\pi} & e^{j2\pi} & e^{j3\pi} \\ 1 & e^{j3\pi/2} & e^{j3\pi} & e^{j9\pi/2} \end{bmatrix} . \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix}$$

$$4\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0+j & -1+0\mathrm{j} & 0-j \\ 1 & -1+0j & +1+0\mathrm{j} & -1+0j \\ 1 & 0-j & -1+0j & 0+j \end{bmatrix} . \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix}$$



IDFT: By Matrix

$$4\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0+j & -1-0j & 0-j \\ 1 & -1-0j & +1-0j & -1-0j \\ 1 & 0-j & -1-0j & 0+j \end{bmatrix} . \begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 11 \\ -3+2j \\ -1 \\ -3-2j \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & +1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix} . \begin{bmatrix} 11 \\ -3+2j \\ -1 \\ -3-2j \end{bmatrix} = \begin{bmatrix} 11-3+2j-1-3-2j \\ 11-3j-2+1+3j-2 \\ 11+3-2j-1+3+2j \\ 11+3j+2+1-3j+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 16 \\ 16 \end{bmatrix}$$

Inverse Discrete Fourier Transform

$$\text{DFT}: F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}, u = 0,1,2,\cdots M-1$$

$$\text{IDFT}: f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}, x = 0,1,2,\cdots M-1$$

Taking complex conjugate:

$$f^*(x) = \frac{1}{M} \sum_{u=0}^{M-1} F^*(u) e^{-j2\pi ux/M} = \frac{1}{M} DFT(F^*(u))$$

$$f(x) = \frac{1}{M} DFT^*(F^*(u))$$

DFT Complexity

$Mult(M)$: Number of complex multiplications in an M point DFT

$Add(M)$: Number of complex additions in an M point DFT

DFT / IDFT Complexity

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}, u = 0,1,2,\cdots M-1$$

Assume that $e^{-j2\pi ux/M}$ are pre-computed.

$$Mult(M) = M^2 = O(M^2)$$

$$Add(M) = M(M-1) = O(M^2)$$

$$F(u)=\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}, u=0,1,2,\cdots M-1$$

Fast Fourier Transform

$$F(u)=\sum_{x=0}^{M-1} f(x)W_M^{ux}, \quad u=0,1,2,\cdots M-1$$

$$W_M=e^{-j2\pi/M} \qquad M=2^n=2K$$

Properties:

$$W_{2K}^{2ux}=W_K^{ux}; \quad W_M^{u+M}=W_M^u; \quad W_{2M}^{u+M}=-W_{2M}^u$$

$$W_M = e^{-j2\pi/M} \qquad M = 2^n = 2K$$

$$W_{2K}^{2ux} = W_K^{ux}$$

Fast Fourier Transform

$$F(u) = \sum_{x=0}^{2K-1} f(x) W_M^{ux}$$

$$= \sum_{x=0}^{K-1} f(2x) W_{2K}^{u(2x)} + \sum_{x=0}^{K-1} f(2x+1) W_{2K}^{u(2x+1)}$$

$$= \sum_{x=0}^{K-1} f(2x) W_K^{ux} + \sum_{x=0}^{K-1} f(2x+1) W_K^{ux} W_{2K}^{u}$$

$$W_M = e^{-j2\pi/M}$$

$$M = 2^n = 2K$$

$$W_M^{u+M} = W_M^u; \quad W_{2M}^{u+M} = -W_{2M}^u$$

Fast Fourier Transform

$$F_{even}(u) = \sum_{x=0}^{K-1} f(2x) W_K^{ux}$$

$$F_{odd}(u) = \sum_{x=0}^{K-1} f(2x+1) W_K^{ux}$$

$$F(u) = F_{even}(u) + F_{odd}(u) W_{2K}^u$$

$$F(u+K) = F_{even}(u) - F_{odd}(u) W_{2K}^u$$

FFT Complexity

$Mult(M) = Mult(n)$: Number of complex multiplications in an $M = 2^n$ point FFT

$Add(M) = Add(n)$: Number of complex additions in an $M = 2^n$ point FFT

$$F_{even}(u) = \sum_{x=0}^{K-1} f(2x) W_K^{ux}$$

$$F_{odd}(u) = \sum_{x=0}^{K-1} f(2x+1) W_K^{ux}$$

$$F(u) = F_{even}(u) + F_{odd}(u) W_{2K}^{u}$$

$$F(u+K) = F_{even}(u) - F_{odd}(u) W_{2K}^{u}$$

FFT

$$n = 1; M = 2^1 = 2; K = 1$$

$$Mult(1) = 1$$

$$Add(1) = 2$$

$$F_{even}(0) = f(0)$$

$$F_{odd}(0) = f(1)$$

$$F(0) = F_{even}(0) + F_{odd}(0) W_2^0$$

$$F(1) = F_{even}(0) - F_{odd}(0) W_2^0$$

$$F_{even}(u) = \sum_{x=0}^{K-1} f(2x) W_K^{ux}$$

$$F_{odd}(u) = \sum_{x=0}^{K-1} f(2x+1) W_K^{ux}$$

$$F(u) = F_{even}(u) + F_{odd}(u) W_{2K}^{u}$$

$$F(u+K) = F_{even}(u) - F_{odd}(u) W_{2K}^{u}$$

FFT

$$n = 2; M = 2^2 = 4; K = 2$$

$$F_{even}(u) = 2\text{-point FFT from } f(0) \;\&\; f(2)$$

$$F_{odd}(u) = 2\text{-point FFT from } f(1) \;\&\; f(3)$$

$$F(0) = F_{even}(0) + F_{odd}(0) W_4^0$$

$$F(1) = F_{even}(1) + F_{odd}(1) W_4^1$$

$$F(2) = F(0+2) = F_{even}(0) - F_{odd}(0) W_4^0$$

$$F(3) = F(1+2) = F_{even}(1) - F_{odd}(1) W_4^1$$

$$Mult(2) = 2Mult(1) + 2$$

$$Add(2) = 2Add(1) + 4$$

FFT Complexity

$$\begin{aligned} Mult(n) &= 2Mult(n-1) + 2^{n-1}; \quad n \geq 1 \\ &= 0; \qquad\qquad\qquad\qquad ; \quad n = 0 \end{aligned}$$

$$\begin{aligned} Add(n) &= 2Add(n-1) + 2^{n}; \quad n \geq 1 \\ &= 0; \qquad\qquad\qquad\qquad ; \quad n = 0 \end{aligned}$$

$$Mult(M) = Mult(n) = \frac{1}{2} M \log_2 M = O(M \log_2 M)$$

$$Add(M) = Add(n) = M \log_2 M = O(M \log_2 M)$$

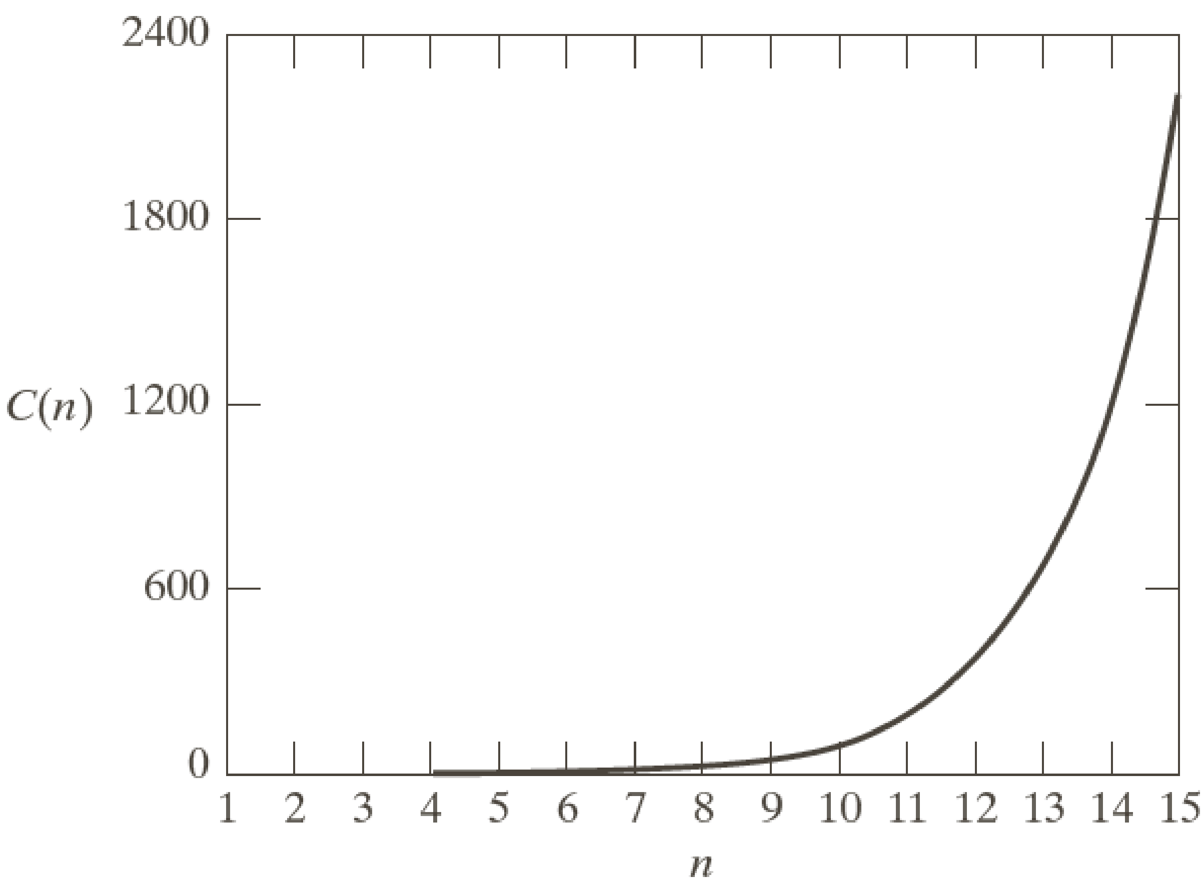


FIGURE 4.67 Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of n.

2-D DFT & Images

Lecture 18-19: 04-Sep-12

Dr. P P Das

Kinect Demo in Lecture 19

2-D Impulse

$$\delta(t,z) = \begin{cases} 0 & t \neq 0, z \neq 0 \\ \infty & t = z = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$$

2-D Sifting Property

$$\int_{\infty}^{\infty}\int_{\infty}^{\infty} f(t,z)\delta(t,z)\,dtdz = f(0,0)$$

$$\int_{\infty}^{\infty}\int_{\infty}^{\infty} f(t,z)\delta(t - t_0, z - z_0)\,dtdz = f(t_0, z_0)$$

2-D Discrete Impulse

$$\delta(x,y) = \begin{cases} 0 & x \neq 0, y \neq 0 \\ 1 & x = y = 0 \end{cases}$$

2-D Discrete Sifting Property

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0) = f(x_0,y_0)$$

2-D Impulse

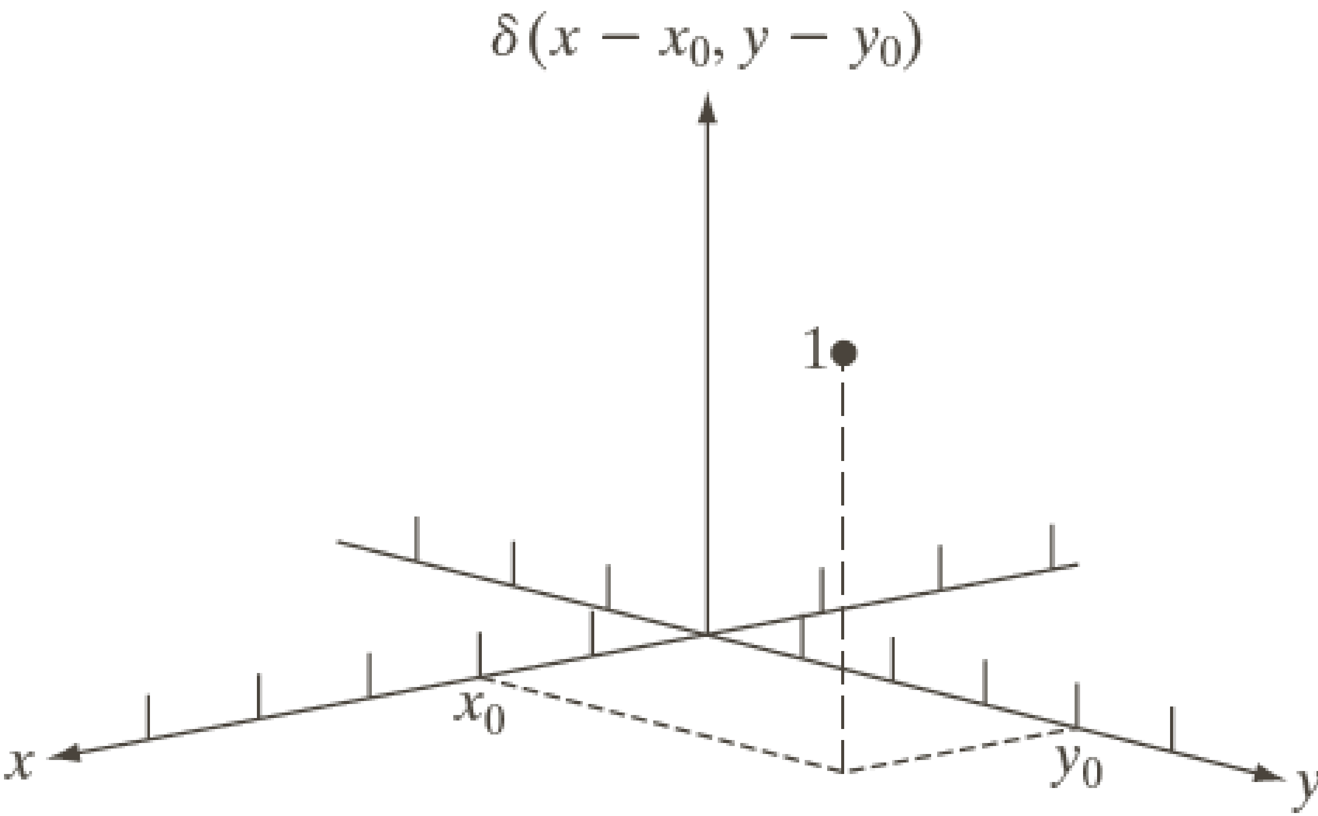


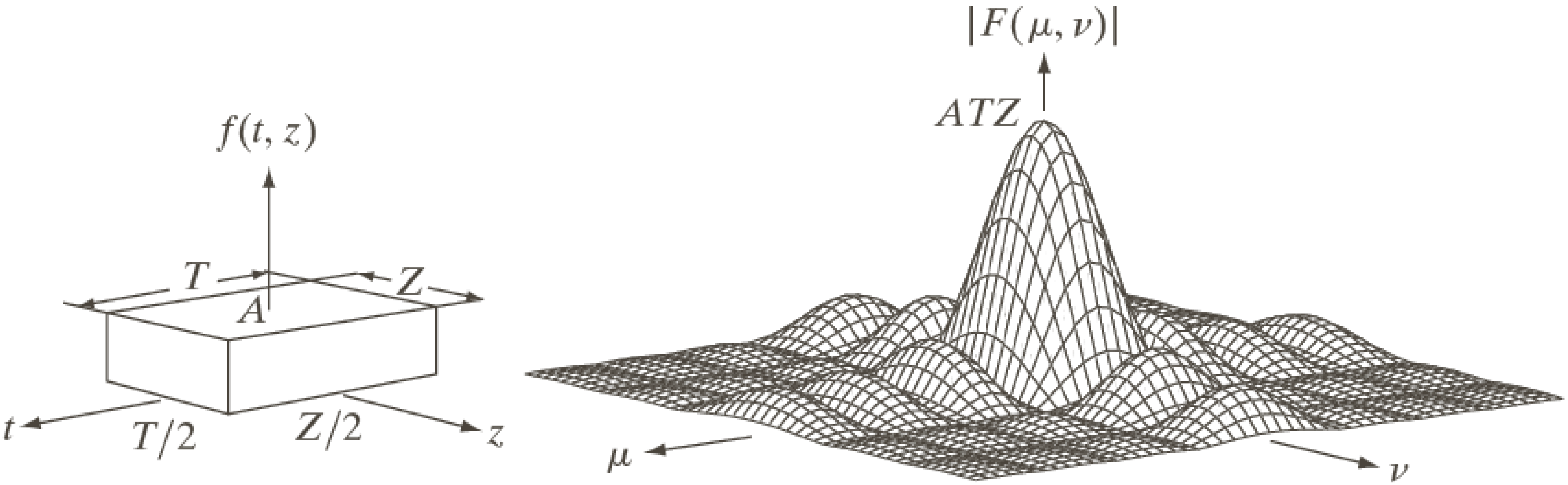
FIGURE 4.12 Two-dimensional unit discrete impulse. Variables x and y are discrete, and δ is zero everywhere except at coordinates (x_0, y_0).

2-D Continuous Fourier Transform Pair

$$F(\mu,\nu) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

$$f(t,z) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t+\nu z)}d\mu d\nu$$

2-D FT: Example



a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

2-D FT: Example

$$F(\mu,\nu)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

$$=\int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} Ae^{-j2\pi(\mu t+\nu z)}dtdz$$

$$=ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

2-D Sampling

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

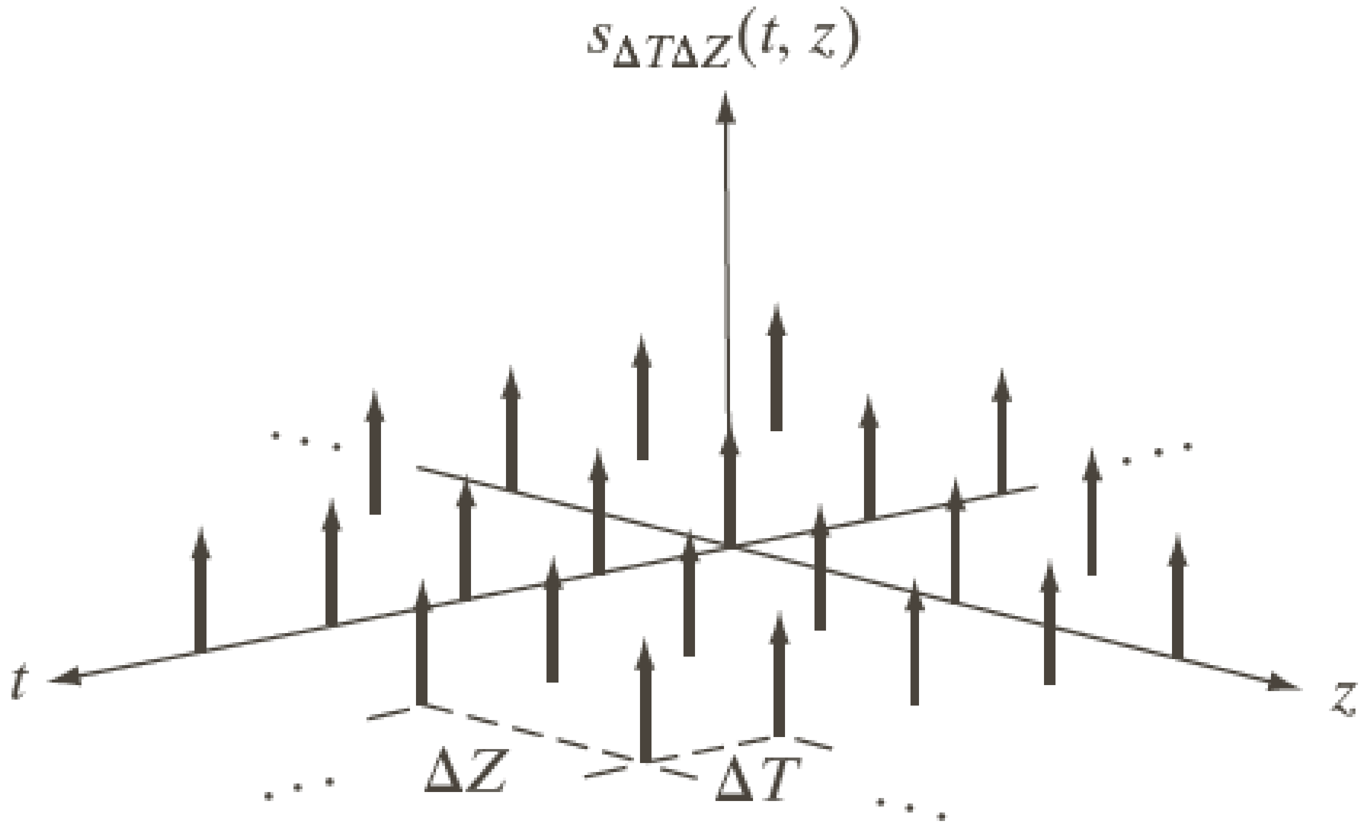
$$f(t,z) \text{ is band-limited}$$

$$F(\mu,\nu) = 0 \text{ for } |\mu| \geq \mu_{\max} \text{ and } |\nu| \geq \nu_{\max}$$

$$\Delta T < \frac{1}{2\mu_{\max}} \qquad \Delta Z < \frac{1}{2\nu_{\max}}$$

2-D Sampling

FIGURE 4.14
Two-dimensional impulse train.



Frequency Aliasing

- Signals sampled below Nyquist rate (under-sampled) have overlapped periods.
- In aliasing, high frequency components masquerade as low frequency components in sampled function. Hence, ‘aliasing’ or ‘false identity’.

Aliasing

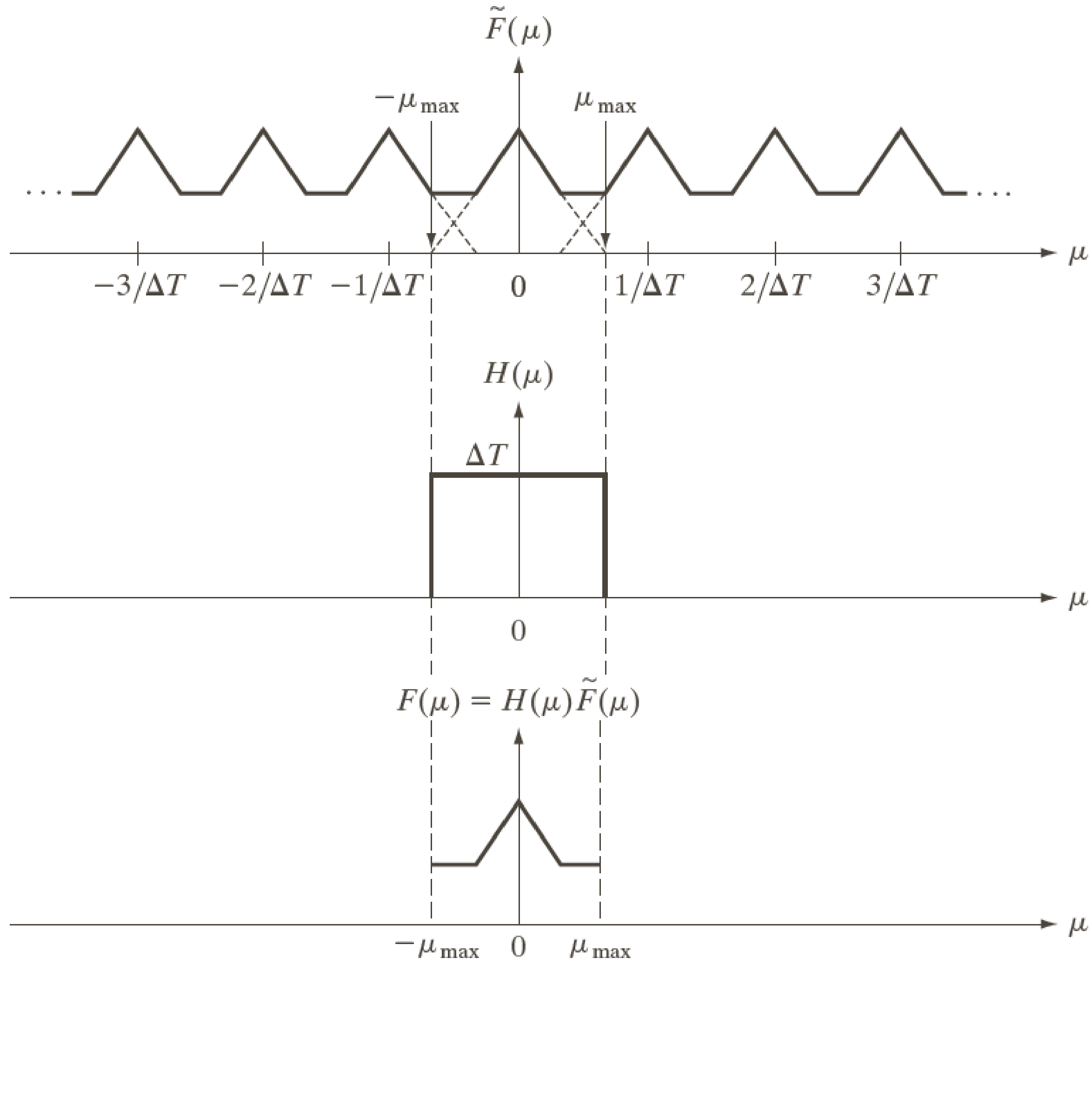


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Aliasing

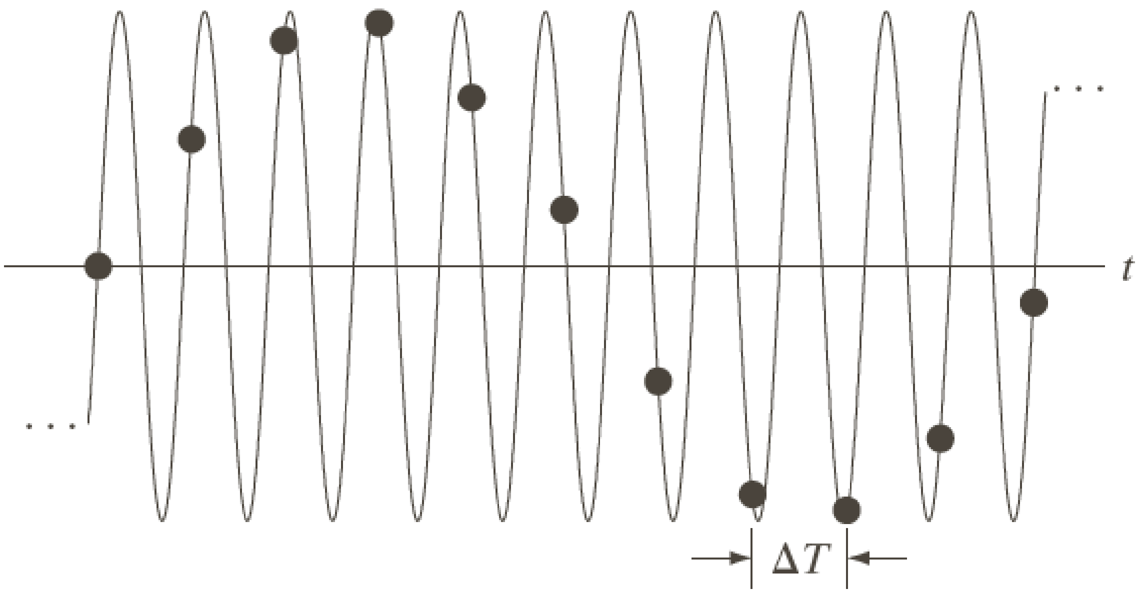


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

$$h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

Frequency Aliasing

- Most band-limited signals have infinite frequency components in sampled signal due to finite duration of sampling.

$$f(t) = \Im^{-1}\{F(\mu)\} = \Im^{-1}\{\tilde{F}(\mu)H(\mu)\}$$

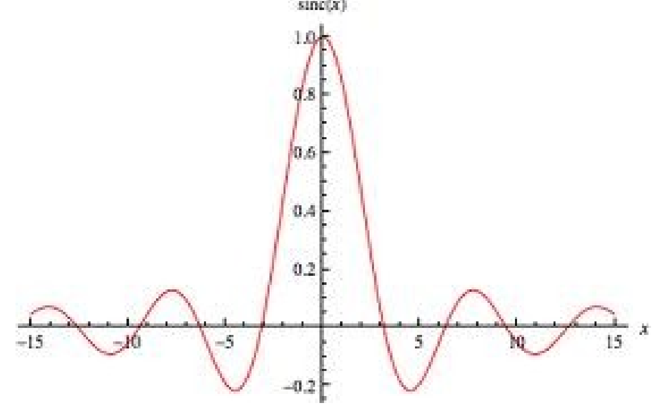
$$= h(t) \bullet \tilde{f}(t)$$

$$= \sum_{n=-\infty}^{\infty} f(n\Delta T)\operatorname{sinc}\left[(t - n\Delta T)/n\Delta T\right]$$

Frequency Aliasing

- Aliasing is inevitable while working with sampled records of finite length
- Aliasing can be *reduced* by smoothing the input function to *attenuate* higher frequencies (defocusing images)
- This is known as ‘anti-aliasing’ and has to be done *before* the function is sampled

Aliasing & Interpolation



$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\,\mathrm{sinc}\left[(t - n\Delta T)/n\Delta T\right]$$

$$f(t) = \begin{cases} f(k\Delta T), t = k\Delta T \text{ as } \mathrm{sinc}(0) = 1 \text{ and } \mathrm{sinc}(m) = 0 \\ \text{interpolation of sum of sinc function}, t \neq k\Delta T \end{cases}$$

Aliasing in Images

- Spatial Aliasing
  - Due to under-sampling in space
- Temporal Aliasing
  - Due to slow time interval between frames in video
  - Wagon-Wheel Effect

Aliasing in Spectrum

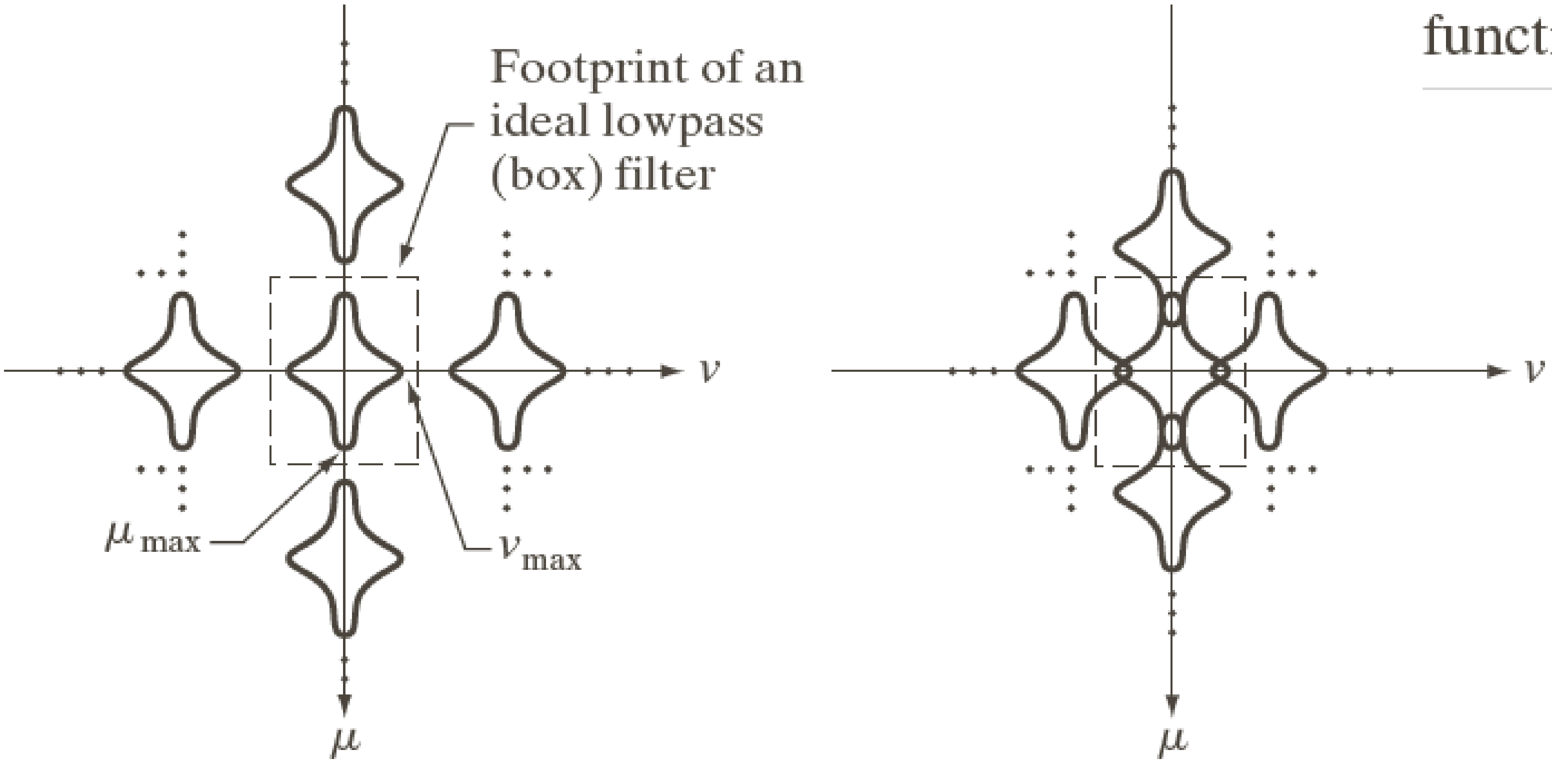
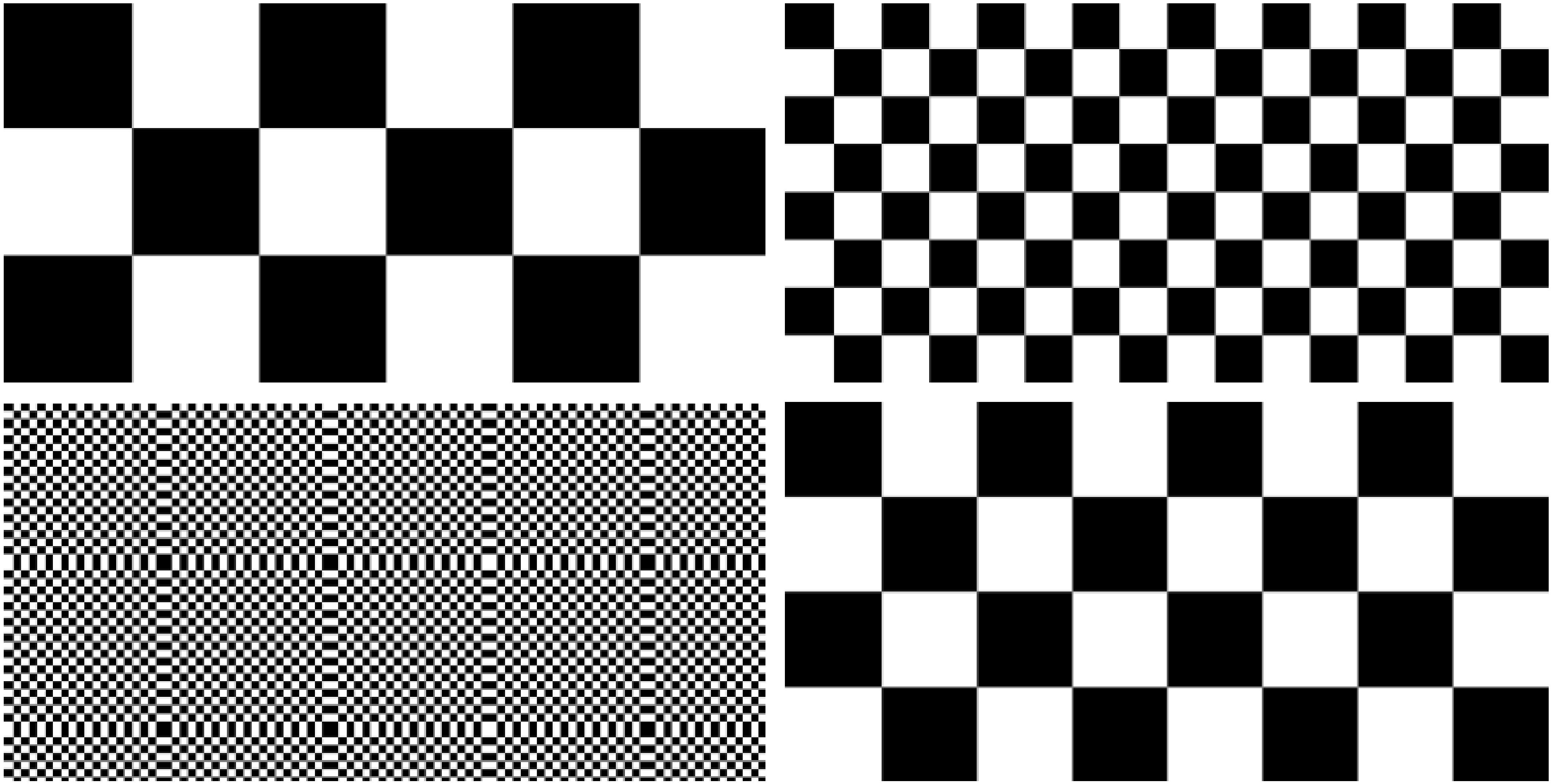


FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.



a	b
c	d

Sampling: 96X96 Pixels

FIGURE 4.16 Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a "normal" image.

Interpolation Techniques

- Nearest Neighbor Interpolation
- Bi-Linear Interpolation
- Bi-Cubic Interpolation

Interpolation & Re-sampling

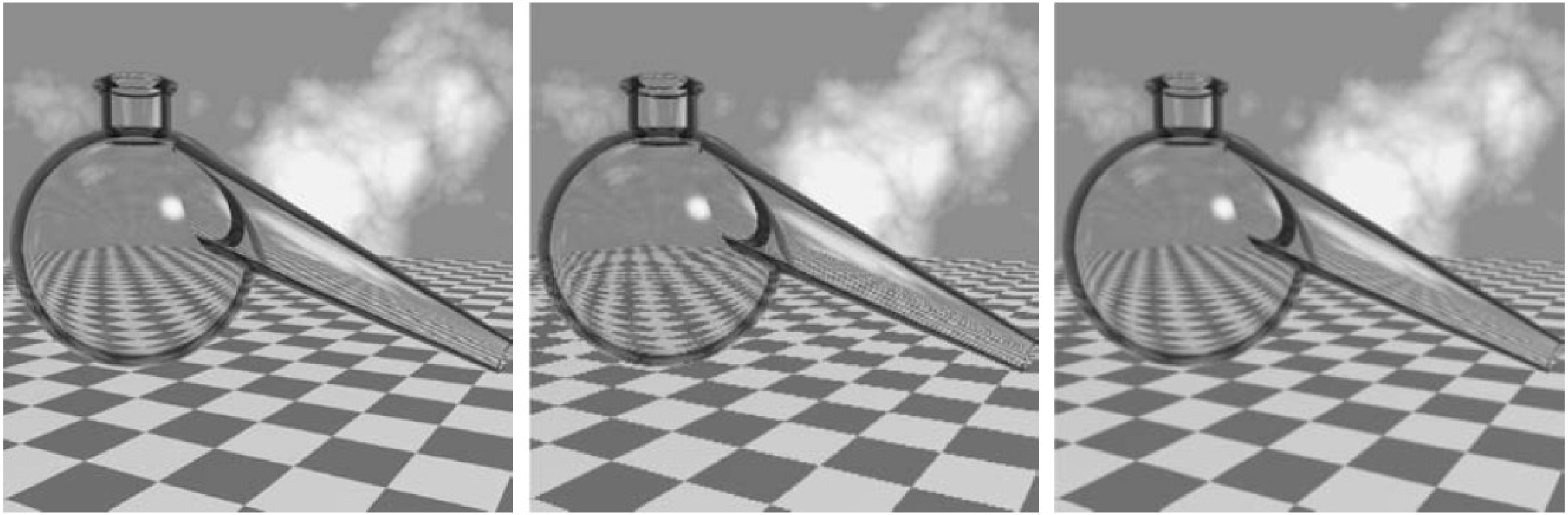
- Zooming
  - Over-sampling
  - Pixel Replication / Row-Column Duplication
- Shrinking
  - Under-sampling
  - Row-Column Deletion
  - Blur slightly before shrinking

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

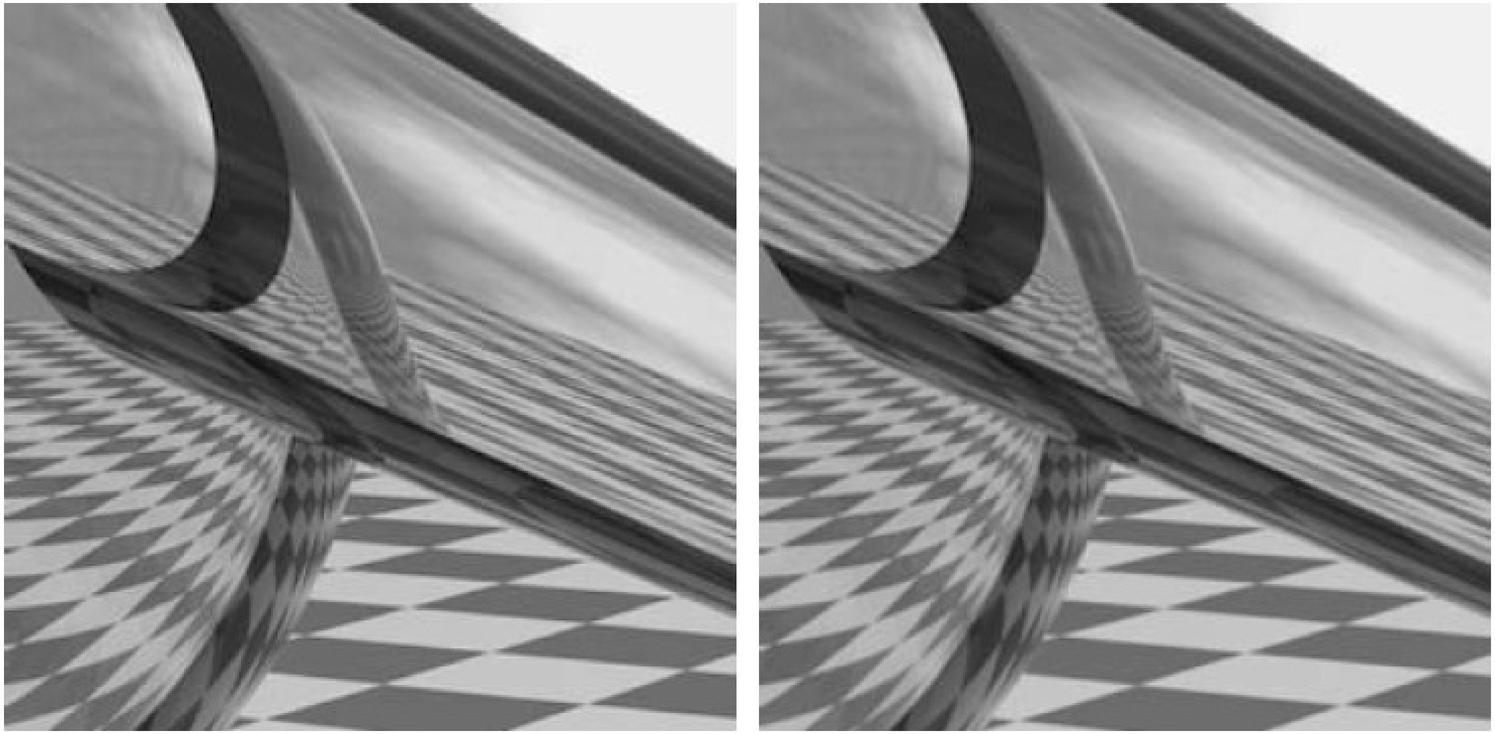
Aliasing Artifact: Jaggies

- Block-like image component
- Common in images with string edge content



a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024×1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5×5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)



a b

FIGURE 4.19 Image zooming. (a) A 1024×1024 digital image generated by pixel replication from a 256×256 image extracted from the middle of Fig. 4.18(a). (b) Image generated using bi-linear interpolation, showing a significant reduction in jaggies.

Aliasing Artifact: Moiré Patterns

- Beat patterns produced between two gratings of approximately equal spacing
- Common in images with periodic or nearly periodic content

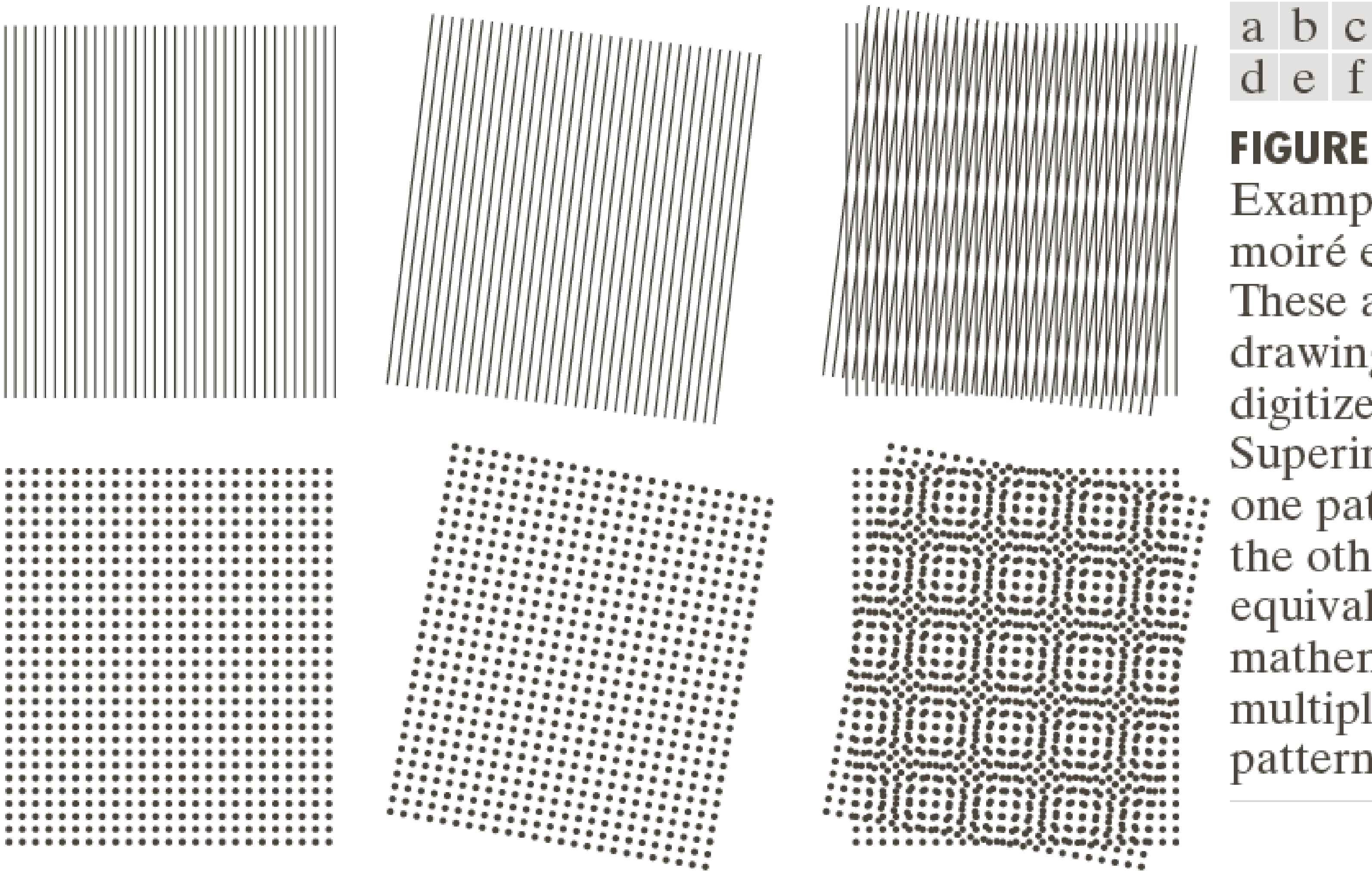


FIGURE 4.20 Examples of the moiré effect. These are ink drawings, not digitized patterns. Superimposing one pattern on the other is equivalent mathematically to multiplying the patterns.

FIGURE 4.21
A newspaper image of size 246×168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the $\pm 45°$ orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

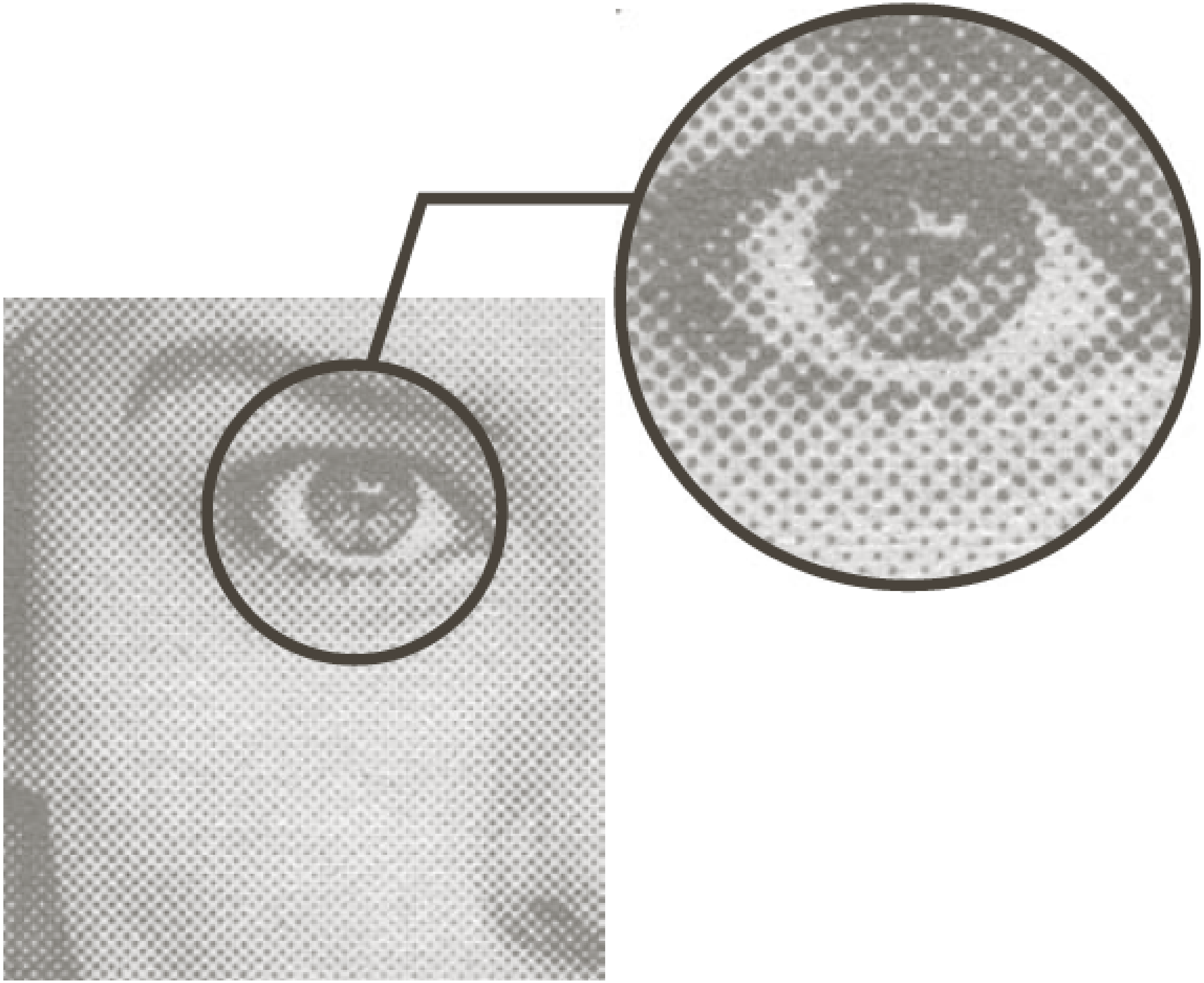


FIGURE 4.22
A newspaper image and an enlargement showing how halftone dots are arranged to render shades of gray.

2-D DFT Properties

Lecture 20: 10-Sep-12

Dr. P P Das

2-D DFT Pair

DFT

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

IDFT

$$f(x) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

2-D: Separability

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$= \sum_{x=0}^{M-1} e^{-j2\pi ux/M} \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi vy/N}$$

$$= \sum_{x=0}^{M-1} F(x,v)e^{-j2\pi ux/M}$$

$$\text{where } F(x,v) = \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi vy/N}$$

Properties of 2-D DFT: Spatial & Frequency Intervals

$$\Delta u = \frac{1}{M \Delta T}$$

$$\Delta v = \frac{1}{N \Delta Z}$$

Properties of 2-D DFT: Translation & Rotation

$$F(u-u_0, v-v_0) \Leftrightarrow f(x,y)e^{j2\pi(u_0 x/M + v_0 y/N)}$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

In polar coordinates :

$$x = r\cos\theta \quad y = r\sin\theta \quad u = \cos\phi \quad v = \sin\phi$$

$$f(r, \theta + \theta_0) \Leftrightarrow F(\varpi, \varphi + \theta_0)$$

Properties of 2-D DFT: Periodicity

$$\begin{aligned} &F(u,v) \\ &= F(u+k_1M, v) \\ &= F(u, v+k_2N) \\ &= F(u+k_1M, v+k_2N) \end{aligned}$$

$$\begin{aligned} &f(x,y) \\ &= f(x+k_1M, y) \\ &= f(x, y+k_2N) \\ &= f(x+k_1M, y+k_2N) \end{aligned}$$

Periodicity: Centering the Transform

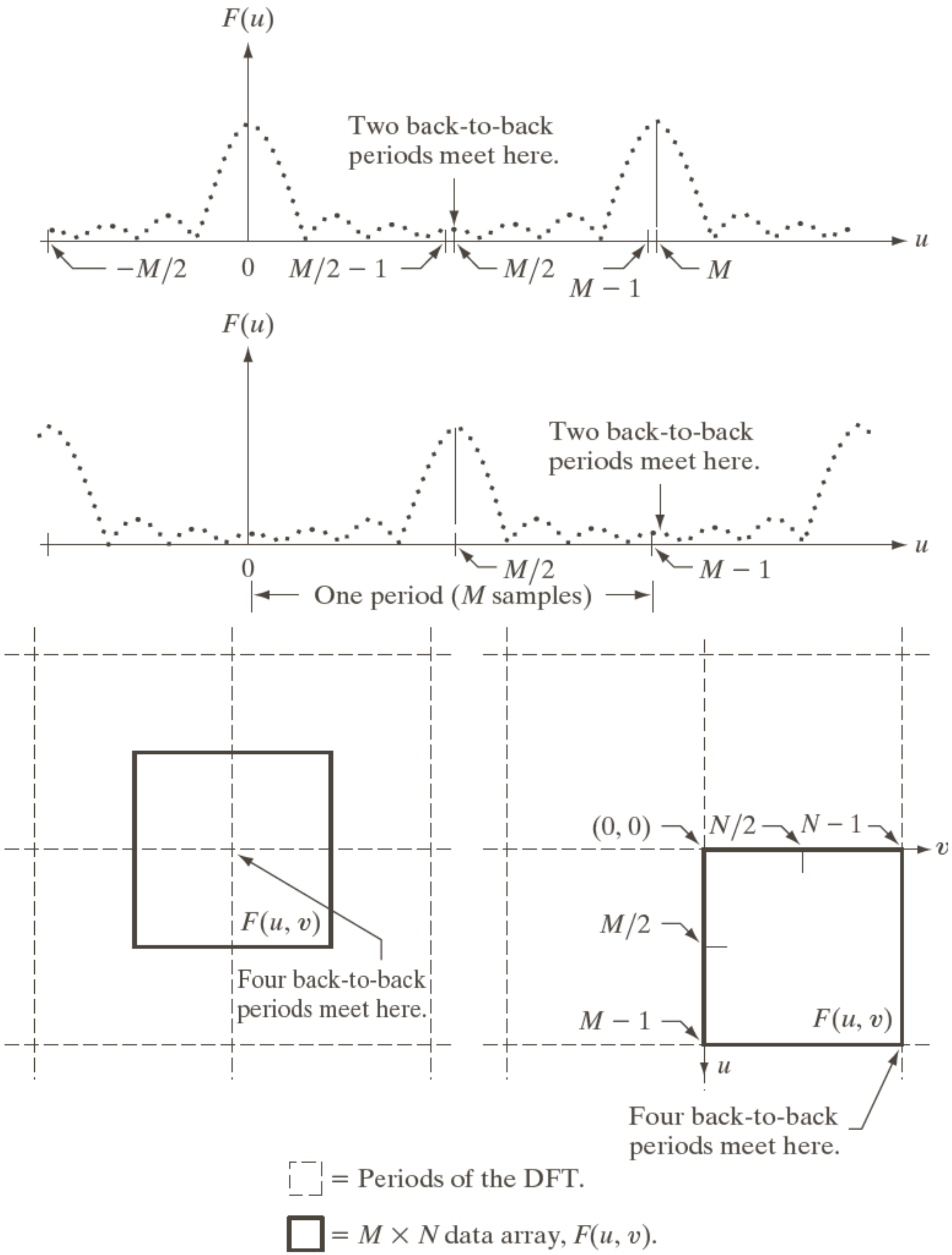
In 1-D

$$f(x)e^{2j\pi(u_0x)/M} \Leftrightarrow F(u-u_0)$$

For $u_0 = M/2$

$$f(x)(-1)^x \Leftrightarrow F(u-M/2)$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$



a
b
c d

FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u, v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A Shifted DFT obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u, v)$. The data now contains one complete, centered period, as in (b).

Properties of 2-D DFT: Symmetry

$$w(x,y) = w_e(x,y) + w_o(x,y)$$

$$w_e(x,y) = \frac{w(x,y) + w(-x,-y)}{2} = w_e(-x,-y)$$

$$w_o(x,y) = \frac{w(x,y) - w(-x,-y)}{2} = -w_o(-x,-y)$$

Properties of 2-D DFT: Symmetry

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} w_e(x,y)w_o(x,y) = 0$$

$$w_e(x,y) = w_e(M-x, N-y)$$

$$w_o(x,y) = -w_o(M-x, N-y)$$

Symmetry

For real $f(x, y)$, FT is conjugate symmetric

$$F^*(u,v) = F(-u,-v)$$

For imaginary $f(x, y)$, FT is conjugate anti-symmetric

$$F^*(u,v) = -F(-u,-v)$$

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u, v)$ and $I(u, v)$ are the real and imaginary parts of $F(u, v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

Fourier Spectrum & Phase Angle

$$F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

$$\text{Fourier Spectrum}: |F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

$$\text{Phase Angle}: e^{j\phi(u,v)} = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

$$\text{Power Spectrum}: P(u,v) = |F(u,v)|^2$$

$$= R^2(u,v) + I^2(u,v)$$

Fourier Spectrum & Phase Angle

For real $f(x, y)$, FT is conjugate symmetric

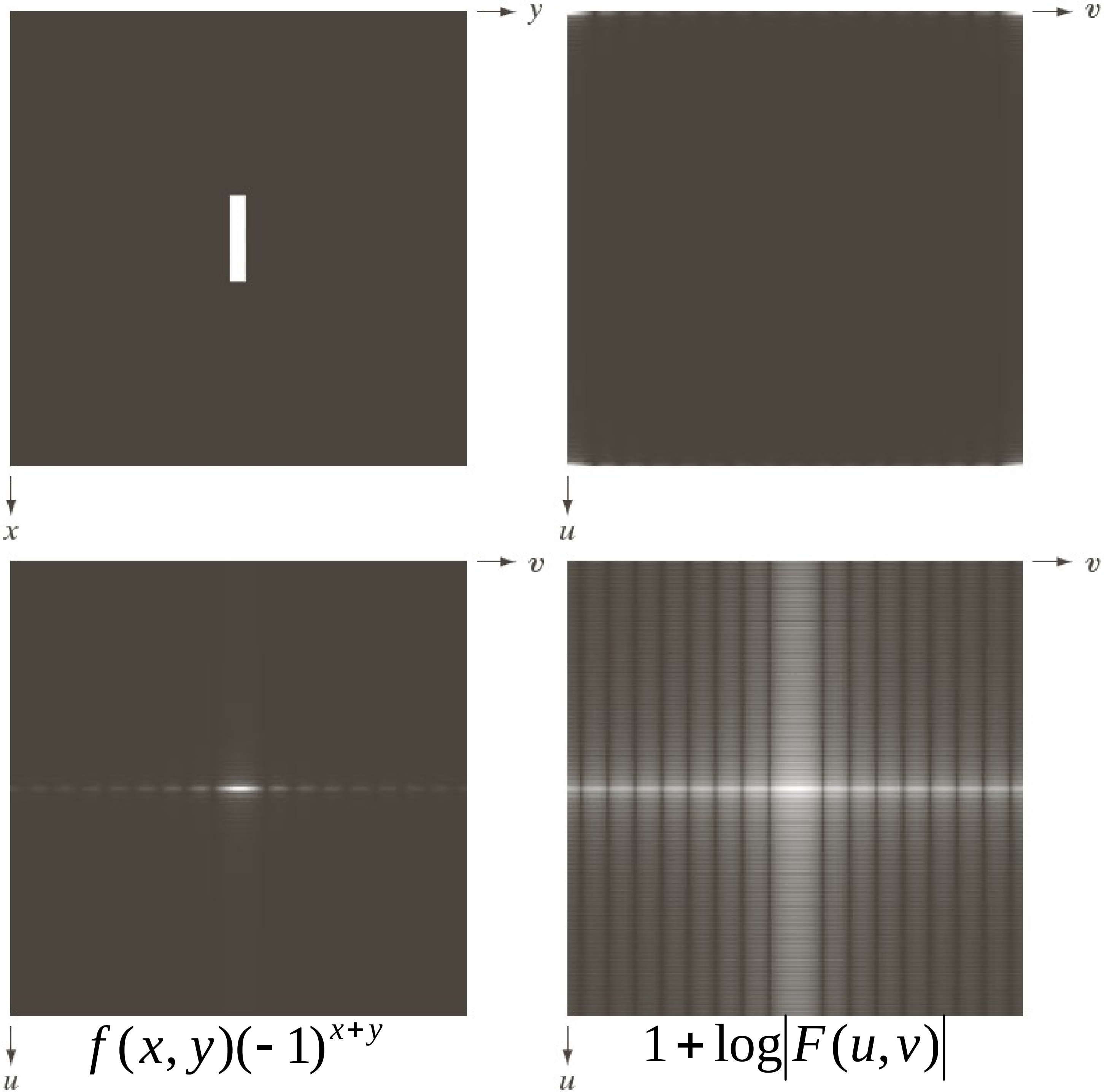
- Spectrum has even symmetry

$$|F(u,v)| = |F(-u,-v)|$$

- Phase Angle has odd symmetry

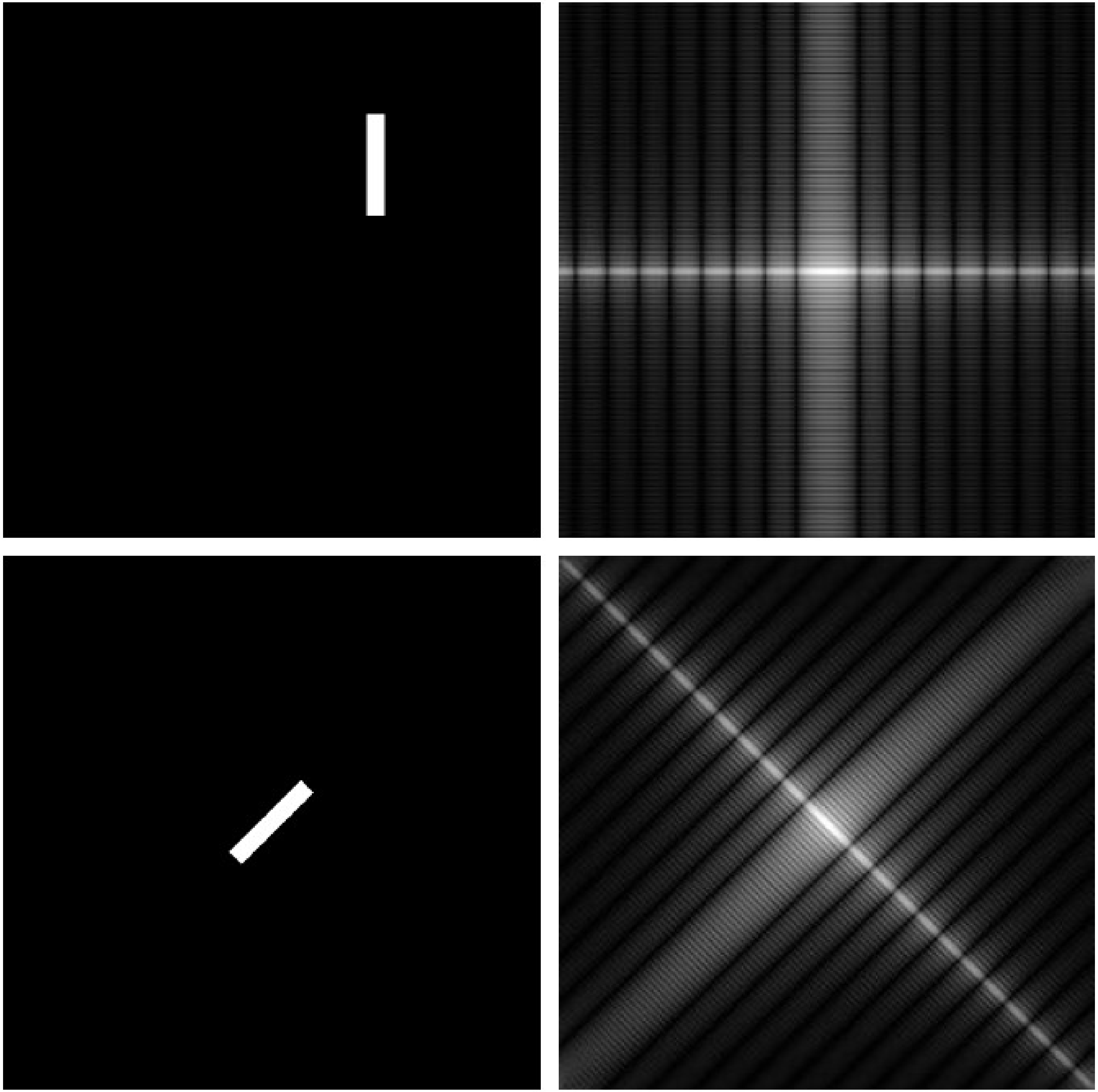
$$\phi(u,v) = -\phi(-u,-v)$$

- $F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\,\overline{f}(x,y)$



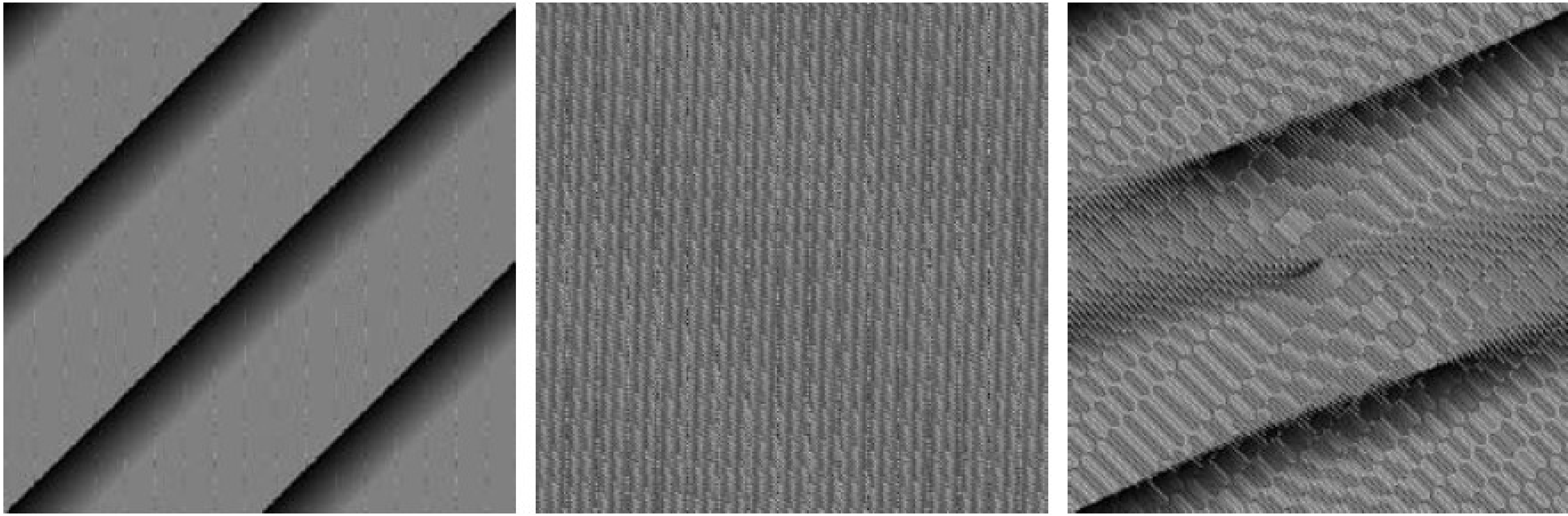
a b
c d

FIGURE 4.24 (a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.



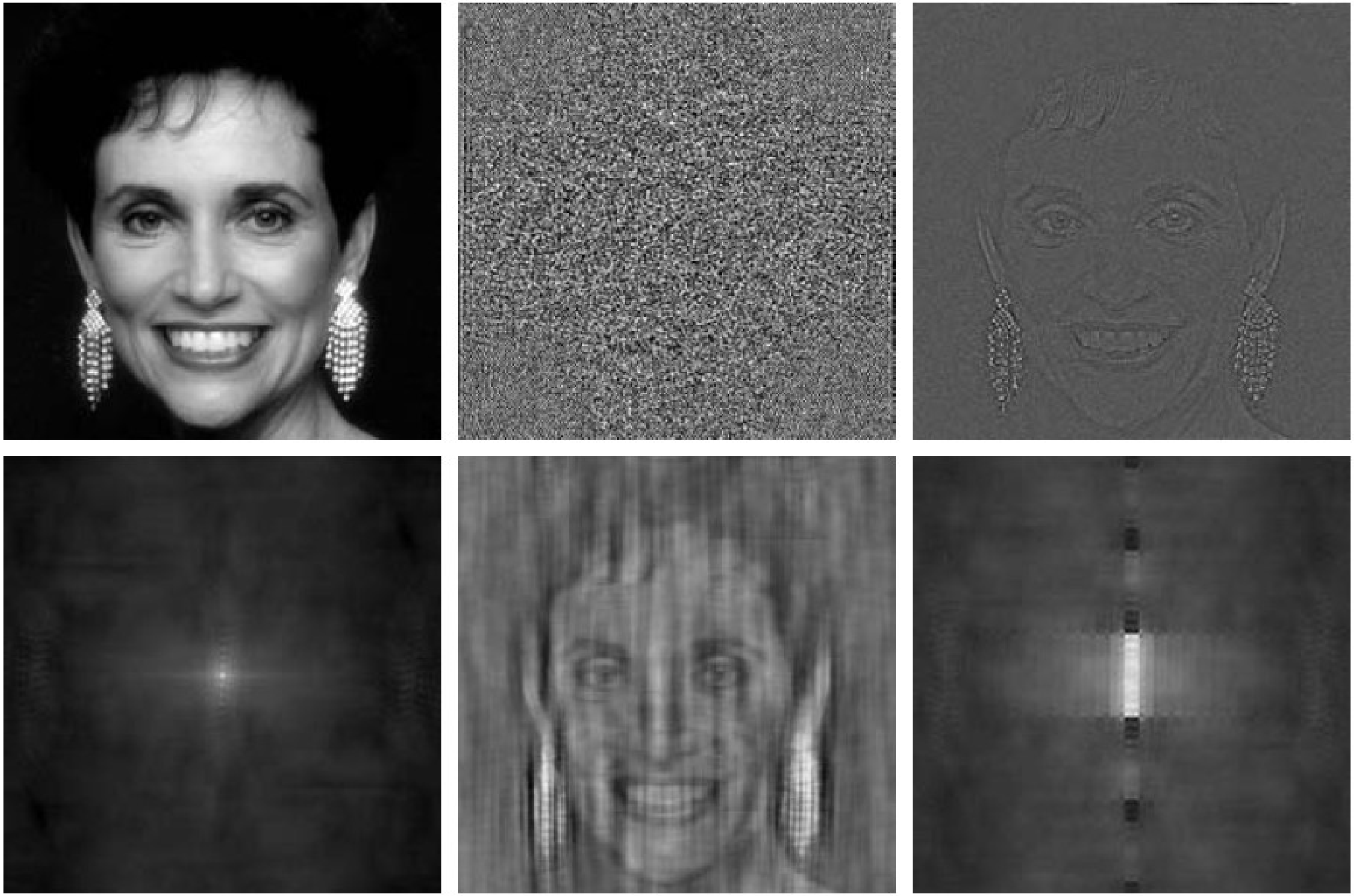
a b
c d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).



a b c

FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).



a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

2-D Convolution

$$f(x,y) \bullet h(x,y) =$$

$$\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

$$f(x,y) \bullet h(x,y) \Leftrightarrow F(\mu) H(\mu)$$

$$f(x,y) h(x,y) \Leftrightarrow F(\mu) \bullet H(\mu)$$

Effect of DFT on Convolution

$$f(x) \bullet h(x) = \sum_{m=0}^{399} f(m) h(x - m)$$

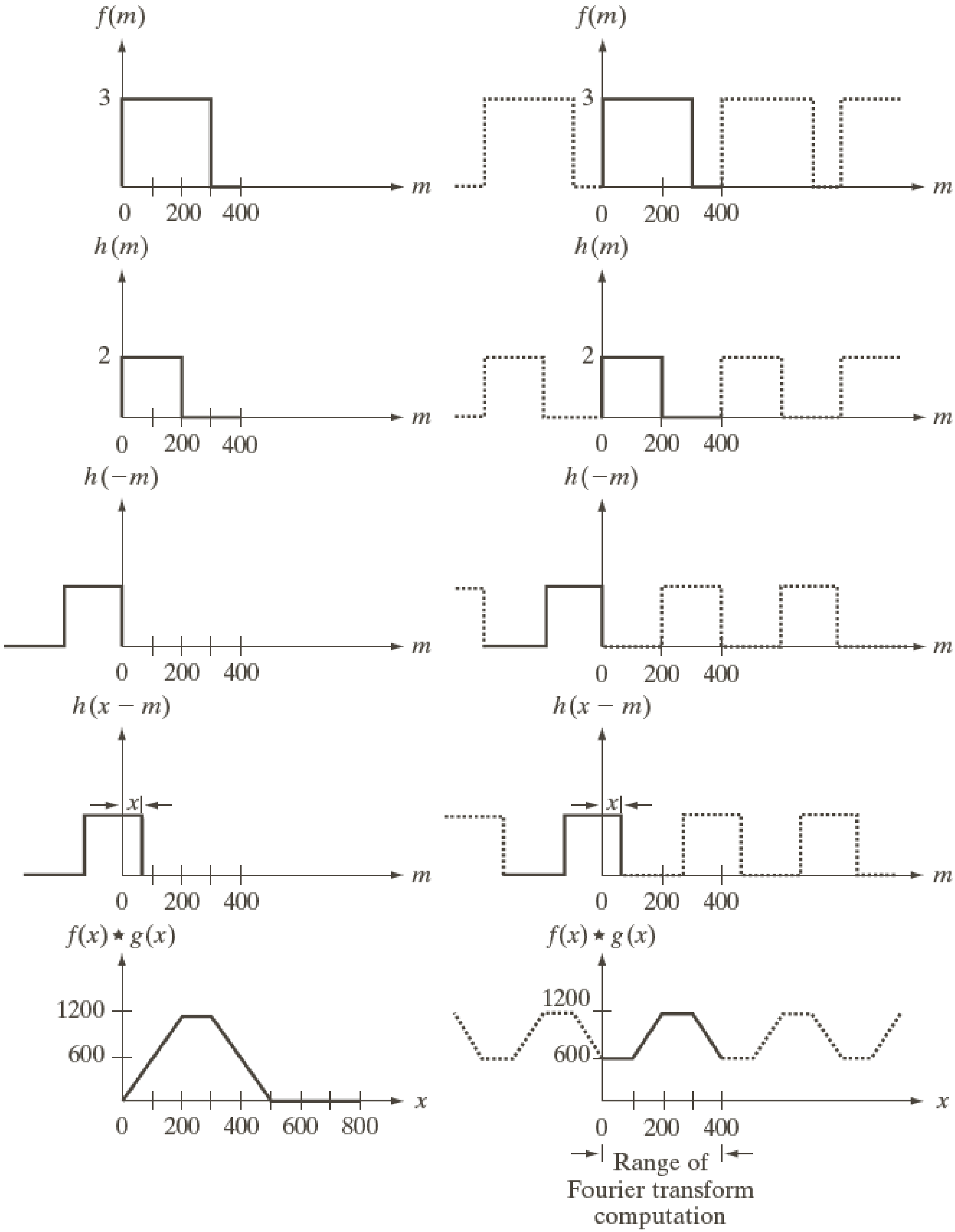


FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

Wraparound Error needs Zero Padding

2-D: Separability

DFT

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M + vy/N)}$$

$$= \sum_{x=0}^{M-1} e^{-j2\pi ux/M} \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi vy/N}$$

$$= \sum_{x=0}^{M-1} F(x,v)e^{-j2\pi ux/M}$$

Thank you