CS60047 Advanced Graph Theory	08-09-2023
Lecture 11	
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This lecture develops the lower bounds for the number of triangles in a general graph (deriving the solution for the Erdős–Rademacher problem of minimizing triangles in a graph) and the resulting proof for Mantel's Theorem. We go on to prove Mantel's theorem using Quadratic Optimization techniques (subsequently proving a special case of the Motzkin-Straus theorem), and then introduce Turán's theorem and prove it using induction on the set of edges. We introduce the concept of the Turán graphs and prove a corollary of the Turán's theorem using the pigeonhole principle, and finally look at some results regarding the maximal graph without a 1-factor.

11.1 Bound on the number of triangles in a simple graph - The Erdős–Rademacher problem

Claim: *For any arbitrary simple graph with n vertices and m edges, the number of triangles is at least* $\frac{4m}{3n}(m - \frac{n^2}{4})$

Proof: Consider a pair of vertices $\{x, y\}$ connected by an edge. Let the notation $d(x)$ represent the degree of vertex x. Now, the expression $d(x) + d(y)$ is the total number of vertices connected to each of the vertices x and y. But, since the set of vertices connected to either of x or y can be at most V, there can only be n distinct vertices at maximum counted in $d(x) + d(y)$. Thus, the expression $d(x) + d(y) - n$ provides a lower bound for the number of vertices counted twice in the set of vertices adjacent to either of x or y. But these vertices can only be counted twice if they are reachable from both x and y, i.e, there exists edges from both x and y to these vertices, which means that these vertices form triangles with the edge xy. Thus, we get that the number of triangles T_{xy} with base xy is bounded by the inequality

$$T_{xy} \geq d(x) + d(y) - n$$

Summing the T_{xy} terms over all edges, we get the sum

$$\begin{aligned}\sum_{xy \in E} T_{xy} &\geq \sum_{xy \in E} (d(x) + d(y) - n) \\ &= \sum_{x \in V} d^2(x) - mn\end{aligned} \tag{1}$$

where the sum $\sum_{xy \in E}(d(x) + d(y))$ basically adds up the degrees of each vertex x a number of $d(x)$ times (as there are $d(x)$ edges with x as one of the end-points), and hence simplifies down to $\sum_{x \in V}(d(x) * d(x)) = \sum_{x \in V} d^2(x)$.

But, since we count the number of triangles using every edge in the graph as a base in the summation, we essentially recount each triangle 3 times, once for each edge making up each side of the triangle in the graph. Thus, the actual number of triangles in the graph

$$T = \frac{1}{3} \sum_{xy \in E} T_{xy} \tag{2}$$

Consider the sum of all $d^2(x)$ for $x \in V$. Using the Cauchy-Schwartz inequality $(\sum_{k=1}^{n}(a_k * b_k))^2 \leq (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)$, we get

$$\begin{aligned} \sum_{x \in V} d^2(x) &= n \sum_{x \in V} \frac{d^2(x)}{n} \\ &= (\sum_{k=1}^{n}(1^2))(\sum_{x \in V} \frac{d^2(x)}{n}) \\ &\geq (\sum_{x \in V}(1 * \frac{d(x)}{\sqrt{n}}))^2 \\ &= \frac{(\sum_{x \in V} d(x))^2}{n} \end{aligned} \tag{3}$$

But, $\sum_{x \in V} d(x) = 2m$, as summing up the degrees of all vertices in a graph is equivalent to counting the number of outgoing edges from each vertex in the graph. Since the given graph is undirected (simple graph), each edge is counted twice (once for each end-vertex of the edge). Thus, equation (3) reduces to

$$\begin{aligned} \sum_{x \in V} d^2(x) &= \frac{(2m)^2}{n} \\ &= \frac{4m^2}{n} \end{aligned} \tag{4}$$

Hence, using (4) the expression (1) becomes

$$\begin{aligned} \sum_{xy \in E} T_{xy} &\geq \sum_{x \in V} d^2(x) - mn \\ &= \frac{4m^2}{n} - mn \\ &= \frac{4m}{n}(m - \frac{n^2}{4}) \end{aligned} \tag{5}$$

And thus, using (5) and (2), we get the bound for T as

$$\begin{aligned} T &= \frac{1}{3} \sum_{xy \in E} T_{xy} \\ &\geq \frac{1}{3} \times \frac{4m}{n}(m - \frac{n^2}{4}) \\ &\geq \frac{4m}{3n}(m - \frac{n^2}{4}) \end{aligned} \tag{6}$$

proving the claim. □

Note: The above result was proved by Moon and Moser [MM62], and, independently, by Nordhaus and Stewart [NS63].

11.2 Mantel's Theorem

Theorem 11.1. *The maximum number of edges in a graph on n vertices with no triangle subgraph is* $\lfloor \frac{n^2}{4} \rfloor$ *[Man07]*

11.2.1 Proof by degree-counting

Consider the graph $G(V, E)$, where $|V| = n, |E| = m$ and there are no triangles in G. Consider a pair of vertices $\{x, y\} \in V$ that are connected by an edge $e \in E$.

Since G is triangle-free, the two vertices x and y can not have any common neighbours (considering that any vertex is not a neighbour of itself) as that leads to the formation of a triangle in the graph. Thus, the union of the neighbour sets of these two vertices can be V in the maximal case, i.e, when the two vertices are connected to every other vertex in G. As the number of elements in such a union is the sum of degrees of x and y (as the intersection of neighbour sets of x and y is an empty set), we have

$$d(x) + d(y) \leq n \tag{1}$$

Now, consider the sum of degrees of the end-vertices of each edge in the graph G. The sum $\sum_{xy \in E}(d(x) + d(y))$ adds up the degrees of each vertex x a number of $d(x)$ times (as there are $d(x)$ edges with x as one of the end-points), and hence simplifies down to $\sum_{x \in V}(d(x) * d(x)) = \sum_{x \in V} d^2(x)$. Thus, we have

$$\sum_{xy \in E}(d(x) + d(y)) = \sum_{x \in V} d^2(x) \tag{2}$$

But from equation (1), we have that $d(x) + d(y) \leq n$ for any edge $xy \in E$. Thus, combining the equations (1) and (2), we get

$$\sum_{x \in V} d^2(x) = \sum_{xy \in E}(d(x) + d(y)) \leq m \times n \tag{3}$$

Since the sum $\sum_{x \in V} d^2(x)$ can be written as $\frac{1}{n}(n \times \sum_{x \in V} d^2(x))$, and the product $n \sum_{x \in V} d^2(x)$ is a product of sums of squares of two series $[n \sum_{x \in V} d^2(x) = (\sum_{x \in V}(1)^2)(\sum_{x \in V} d^2(x))]$, we can apply the Cauchy-Schwartz inequality to $n \sum_{x \in V} d^2(x)$, and state that

$$n \sum_{x \in V} d^2(x) = (\sum_{x \in V}(1)^2)(\sum_{x \in V} d^2(x)) \geq (\sum_{x \in V}(1 * d(x)))^2$$

which readily gives us the result

$$n \sum_{x \in V} d^2(x) \geq (\sum_{x \in V} d(x))^2$$

From the Handshake lemma ($\sum_{v\in V} d(v) = 2|E|$), we can then finally state that

$$n\sum_{x\in V} d^2(x) \geq 4m^2$$

or that

$$\sum_{x\in V} d^2(x) \geq \frac{4m^2}{n} \tag{4}$$

Thus, from equations (4) and (3), we can say that

$$\frac{4m^2}{n} \leq \sum_{x\in V} d^2(x) \leq mn$$

which on rearranging, gives us

$$m \leq \frac{n^2}{4} \tag{5}$$

Finally, since m is an integer and n need not always be even, for the inequality to hold m should be less than or equal to the greatest integer value not greater than $\frac{n^2}{4}$, i.e,

$$m \leq \lfloor\frac{n^2}{4}\rfloor \tag{6}$$

Thus, the maximum number of edges in an arbitrary graph on n vertices can be at most $\lfloor\frac{n^2}{4}\rfloor$ □

11.2.2 Proof by edge-weight maximization

Consider the graph $G(V, E)$, with $|V| = n, |E| = m$ and let there be no triangles in G. Now, to all vertices $v \in V$, let us assign a weight $w(v)$ such that $\sum_{v\in V} w(v) = 1$. Consider the sum $S = \sum_{xy\in E} w(x)w(y)$.

For the sum S, assigning the weight $\frac{1}{n}$ to every vertex, we get

$$S_1 = \sum_{xy\in E} (\frac{1}{n} * \frac{1}{n}) = m * (\frac{1}{n^2}) = \frac{m}{n^2} \tag{1}$$

and since this sum can always be attained, the maximum sum S_{max} will be greater than or equal to the sum S_1.

Now, let there be two vertices x, y which are not connected by an edge but have non-zero weights $w(x), w(y)$. Let the sum of weights of all adjacent vertices of x and y be $W(x)$ and $W(y$ respectively. Without loss of generality, we can choose the two vertices in such a way that $W(x) \geq W(y)$. Then, the operation $(w(x), w(y)) \to (w(x) + t, w(y) - t)$ for some $t > 0$ will produce new weights $w'(x), w'(y)$ which will follow the relation

$$w'(x) \times W(x) + w'(y) \times W(y) \geq w(x) \times W(x) + w(y) \times W(y) \tag{2}$$

as $w'(x)W(x)+w'(y)W(y) = (w(x)W(x)+tW(x))+(w(y)W(y)-tW(y))$, and $t(W(x)-W(y)) \geq 0$ as $W(x) \geq W(y)$. Hence, we can shift all the weight $w(y)$ of some vertex y to a non-adjacent vertex x and not get a lesser value of the sum S (as the sum S will remain the same for the graph induced by $V - X$ because there is no change in the weights of vertices inn this sub-graph, where X is the set of vertices adjacent to x and y, and the sum S in the induced graph of X does not decrease by this operation). Thus, this operation can be performed until we are left with one pair of adjacent vertices (there can not be three or more such vertices as that will form triangles in the graph, and we are given graph with no triangles). Hence, the sum S is maximized when all the weight occurs in a pair of adjacent vertices, and all other vertices have a weight of 0. Now, the sum S can be re-written as

$$S = w(x)w(y)$$

for the only edge xy with both vertices x, y having non-zero weights.

Now, consider that w(x) is some random value between 0 and 1. Since $w(x)$ and $w(y)$ are the only non-zero weights and $\sum_{v \in V} w(v) = 1$, we have $w(y) = 1 - w(x)$ and the sum $S = w(x) - w^2(x)$. This is a quadratic function in $w(x)$, which attains its minimum at the value $w(x) = \frac{1}{2}$. Thus, the maximum sum S possible in the triangle-free graph G is $S = \frac{1}{4}$.

But, $S \geq S_1$ as S_1 is an assured sum and always occurs in the graph G. Thus, we have

$$\frac{1}{4} \geq \frac{m}{n^2}$$

which can be re-written as

$$m \leq \frac{n^2}{4} \tag{3}$$

And as m is a positive integer (it is the number of edges in the graph and so can not be a decimal value) but n need not be even, the inequality is satisfied only if m is less than or equal to the greatest integer less than or equal to $\frac{n^2}{4}$, i.e,

$$m \leq \lfloor \frac{n^2}{4} \rfloor \tag{4}$$

Thus, the bound $m \leq \lfloor \frac{n^2}{4} \rfloor$ is proved using the Quadratic Programming and weight-shifting techniques. □

Note: The above theorem was proved more generally (for graphs which have a k-complete maximal sub-graph, the maximum value of the sum S as defined above would be $\frac{1}{2}(1 - \frac{1}{k})$) by Motzkin and Straus [MS65]

11.3 Turán Graphs

Definition 11.1. *A Turán graph is a graph $T(n, r)$ that is complete and multipartite, i.e, with n vertices partitioned into r subsets of as equal sizes as possible, and edges connecting two vertices if they are not in the same partition.*

11.3.1 Example:

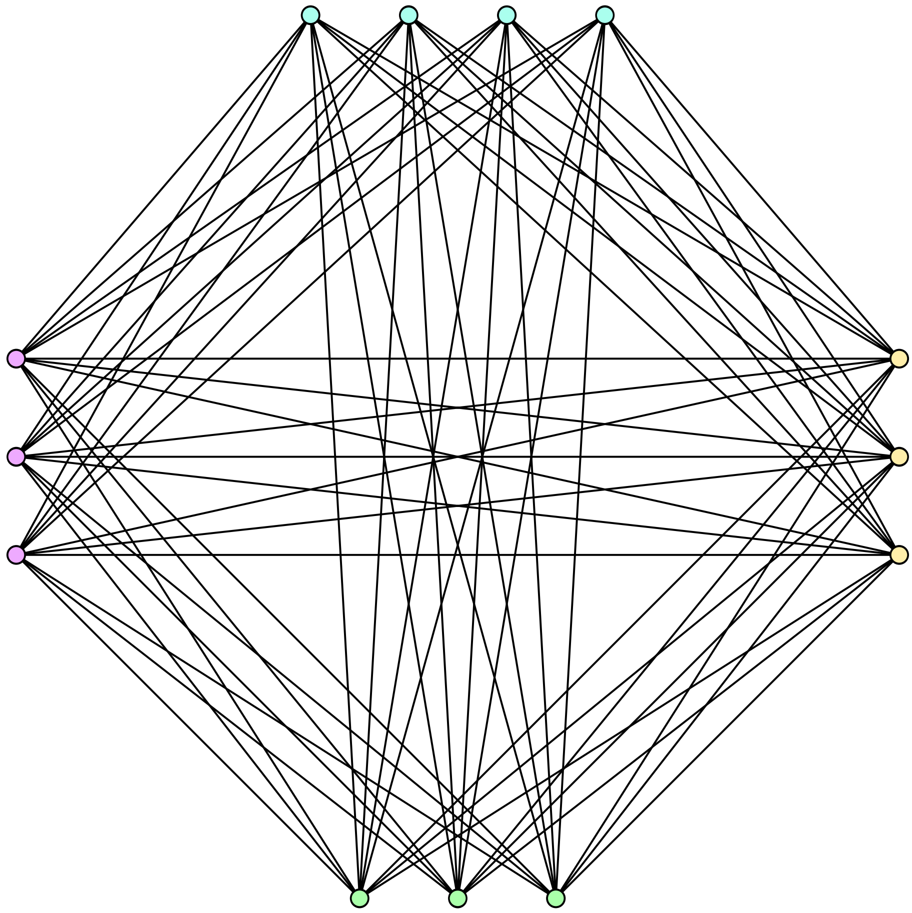

Figure 1: The Turán graph $T(13, 4)$

11.4 Turán's Theorem

Theorem 11.2. *Every graph G on n vertices that does not contain a K_{r+1} as a sub-graph will have at most as many edges as the Turán graph $T(n, r)$. For a fixed r, the Turán graph $T(n, r)$ has $|E| \leq \frac{r-1}{r}\frac{n^2}{2}$.*

Proof: The case of $r = 2$, i.e, the maximum number of edges in a graph without a K_3 (a triangle) was proved to attain the upper bound of edges as $\lfloor\frac{n^2}{4}\rfloor$, where n is the number of vertices in the graph G by W. Mantel [Man07]. And it is easy to see that the bipartite graph $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor} = T(n, 2)$ has exactly $\lfloor\frac{n^2}{4}\rfloor$ edges, showing the validity of the theorem for $r = 2$.

Now, consider the general case of an edge-maximal graph G having no K_{r+1}. Since G is edge-maximal, it has at least 1 K_r as an induced sub-graph, because assuming the contrary would mean that adding 1 edge would not give us a K_{r+1}, as K_{r+1} is built by adding r edges from each vertex of the preceding K_r to a new vertex and there would be $r+1 K_r$'s in the graph of K_{r+1}. Consider the partitioning of the vertex set $V(G)$ as follows:

- The set A of vertices forming the r-clique K_r.
- The set B of vertices of all vertices not in A, i.e, $B = V(G) \setminus A$.

Let $T(n,r)$ represent the Turán graph on n vertices, i.e, the n-vertex graph that is maximal, r-partite and balanced.

Now, the number of edges E in G is the sum of the number of edges E_A having both vertices in the set A on G, the number of edges E_B having both vertices in the set B on G and the number of edges E_{AB} having one end in A and the other in B. Since A is an r-clique,

$$E_A = \binom{r}{2} \tag{1}$$

And as B is a sub-graph obtained by deleting vertices of G and G does not contain an $(r+1)$-clique, B also does not contain an $(r+1)$-clique. So, by the induction hypothesis [that every graph G with less than n vertices has at most as many edges as the corresponding Turán graph $T(x,r)$], the number of edges with both vertices in B can be at most the number of edges in the Turán graph $T(n-r,r)$. Thus,

$$E_B \leq t(n-r,r) \tag{2}$$

where $t(n-r,r)$ is the number of edges in the Turán graph of $(n-r)$ vertices not containing a K_{r+1}.
Finally, the number of edges E_{AB} can at most be the product $(r-1)|B|$, as all r vertices in A can not connect to any one vertex in B (if they do, that forms a K_{r+1} clique, which is forbidden). Thus,

$$E_{AB} \leq (r-1)|B| = (r-1)(n-r) \tag{3}$$

Adding equations (1), (2) and (3), we get

$$\begin{aligned} E &= E_A + E_B + E_{AB} \\ &\leq \binom{r}{2} + t(n-r,r) + (r-1)(n-r) \\ &= (r-1)[(n-r) + \frac{r}{2}] + \frac{r-1}{r}\frac{(n-r)^2}{2} \\ &= (r-1)[n - \frac{r}{2} + \frac{(n^2-2nr+r^2)}{2r}] \\ &= (r-1)[n - n - \frac{r}{2} + \frac{n^2}{2r} + \frac{r}{2}] \\ &= (r-1)[\frac{n^2}{2r}] \end{aligned} \tag{4}$$

thus establishing the induction step. Hence, the bound on the number of edges $E \leq (r-1)\frac{n^2}{2r}$ for K_{r+1}-free n-vertex graphs is verified. □

Corollary 11.2.1. *Let $G(V,E)$ be a graph with $|V| = n = mk$ vertices and more than $|E| = e = \binom{k}{2}m^2$ edges, then G has a sub-graph that is a $(k+1)$-clique, i.e, K_{k+1} is a sub-graph of G.*

Proof: Let us assume that G has no sub-graph forming K_{k+1}. Let the subset of vertices forming the k-clique K_k be A, and let $B = V(G) \setminus A$. Let the graph G' be formed as the complementary graph of the induced subgraph of A on G. Let the induction hypothesis be that every graph with less number of vertices than n follow the stated result

In B, there are only $k(m-1)$ vertices (as the set A has exactly k vertices), each of which can not have more than $k-1$ edges connecting to the set A in G (because, if there exists a vertex with k edges to the k-clique A, we can form a clique of greater size $k+1$ using this vertex). Thus, the number of edges in G' has to be more than the difference of the least bound on edges in G and the number of edges in the induced subgraph of A, i.e,

$$E(G') > \binom{k}{2}m^2 - k(m-1) \times (k-1) - \binom{k}{2} \tag{1}$$

[where the product $k(m-1) \times (k-1)$ is the greatest number of edges with one end in A and the other in B and the factor $\binom{k}{2}$ is the number of edges with both edges in A]

Rewriting the above expression, we end up with

$$E(G') > \binom{k}{2}m^2 - k(k-1)(m-1) - \binom{k}{2}$$

But, $\binom{k}{2} = \frac{k(k-1)}{2}$. So, $k(k-1) = 2\binom{k}{2}$. Thus equation (1) becomes,

$$\begin{aligned} E(G') &> \binom{k}{2}[m^2 - 2(m-1) - 1] \\ &= \binom{k}{2}[m^2 - 2m + 2 - 1] \\ &= \binom{k}{2}[m^2 - 2m + 1] = \binom{k}{2}(m-1)^2 \end{aligned} \tag{2}$$

But, if $|V(G')| = k(m-1) < |V(G)|$ and $|E(G')| > \binom{k}{2}(m-1)^2$, by the induction hypothesis the sub-graph G' contains a $(k+1)$-clique, i.e, K_{k+1} is a sub-graph of G'. But then, as G can be formed by adding the vertices and edges in the induced graph of A and we do not delete any vertices or edges in G', the sub-graph forming K_{k+1} in G' should also be present in G, hence leading to the conclusion that G contains a K_{k+1}, which is a contradiction.

Thus, our assumption that G contains no K_{k+1} is incorrect, and every G with $|V(G)| = mk$ vertices and more than $|E| = e = \binom{k}{2}m^2$ edges has a $(k+1)$-clique. □

11.4.1 Number of edges in a Turán graph $T(n, r)$ when r does not divide n

Consider a Turán graph $T(n, r)$ of n vertices and r partitions. Let $n = pr + s$. Since the Turán graph has partitions whose sizes are as nearly equal as possible, $T(n, r)$ will have s partitions of size $p + 1$ and $r - s$ partitions of size p.

The edges within a partition of size x will be $\binom{x}{2}$, and so the total number of edges E_p possible within all these partitions of vertices is

$$E_p = s\binom{p+1}{2} + (r-s)\binom{p}{2} \tag{1}$$

Now, the graph $T(n,r)$ does not have any edges within a partition, but has all edges between any two partitions. So, the number of edges in the graph $T(n,r)$ is the number of edges within each partition subtracted from the total number of edges possible in an n-vertex graph.

$$E(T(n,r)) = \binom{n}{2} - E_p = \binom{n}{2} - s\binom{p+1}{2} - (r-s)\binom{p}{2} \tag{2}$$

Simplifying this expression, we get

$$\begin{aligned} E(T(n,r)) &= \binom{n}{2} - \frac{sp(p+1)}{2} - \frac{(r-s)p(p-1)}{2} \\ &= \binom{n}{2} - \frac{p}{2}[sp+s+rp-sp-r+s] \\ &= \binom{n}{2} - \frac{p}{2}[2s+rp-r] \end{aligned} \tag{3}$$

But, $n = pr + s$. So,

$$\begin{aligned} E(T(n,r)) &= \frac{n(n-1)}{2} - \frac{p}{2}[n+s-r] \\ &= \frac{n^2-n-np-sp+pr}{2} \\ &= \frac{n^2-np-sp+s}{2} \\ &= \frac{p^2r^2+s^2+2prs-p^2r-ps-ps-s}{2} \\ &= \frac{p^2r(r-1)+2ps(r-1)+s(s-1)}{2} \\ &= \frac{(r-1)(p^2r+2ps)}{2} + \binom{s}{2} \\ &= \frac{(r-1)(p^2r^2+2prs)}{2r} + \binom{s}{2} \\ &= \frac{(r-1)[(pr+s)^2-s^2]}{2r} + \binom{s}{2} \\ &= \frac{(r-1)(n^2-s^2)}{2r} + \binom{s}{2} \end{aligned} \tag{4}$$

which is the required result. □

11.5 Results on the maximal graph without a 1-factor

Let the graph $G'(V, E)$ be a special graph that is edge-maximal and does not have a 1-factor (perfect matching). Due to its edge-maximality, adding any one edge to the graph G' will introduce a 1-factor to G'.

Theorem 11.3. *If S is a bad set in the graph G', then all the components of $G' \setminus S$ are complete.*

Proof. Assume for the sake of a contradiction that some component $C \subseteq G' \setminus S$ dos not induce a complete sub-graph in G'. Then, there exists a pair of vertices $u, v \in C$ which is not connected by an edge in G'.

Consider the graph G'' with the same set of vertices V as G' but with the set of edges $E(G') + uv$. Since we are not changing the number of vertices, for the same bad set S the cardinality of S does not change and the components of $G'' \setminus S$ retain their parity from G', i.e, odd components in $G' - S$ are odd components in $G'' - S$, and the same holds for the even components. Since the added edge is not between two components in $G' \setminus S$, there is no change in the number of vertices in each component. So, the bad set in G' has to be a bad set in G'' also (as $o(G'' \setminus S) = o(G' \setminus S) > |S|$).

But, G' is a maximal graph without a perfect matching, which means that adding an edge uv to generate G'' should introduce a perfect matching in G''. But the set S in G'' violates Tutte's condition as shown above, which should mean that there is no perfect matching in G''. Thus, we arrive at a contradiction, and so our initial assumption that there is some component $C \subseteq G' \setminus S$ that does not induce a complete sub-graph in G' is wrong.
Thus, every component of $G' \setminus S$ induces a complete sub-graph in G'. Thus, proved. □

Theorem 11.4. *If S is a bad set in the graph G', then all the vertices $s \in S$ are connected to all vertices $v \in V \setminus \{s\}$.*

Proof. As the sets S and $G' \setminus S$ are mutually exclusive and exhaustive, any vertex v must either lie in S or in $G' \setminus S$ but not both.

Taking any vertex $v \in S$, since S induces a complete sub-graph in G' every other vertex in S is connected to the vertex v. As v is an arbitrary vertex, this property holds for all vertices in the set S.

Now consider a vertex $u \in S$ not connected to some vertex $v \in G' \setminus S$. Consider the graph G'' with the same set of vertices as G', but with the set of edges $E(G') + uv$. For the same set of vertices S, since we are not changing the number of vertices, the cardinality of S remains the same and the components of $G'' \setminus S$ retain their parity from G', i.e, odd components in $G' - S$ are odd components in $G'' - S$, and the same holds for the even components. Since the added edge is not between two components in $G' \setminus S$, there is no change in the number of vertices in each component. So, the bad set in G' has to be a bad set in G'' also (as $o(G'' \setminus S) = o(G' \setminus S) > |S|$).
But, G' is a maximal graph without a perfect matching, which means that adding an

edge uv to generate G'' should introduce a perfect matching in G''. But the set S in G'' violates Tutte's condition as shown above, which should mean that there is no perfect matching in G''. Thus, we arrive at a contradiction, and so our initial assumption that there is a vertex in S not connected to some vertex in $G' \setminus S$ is wrong. Thus, every vertex in S is connected to every vertex in $G' \setminus S$

As every vertex in S is connected to all vertices in both the sets S and $G' \setminus S$, and $S \cup (G' \setminus S) = V(G')$ we see that every vertex in the graph G' is connected to all the vertices in the set S (considering that a vertex is always connected to itself).
Thus, proved. □

Theorem 11.5. *Let V_1 be the set of vertices connected to all other vertices in the graph G'. Let $V_2 = V(G') \setminus V_1$. If $a, b, c \in V_2$, and a is adjacent to b and b is adjacent to c, then a is adjacent to c. Thus, adjacency is an equivalence relation in such a graph G', so V_2 is partitioned into complete sub-graphs.*

Proof. Assume for the sake of a contradiction that in such a triplet of vertices a, b, c, a is not adjacent to c. There has to exist a fourth vertex d not adjacent to b, as b would be in the set V_1 otherwise (since b is adjacent to both a and c, there has to be some other vertex d not adjacent to b). This means that edges ac and bd are not in $E(G')$. So, the graph $G' + ac$ should have a 1-factor F_1 (by the edge-maximality of G'), and similarly $G' + bd$ should have a 1-factor F_2. In F_1, ac is a perfect matching edge, as otherwise G' would have had a perfect matching in itself, which is a contradiction. Similarly, bd is a matched edge in F_2

Now, let $F = F_1 \cup F_2$. The union F has some edges common between F_1 and F_2, and some circuits. Since ac and bd are distinct edges, they can not be edges common among F_1 and F_2 and so they must lie on some circuits (ac on circuit C_1 and bd on circuit C_2). Two cases arise:

1. The two circuits C_1 and C_2 are distinct. This would mean that we can generate a new matching F_3 from F_1 and F_2 by alternating the edges of the circuit C1 while maintaining the rest of F_1, i.e, switching the edges matched in F_1 to those matched in F_2 in the cycle C_1, while keeping the rest of the matching edges the same as in F_1. This would entirely remove the matching edge ac from the matching F_3 while still keeping it a maximum matching because alternating edges along a circuit leads to the same number of matching edges. But this would men that the graph G' without edge ac also has the matching F_3, which is a contradiction as G' does not have a 1-factor.

2. The two circuits C_1 and C_2 are the same. Then, let the cycle be called C, and let us develop an ordering in the cycle starting from the vertex b as $b \to d \to ... \to a \to c \to ... \to b$. This can be done without loss of generality, as we can choose the vertices a and c for such an arbitrary edge ac such that the ordering is valid.
   Consider now, the path $P = b \to d \to ... \to a$. This is a path from b to a starting and ending at an edge from F_2, as the next edge ac is only found in F_1. But then, the edge $ab \in E(G')$. So the path P and the edge ab together (let it be the cycle

K) will form a cycle of alternating edges (as ab is not in F_2 because b is already matched by the edge bd). Thus, replace the edges of cycle K that is in F_2 by their alternating edges, i.e, the edges of K not in F_2. In essence, we drop the edge $bd \notin G'$ and add the edge $ab \in G'$ and replace all other matching edges by their alternating edges in K. Since it is an alternating cycle, the number of matched edges in the cycle remains the same, and the matching thus formed is still a maximum, perfect matching. But, the alternate edges do not contain the edge bd, which means that the newly formed matching exists wholly in the graph G'. Thus, the graph G' itself has a perfect matching, which is a contradiction.

As all cases of such an edge not existing leads to a contradiction with the fact that G' does not have a 1-factor, given 3 vertices $a, b, c \in V_2$, if a is adjacent to b and b is adjacent to c, then a is adjacent to c. And as a vertex is adjacent to itself and adjacency is symmetric in undirected graphs (and G' is undirected), adjacency is an equivalence relation for such graphs. Thus, because of transitivity, the partitions of vertices in V_2 that are connected to each other form cliques, and so the relation of adjacency in G' partitions V_2 into complete sub-graphs. □

Note: Such a maximal graph G' can be used for a proof of the Tutte's theorem as proved by L. Lovász in 1975 [Wes01; Die17]

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