CS60047 Advanced Graph Theory	11-10-2023
Lectures 20 - 21	
Lecturer: Prof. S. P. Pal	*Scribe: Anirudh M*

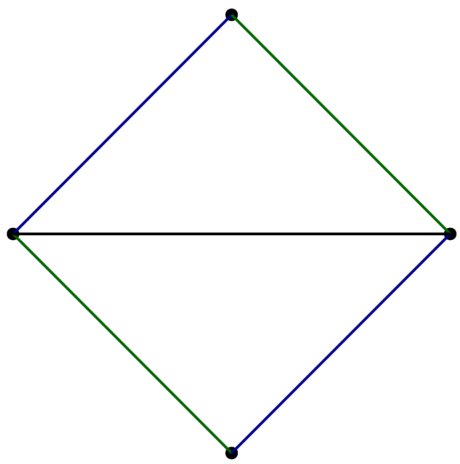
Lecture Summary

These lectures first build up the intuition for a proper edge-colouring drawing inspiration from a proper vertex-colouring, subsequently focusing on results bounding the *chromatic number* $\chi(G)$ and the *chromatic index* $\chi'(G)$, also known as the edge-colouring number. To this extent, we first look at the edge-colourings of a few trivial graphs, and proceed to prove a well-known result of Kőnig. We then jump back to vertex-colourings and derive a result from the colouring strategy proposed by Wilf and Szekeres, subsequently finding a way to prove Brooks' theorem. We finally look at L. Lovász's construction in the (weak) perfect graph theorem to round out the discussion on chromatic numbers and indices.

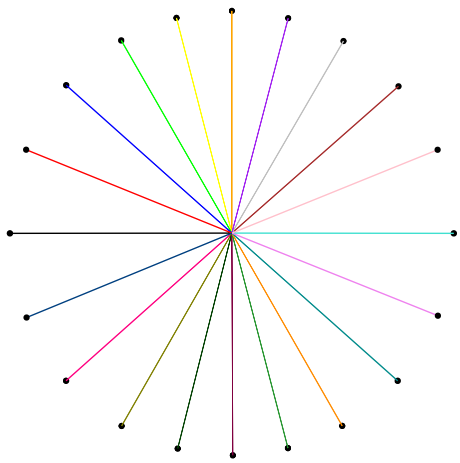
Preliminaries and Notations

Consider an undirected graph $G(V, E)$. We define the maximum vertex degree of G as $\Delta(G)$ and the minimum vertex degree as $\delta(G)$. A proper vertex-colouring of G is a *labelling* of all the vertices of G such that no two adjacent vertices in G have the same label assigned to them. A proper edge-colouring of G is a labelling of all the edges of G such that no two edges sharing a vertex are assigned the same label. As such, the minimum number of colours needed to achieve a proper vertex-colouring of G is known as its *chromatic number* $\chi(G)$, and the minimum number of colours needed to realize a proper edge-colouring of G is known as its *chromatic index* $\chi'(G)$.

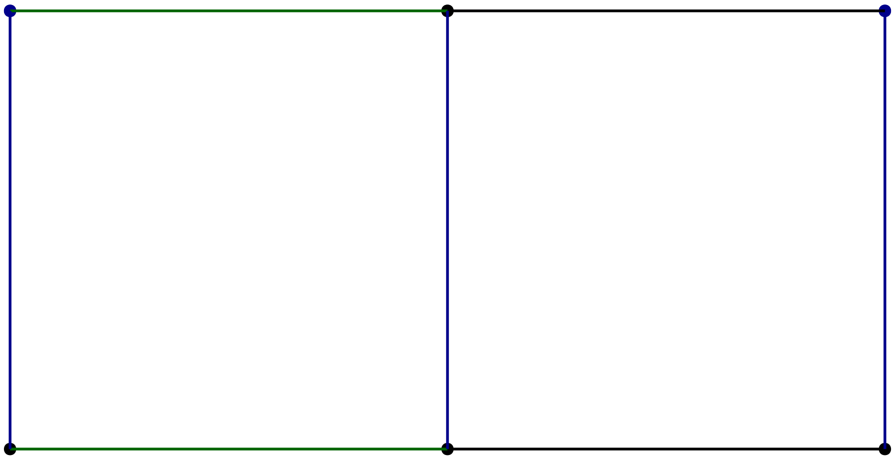
20.1 Colouring results on simple, trivial graphs



(1) A kite graph. $\chi'(G) = 3$, $\chi(G) = 3$



(2) A star graph of 21 vertices. $\chi'(G) = 20$, $\chi(G) = 2$



(3) A simple graph. $\chi'(G) = 3$, $\chi(G) = 3$

20.2 Kőnig's line colouring theorem

Theorem 20.1. *The chromatic index of a bipartite graph is always equal to the maximum degree of its vertices. Mathematically, for a bipartite graph G, $\chi'(G) = \Delta(G)$*[1].

20.2.1 Proof by Induction

Proof. [BM01; Die17; Kőn16] The theorem follows naturally by induction on the number of edges $|E(G)|$. Let the inductive hypothesis be that a bipartite graph G' with $|E(G')| < |E(G)|$ satisfies the condition $\chi'(G') = \Delta(G')$. For a graph G with no edges, $\chi'(G) = 0 = \Delta(G)$ trivially, as there are no edges to be coloured. Thus, the assertion holds.

For a graph G with $|E(G)| \geq 1$, let $\Delta := \Delta(G)$. Pick an arbitrary edge $xy \in E(G)$, and let G' be the sub-graph generated by removing the edge xy from G, i.e, $E(G') = E(G) \setminus \{xy\}$. Since G' remains bipartite, —$E(G')| = |E(G)| - 1 < |E(G)|$ and $\Delta(G') \leq \Delta$ due to the removal of an edge, G' has a $\Delta(G')$-edge-colouring (and hence a Δ-edge-colouring) by the induction hypothesis. Consider the Δ-edge-colouration of G' with the colours $C = \{1, 2, ..., \Delta\}$. Let the edges coloured by the colour $\alpha \in C$ be henceforth referred to as α-edges and so on for the other $\Delta - 1$ colours.

In G', as the edge xy has been deleted, the maximum possible degrees of the vertices x and y is $\Delta - 1$ because the graph $G = G' + xy$ has a maximum vertex degree Δ. So, we can have at most $\Delta - 1$ edges incident upon each of the vertices x and y. Thus, there exists a pair of colours α and β such that $\alpha, \beta \in C$, no α-edge is incident upon x and no β-edge is incident upon y. As α and β are not necessarily distinct, two cases arise.

[1] The original article of König states this result as follows: If each vertex of a bipartite graph is incident to at most k edges, then it is possible to assign each edge of the graph an index between 1 and k so that any two adjacent edges have different indexes. ([Kőn16], translated by Ágnes Cseh, Hungarian Academy of Sciences.

In the case where $\alpha = \beta$, we can trivially colour the edges of G with Δ colours by using the Δ-edge-colouration of G' for all edges in $E(G') = E(G) \setminus \{xy\}$ and the colour α for xy. Indeed, this produces a valid colouring as α is a colour not appearing in any edge incident on either of the vertices x or y in G'. Thus, $\chi'(G) = \Delta(G)$ in this case.

For the other case where $\alpha \neq \beta$, let us assume that there is a β-edge incident upon x (if not, then by choosing the colour β as our required colour α, we can reduce this case back to the earlier case of $\alpha = \beta$).

Consider the maximal walk W starting from the vertex x, such that W consists of alternating α-edges and β-edges. Since there is no α-edge incident on x, the first edge of such a walk W is always a β-edge. Such a walk always contains at least 2 vertices (as there always exists a β-edge by virtue of the graph's construction) and is always a path. Assume to the contrary, that there exists such a walk of alternating α and β-edges in G' that forms a cycle. The vertex x can not be a part of this cycle because it does not have an α-edge incident on it, and thus has only one edge in W. So the vertices forming such a cycle have to be attached to some chain of alternating edges ending at x. Let such a chain be incident on a vertex v_1 on the cycle. Since the chain is not a part of the cycle but v_1 has to have exactly 2 neighbours in the cycle, there are exactly 3 edges incident on v_1. But the walk contains edges of only 2 colours α and β. Thus, by the pigeonhole principle, at least 2 of the 3 incident edges have to have the same colour in G', which means that the Δ-edge-colouration of G' considered was not proper, hence leading to a contradiction.

Now, any such path W can not contain the vertex y. Assume to the contrary, that W is such a path from x to y. Then, the edge leading to y has to be an α-edge (by virtue of the construction), and as the edge beginning from x was a β-edge, the length of such a path is even. But then, by the construction there would be a cycle of odd length in G (the path of even length from x to y along with the edge xy itself, which is not a part of the path), which means that G has $\chi(G) \geq 3$ (since it contains an induced sub-graph which has $\chi = 3$), and so G is not bipartite, which is clearly a contradiction. Finally, in this maximal path W, we recolour all the edges by swapping the colour α with β and vice-versa to obtain a new Δ-edge-colouring of G', which is still valid as:

- For the two end vertices x and z, by the construction of the walk x misses an α-edge and z misses a β-edge in G', and so swapping colours would not lead to a conflict in these vertices.
- For any internal vertex v in W, as only two edges incident upon them can possibly be coloured with α and β in a proper colouration of G', swapping the colours α and β would again result in one edge coloured α and the other coloured β, which is also not a conflict.

Thus, the new Δ-edge-colouring of G' is valid, with both vertices x and y missing a β-edge. Hence, we can use the new Δ-edge-colouration of G' (of all edges in $E(G)\setminus\{xy\}$) and the colour β for xy to colour the edges of G using Δ colours. As G is Δ-edge-colourable in every possible case of the construction, for every bipartite graph G $\chi'(G) = \Delta(G)$. Thus, proved. □

20.2.2 Proof using Hall's Theorem

Proof. [Wes01] To prove the theorem, we first establish that a k-regular bipartite graph G' has $\chi'(G') = \Delta(G')$.

Let $G'(A \cup B, E')$ be a k-regular bipartite graph. Consider any subset of vertices $S \subseteq A$. Let $N(S)$ be the set of vertices adjacent to any vertex in S, and $E_N(S)$ denote the set of edges incident on any vertex in S. Since G' is k-regular, $|E_N(S)| = k|S|$ as there are k distinct edges incident on each vertex in S. Now consider the set of edges incident on the vertex set $N(S)$. As $E_N(N(S))$ contains all the edges incident on S, along with some edges incident on $N(S)$ but not on S, we can say that $|E_N(N(S))| \geq |E_N(S)|$. But $E_N(S) = k|S|$, so $k|N(S)| \geq k|S|$, or $|N(S)| \geq |S|$ for all $S \subseteq A$. As a similar argument can be given to the partite B, Hall's condition holds for both the partites A and B. Hence, there exists a perfect matching in G'. This result can be generalized to state that every k-regular bipartite graph has a perfect matching.

Let the set of edges of a perfect matching in such a k-regular bipartite graph G' be M. Since every vertex is covered by the edges in M exactly once, the graph $G''(A \cup B, E' \setminus M)$ is $(k-1)$-regular as every vertex in G'' has exactly one edge less than in G'. So, we use induction on k. Let the induction hypothesis be that bipartite graphs which are m-regular have $\chi'(G') = m$, if $m < k$. Trivially, a 0-regular graph has $\chi'(G') = 0$, establishing the base case of induction. Since the graph G'' is $(k-1)$-regular and $k - 1 < k$, by the induction hypothesis $\chi'(G'') = k - 1$. Now, let the edges in the matching M be coloured with the colour k, which is possible as a matching is an independent set of edges. Thus, the $(k-1)$-edge-colouring of $E(G') \setminus M$, along with the colour k for edges in M generate a k-edge-colouring for G', completing the inductive step. Thus, $\chi'(G') = k$ for any k-regular graph G'.

Now, any bipartite graph G can either be $\Delta(G)$-regular or not. If G is $\Delta(G)$-regular, by our previous result $\chi'(G) = \Delta(G)$, thus proving the theorem. If not, then we show that there always exists some $\Delta(G)$-regular bipartite graph H that contains G as a sub-graph. Such a graph can be constructed in many ways, 2 of which are expanded upon below.

1. Equalize the number of vertices in the two partites of G by adding some artificial vertices to the smaller partite, if necessary. If the resulting graph is not $\Delta(G)$-regular, there has to exist at least 1 vertex in each partite that has a degree strictly less than $\Delta(G)$. Add an edge between these two vertices, and continue to add such edges until the graph becomes $\Delta(G)$-regular. The final graph we end up with is a $\Delta(G)$-regular graph H.

2. Take a copy of G, let it be $G'(A' \cup B', E')$. For each vertex $v \in A \cup B$ having a degree less than $\Delta(G)$, add an edge between v and its copy $v' \in A' \cup B'$. The newly generated graph is bipartite with parts $A \cup B'$ and $B \cup A'$. Repeat this process as many times as necessary. In each iteration, the minimum degree of the graph has to increase by 1 as we add 1 edge to all such vertices that do not have the maximum

degree, while the maximum degree itself remains constant, and so in $\Delta(G) - \delta(G)$ iterations we would arrive at a graph H that is $\Delta(G)$-regular.

Since H is $\Delta(G)$-regular, by the above results $\chi'(H) = \Delta(G)$. Then, as G is a subgraph of H, G can be coloured with at most $\Delta(G)$ colours, i.e, $\chi'(G) \leq \Delta(G)$. But, since at least $\Delta(G)$ vertices share a vertex (by definition), $\chi'(G) \geq \Delta(G)$. Therefore, for every bipartite graph G, $\chi'(G) = \Delta(G)$. Thus, proved. □

Notes:

- A proper edge colouring of a graph can also be characterized as the partition of the graph into edge-disjoint matchings. So, the above theorem implies a k-regular bipartite graph G' can be partitioned into exactly k edge-disjoint perfect matchings (Theorem B in [Kőn16]). It further implies that every regular bipartite graph always has a perfect matching (Theorem A in [Kőn16]).
- The above theorem is valid even for bipartite multigraphs, i.e, graphs with multiple edges between the same pair of vertices. This is because we do not disallow for the presence of multiple edges in the graph in the above proofs.
- The above theorem is equivalent to stating that the *line graph* $L(G)$ of a bipartite graph G are perfect. A short proof is as follows: In $L(G)$, a clique corresponds to a set of edges sharing a common vertex in G. So the maximum size of a clique in $L(G)$ is exactly the maximum vertex degree in G, i.e, $\omega(L(G)) = \Delta(G)$. We also know that the chromatic number $\chi(L(G))$ of $L(G)$ is the same as the chromatic index $\chi'(L(G))$ of G. Perfectness follows[2] .

20.3 The Szekeres-Wilf Result

20.3.1 A Constructive Proof

Theorem 20.2. *Any graph G satisfies the property*

$$\chi(G) \leq max\{\delta(H) \mid H \subseteq G\} + 1$$

Proof. [SW68; Die17] Let $G(V, E)$ be an arbitrary connected graph. Let $n = |V|$ and $k = \underset{G' \subseteq G}{max}\{\delta(G')\}$ over all induced sub-graphs G' of G. Consider a vertex v_n of degree not greater than k in G. This vertex always exists, as the vertex with the minimum degree in G has to have a degree not greater than k due to its construction. Let the graph H_{n-1} be the sub-graph induced in G by the vertex set $V \setminus \{v_n\}$.

[2] To show that the complement of the line graph of a bipartite graph is perfect, we can use Kőnig's matching theorem, which states that the minimum size of a vertex cover of G is equal to the size of the maximum matching in G. Since a clique in the complement line graph $\bar{L}(G)$ is a matching in G (a matching in G is an independent set in $L(G)$) and a vertex colouring of $\bar{L}(G)$ is a partition of the edges of G such that each partition shares a common vertex, perfection follows by the aforementioned theorem.

As $H_{n-1} \subseteq G$, the vertex of minimum degree in H_{n-1} also has to have a degree not greater than k due to its construction. Let v_{n-1} be the vertex of minimum degree in H_{n-1}, and H_{n-2} be the sub-graph induced in G by the vertex set $V \setminus \{v_n, v_{n-1}\}$. Repeating this process until V gets exhausted, we get an enumeration of all the vertices of G. Let the set $V' = \{v_1, v_2, ..., v_n\}$ be this enumeration.

For an arbitrary element $v_i \in V'$, in the induced sub-graph $H_i := G[v_1, v_2, ..., v_i]$ the vertex v_i can have at most k neighbours, as v_i is the vertex with the least degree in H_i and the least degrees of any induced sub-graph of G is at most k. Let our inductive hypothesis be that H_i is $(k+1)$-colourable. In H_{i+1} then, v_{i+1} still can have at most k neighbours in $\{v_1, v_2, ..., v_i\}$. As we have a $(k+1)$-colouring of H_i and the neighbours of v_{i+1} can cover at most k colours in the $(k+1)$-colouring, we can use the leftover colour for the vertex v_{i+1}, thereby generating a $(k+1)$-colouring for H_{i+1} too. This establishes the inductive step.

Thus, H_{i+1} can be coloured with at most $k+1$ colours if H_i is $(k+1)$-colourable. As this holds for all induced sub-graphs, choosing $i = n-1$, $H_n = G$ can be coloured with at most $k+1$ colours, i.e, $\chi(G) \leq max\{\delta(G') \mid G' \subseteq G\} + 1$. Thus, proved. □

Note: The original proof of this result by Szekeres and Wilf [SW68] removed sets of *stars* of vertices from G where all vertices have a degree not greater than the aforementioned value of k, instead of singular vertices like the construction above. Indeed, this is intuitively a valid construction as we can separate each such star into individual vertices and remove them one by one from G, thus arriving at our above construction[3] .

20.3.2 Non-Regular Graphs – a Notable Result

Claim 20.1. *For a connected non-regular graph G,*

$$\chi(G) \leq \max_{G' \subseteq G}\{\delta(G')\} + 1 \leq \Delta(G)$$

Proof. Consider a connected graph G that is not regular. Let H be an arbitrary induced proper sub-graph of G. Let the vertex $v_\delta(G)$ be one of the vertices with the minimum degree $\delta(G)$ in G, and $v_\Delta(G)$ be one of the vertices with the maximum degree in G. We prove the above claim by showing that for all possible induced sub-graphs H of G, $\delta(G) + 1 \leq \Delta(G)$.

[3] More formally, let G_0 be the empty graph. We can verify the validity of the construction by induction on the number of steps needed to re-construct G from the stars. In step q, add the star S_q to G_{q-1} and the required edges to form G_q. The last graph so obtained has to be the original graph G, and we can see that in each step, the degrees of each vertex in S_q is at most k. Let the vertices of S_q be uncoloured. Since the graph G_{q-1} is $(k+1)$-colourable by the inductive hypothesis, and we have at most k neighbours for each vertex of S_q, these vertices can be coloured with some colour in the $(k+1)$-colouring of G_{q-1} that is not used by any of its neighbours, establishing the inductive step. Since we have not assumed any ordering of the stars to establish the inductive step, this works for any arbitrary ordering of addition of stars to the graph.

In G, we claim that $\delta(G) < \Delta(G)$. Assume to the contrary, that G has $\delta(G) = \Delta(G) = \Delta$ but is not regular. Then G has at least one vertex v that does not have degree Δ. If $deg(v) \neq \Delta$, it can either be more than or less than Δ. In the former case, $v_\Delta(G)$ is such a vertex v. So, $\Delta(G) = deg(v_\Delta(G))$. But, $deg(v_\Delta(G)) \neq \delta(G)$, and so $\Delta(G) > \Delta = \delta(G)$ which is a contradiction. In the latter case, $v_\delta(G)$ is such a vertex v. As $\delta(G) = deg(v_\delta(G))$ and $deg(v_\delta(G)) \neq \Delta(G)$, $\delta(G) < \Delta = \Delta(G)$ which is a contradiction. As all cases of our assumption lead to a contradiction, our assumption is wrong and $\delta(G) < \Delta(G)$ for a non-regular graph G. As $\delta(G)$ and $\Delta(G)$ are whole numbers, this can also be stated as

$$\delta(G) + 1 \leq \Delta(G) \tag{1}$$

In H, $\Delta(H) \leq \Delta(G)$ because edges can not be added to a vertex in a sub-graph. Now, H need not be non-regular, as adding a single edge to an arbitrary vertex in a regular graph X makes a non-regular graph Y. So, two cases arise when considering such an arbitrary H.

Case 1: *When H is not regular*

It follows from the above result that in such a choice of H, $\delta(H) < \Delta(H)$. But, as we have $\Delta(H) \leq \Delta(G)$, for any induced sub-graph H that is not regular we arrive at the result that $\delta(H) < \Delta(G)$, which can be rewritten as

$$\delta(H) + 1 \leq \Delta(G) \tag{2}$$

Case 2: *When H is regular*

If H is regular, $\delta(H) = \Delta(H)$. Now, we claim that $\Delta(H) < \Delta(G)$ strictly. Assume to the contrary, that $\Delta(H) = \Delta(G)$ $(= \Delta)$ when H is a proper induced sub-graph of G. Since H is regular, every vertex in H has the same degree $\Delta(H)$. Now, if $\Delta(H) = \Delta(G)$, every vertex in G can have at most Δ neighbours. But every vertex in G that belongs to H already has Δ neighbours in H, and so can not have any more neighbours in G. So, any vertex $v \in V(G) \setminus V(H)$ can not have any edges incident to $V(H)$, i.e, the vertex sets $V(H)$ and $V(G) \setminus V(H)$ are disconnected. As there exists such a non-empty set of vertices (because H is a proper sub-graph of G) and the graph G is connected, we arrive at a contradiction. So, $\delta(H) \leq \Delta(H) < \Delta(G)$. Thus, for any regular induced sub-graph H of G,

$$\delta(H) + 1 \leq \Delta(G) \tag{3}$$

Now, for every proper induced sub-graph H of G, from (2) and (3) we can say that $\delta(H) \leq \Delta(G) - 1$. And by (1) we get that $\delta(G) \leq \Delta(G) - 1$. Thus,

$$\delta(G') + 1 \leq \Delta(G) \qquad \forall \quad G' \subseteq G \tag{4}$$

But from Theorem 20.2, $\chi(G) \leq \underset{G' \subseteq G}{max}\{\delta(G')\} + 1$. Thus, from (4)

$$\chi(G) \leq \underset{G' \subseteq G}{max}\{\delta(G')\} + 1 \leq \Delta(G)$$

Hence, proved. □

Note: This result can be used to trivially explain the result of Brooks for connected graphs which are not Δ-regular.

20.4 Brooks' Theorem

Theorem 20.3. *Let G be a connected graph. If G is neither complete nor is an odd cycle*[4]*, then*

$$\chi(G) \leq \Delta(G)$$

Proof. [BM01; Wes01] Consider a connected graph G which is neither a complete graph nor an odd cycle. Let the maximum vertex degree in G be $\Delta := \Delta(G)$. Such a G can either be regular (in which case it is Δ-regular) or not. If G is not regular, using Claim 20.1 we can say that $\chi(G) \leq \Delta(G)$[5]. It is also trivial to prove for a path or a cycle, i.e, for graphs with $\Delta(G) \leq 2$ as $\chi(G) = 2$ for even cycles which are 2-regular, and paths are not regular graphs. Regular, connected graphs with $\Delta(G) = 1$ cannot exist outside of the trivial 2-vertex graph, which is complete (K_2) and so is an exception to our theorem. So, it suffices to prove the validity of the theorem for regular graphs with $\Delta(G) \geq 3$.

For a Δ-regular graph there may or may not be a cut-vertex, and there may exist a complete sub-graph of G that disconnects G if it is removed. If there is a cut-vertex or a complete sub-graph as described above, let the components of the graph that form after such a removal be $\{G_1, G_2, ..., G_k\}$ and the removed vertex set be S. Now, the induced sub-graphs $G[V(G_1) \cup S], G[V(G_2) \cup S], ..., G[V(G_n) \cup S]$ can not be regular, as G is regular and we are removing some non-zero number of edges incident on S. Thus, using Claim 20.1 we can again colour all these sub-graphs individually using at most Δ colours. To join them, permute the colourings of each of these sub-graphs such that the assignment of colours to S is the same in all the sub-graphs. So, we can provide a valid Δ colouring of G if there is a cut-vertex, or a complete sub-graph that disconnects G.

Now, the only class of graphs to be seen are those without a cut-vertex or such a complete sub-graph. These graphs have to be at least 2-connected (due to the lack of a cut-vertex). Consider now those graphs which are exactly 2-connected. In these graphs, by definition there exists a pair of vertices $\{v_n, v'_n\}$ which disconnects G upon removal. On removing both vertices from G, we get an induced sub-graph G' that is disconnected into at least two blocks, two of which are G_1 and G_2. Since G was 2-connected, there has to exist at least one vertex $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that they are neighbours of

[4] [Bro87] The original formulation is as follows: Let N be a network (or linear graph) such that at each node not more than n lines meet (where $n > 2$), and no line has both ends at the same node. Suppose also that no connected component of N is an n-simplex. Then it is possible to colour the nodes of N with n colours so that no two nodes of the same colour are joined. Here an n-simplex refers to a complete graph of n-vertices.

[5] [BM01; Wes01] Proves the same without using the Szekeres-Wilf theorem. Take an arbitrary vertex of degree less than $\Delta(G)$ as v_n. Since the graph is connected, there has to exist a spanning tree for the graph, which can be proved by induction on the number of edges. Build such a spanning tree starting from v_n, assigning indices in decreasing order. Greedily colour vertices of the spanning tree starting from v_1. As each vertex in the tree has at most $\Delta(G) - 1$ vertices with a lower index (there has to be at least one vertex with a higher index connected to it by the property of a tree), the vertex ordering thus made is $\Delta(G)$-colourable. This colours the original graph too as we are taking a topological ordering of the vertices and not deleting any edges.

v_n (as there has to exist a path from every vertex in G_1 to every vertex in G_2 through v_n and through v'_n). Since G_1 and G_2 are 2-connected graphs, they do not have cut-vertices, and thus the graphs $G_1 \setminus \{v_1\}$ and $G_2 \setminus \{v_2\}$ are connected. Thus, we have a triplet of vertices (v_n, v_1, v_2) such that $G \setminus \{v_1, v_2\}$ is connected, there is no edge between v_1 and v_2 and v_n is adjacent to the other two vertices.

For graphs which are at least 3-connected, there has to exist a pair of vertices v_1 and v_2 in G such that v_2 is not adjacent to v_1 but both vertices have a common neighbour v_n. The proof of the claim is as follows: G is not a complete graph and so there exists a pair of vertices v_1 and v_2 which are not adjacent. Now, consider the maximum clique in G to be W (W may well be a single edge). Then, since G is not complete, W can not be the whole graph G, so there will be some vertex v_1 not in W. But G is connected, and so v_1 has to be connected to some vertex $v_n \in W$. Since $W \cup \{v_1\}$ is not a clique, there has to exist some vertex $v_2 \in W$ such that v_2 is not adjacent to v_1. Since W is an induced sub-graph of G, we have proved the existence of such a triplet of vertices $\{v_n, v_1, v_2\}$ such that $G \setminus \{v_1, v_2\}$ is connected, there is no edge between v_1 and v_2 and v_n is adjacent to the other two vertices (the fact that $G \setminus \{v_1, v_2\}$ is connected can be easily seen from the fact that G is at least 3-connected).

Now, for graphs G having such triplets of vertices $\{v_n, v_1, v_2\}$ consider the spanning tree of $G \setminus \{v_1, v_2\}$ built with v_n as the rooted vertex. Label vertices such that those vertices occurring at a greater distance from v_n than some vertex v_i receives a label lesser than i. Consider the colouring of the ordering of vertices $\{v_1, v_2, ..., v_n\}$. Since v_1 and v_2 are not adjacent, they receive the same colour in a greedy colouring. For any vertex $v_i \notin \{v_1, v_2, v_n\}$, since G is k-regular for some $k > 2$, v_i is adjacent to at most $k-1$ vertices with a label lesser than i because it was labelled through a spanning tree. Finally to colour v_n, since v_1 and v_2 receive the same colour all k neighbours of v_n can be coloured with $k-1$ colours, and so v_n can be coloured if we have at least k colours. Thus, to colour such graphs, we only need a maximum of $\Delta(G)$ colours.

Thus, we have proved that for every graph that is not complete nor an odd cycle, we need only a maximum of $\Delta(G)$ colours to properly colour its vertices, i.e, $\chi(G) \leq \Delta(G)$. □

20.5 Perfect Graphs and their Constructions

Definition 20.1. *A graph G is said to be perfect if for every induced sub-graph $H \subseteq G$,*

$$\chi(H) = \omega(H)$$

Berge's conjecture (of the weak perfect graph theorem) states that "The complement graph of a perfect graph is perfect as well". This conjecture was reduced by Fulkerson[Ful72] to the following problem using the theory of anti-blocking polyhedra[6]:

[6] Anti-blocking polyhedra are pairs of matrices such that a *min-max* equality, which is the dual of the max-flow min-cut equality of a blocking pair of matrices, holds. As such, they are the solutions to the dual problems of blocking pairs of matrices.

Duplicate an arbitrary vertex of a perfect graph, and join the original and duplicate vertices by an edge. The resulting graph is also perfect.

To prove this, L. Lovász proved the stronger result stated below, thus leading to a proof of Berge's weak perfect graph conjecture.

Claim 20.2. *On replacing an arbitrary vertex in a perfect graph by an arbitrary perfect graph, the graph so generated is a perfect graph.*

Before we prove this, we need an important result.

Theorem 20.4. *A graph is perfect if and only if there exists an independent set that meets all the maximum cliques in every induced sub-graph of the graph.*

Proof. Consider a graph G. Let the maximum clique size in G be $\omega := \omega(G)$and $\chi(G)$ be the colouring number of G. By definition, we know that $\chi(G) = \omega$. Consider a $\chi(G)$ colouration of G and let X be a colour class in this colouration. Consider the set of vertices forming an arbitrary maximum clique in G to be W.

To prove the "only if" part, we can show the existence of such an independent set. Let G be perfect, and assume for the sake of a contradiction that W misses the colour class X. Then W is colourable with $\chi(G) - 1$ colours, and as W is a maximum clique of size ω we can say $\omega < \chi(G)$, thus arriving at a contradiction. Thus, G has some independent set meeting all its maximum cliques. Since, every induced sub-graph of G is perfect by definition, the result follows for every induced sub-graph as the choice of G is arbitrary. Hence, if G is perfect, all its induced sub-graphs have an independent set meeting all its maximum cliques.

To prove the "if" part, we use induction on the number of vertices in an induced sub-graph G' of G. Let the inductive hypothesis be that a graph G'' with less vertices than G' having an independent set meeting all its maximum cliques satisfies $\chi(G'') = \omega(G'')$. Let S be an independent set meeting all maximum cliques in G'. In the induced sub-graph $G'' = G' \setminus S$, by the claim there exists an independent set meeting all the maximum cliques and so by the inductive hypothesis, $\chi(G'') = \omega(G'')$. Now, since only one vertex from a maximum clique can be a part of the independent set S,

$$\omega(G'') \leq \omega(G') - 1 \tag{1}$$

As an independent set can be coloured with the same colour,

$$\chi(G'') + 1 \geq \chi(G') \tag{2}$$

But since $\chi(G'') = \omega(G'')$, equations (1) and (2) result in

$$\chi(G') - 1 \leq \chi(G'') = \omega(G'') \leq \omega(G') - 1 \tag{3}$$

Hence, proving that $\chi(G') = \omega(G')$. Since G is itself an induced sub-graph of G, the result is valid for the graph G too. Thus, as $\chi(G) = \omega(G)$ and for all proper induced sub-graphs G' $\chi(G') = \omega(G')$, G is perfect. Thus, proved. □

Now, we prove Lovász's claim, as in [Lov72].

Proof. Consider a perfect graph G, and let $\chi := \chi(G)$ be its colouring number. Let an arbitrary vertex v be replaced by another perfect graph G' to form H. By "replace", we mean the following two steps:

- Delete the vertex v and all edges incident on v from G.
- Add the vertex set of G' to G, and add edges from every other vertex $v' \in G \setminus \{v\}$ to every vertex in G' if the edge $vv' \in E(G)$. Add all the edges in G' to G.

Since all induced sub-graphs of a perfect graph are also perfect, every induced sub-graph of G and every induced sub-graph of G' are perfect. As any induced sub-graph of H can be seen as the replacement of some vertex of some induced sub-graph of G with some induced sub-graph of G', their perfectness can be proved by proving that $\chi(H) = \omega(H)$ for an arbitrary perfect graph G'.

To prove the claim, it is enough to prove that there always exists an independent set T meeting all the $\omega(H)$-cliques (by the Theorem 20.4 then, H is perfect). Let $k = \omega(H)$, $m = \omega(G)$, $n = \omega(G')$ and p is the maximum size of a clique in G containing the vertex v. Then, k is either the size of a complete sub-graph not meeting G', or contains a maximum clique in G' and some complete sub-graph of G because any vertex adjacent to G' is connected to all vertices in G' by the construction. Thus,

$$k = max\{m, n + p - 1\} \tag{1}$$

As G is m-colourable due to its perfectness, consider such a colouration of G. Let the colour class containing v in this colouration of G be X. As G' is n-colourable due to its perfectness, let one colour class of G' of this n-colouration be Y. We can safely say that $T = (X \setminus \{x\}) \cup Y$ is an independent set in H, as $X \setminus \{x\}$ has no vertices adjacent to x in G, and hence has no vertices adjacent to any vertex of G' in H. We can also show that T meets all the k-cliques in H.

Let C be a k-clique in H. C may or may not meet G' as discussed above. Now, if C meets G', it contains some maximum clique in G'. Since T contains Y as a subset and Y has to meet all the maximum cliques in G' by Theorem 20.4, T thus meets all such C. If C does not meet G', then it has to be an m-element clique of G as it is a maximum clique of H. But X is an independent set meeting all the m-cliques in G and T contains X as a subset. Therefore, T meets all such C too. Thus, T meets all the k-cliques in H.

Since every induced sub-graph of H can be constructed from an induced sub-graph of G and an induced sub-graph of G', this construction of an independent set T meeting all the k-cliques is valid for every such sub-graph. So by Theorem 20.4, as there exists an independent set T meeting the maximum cliques in every induced sub-graph of H, H is perfect. Thus, proved. □

Notes:

- Since the trivial 2-vertex graph is perfect, replacing a vertex by a K_2, i.e, extension by a vertex of a perfect graph also leads to a perfect graph.
- Since a complete graph is always perfect, replacing a vertex by a complete graph also results in a perfect graph, and this result is used to prove the weak perfect graph theorem.

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