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H(d)-approximation (for SC)

$$d \triangleq \max_{i=1,2,\ldots,n} |S_i| \qquad S_i \in \mathcal{F}_J$$

$$\min \sum_{j=1}^{n} w_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad (i=1,2,\ldots,m) \qquad (MC)$$

$$0 \leq x_j \leq 1 \quad (j=1,2,\ldots,n)$$

$$x_j \text{ integer} \quad (j=1,2,\ldots,n)$$

If $b_i = 1$ for all i, then MC becomes SC.

$$I = \{1,2,\ldots,m\}$$

Sets $S_j \in I$, $j \in J = \{1,2,\ldots,n\}$

w_j weight of S_j.

$\{a_{ij}\}_{i=1}^{m}$ – characteristic vector of S_j (0–1 vector).

Dual for SC

$$\max \sum_{i=1}^{m} y_i$$

$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \leq w_j, \quad \{j=1,2,\ldots,n\}$$

$$y_i \geq 0, \quad i \in \{1,2,\ldots,m\}.$$

(2)

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Let C be any cover and let C^G be a cover given by the greedy (heuristic).

It is enough to show that

$$\sum_{j \in C} H(d) w_j = \sum_{j=1}^{n} H(d) w_j x_j \geq \sum_{j \in C^G} w_j$$

[C has characteristic vector $\{x_j\}$] — (1)

The greedy cover is no more than $H(d)$ times any cover, whatsoever.

$\Updownarrow$

Let C^* be an optimal cover.

The statement (1) can hold if we show that there is a "feasible" solution $\{y_i\}$ of the dual with $\sum_{i=1}^{m} y_i = \sum_{j \in G^c} w_j$ — (3)

and $\sum_{i=1}^{m} a_{ij} y_i \leq H(|S|_j) w_j$, $j = 1, 2, \ldots, n$. — (2)

This is so as (1) can hold

$$\left(\sum_{j \in C} H(d) w_j\right) = \sum_{j=1}^{n} H(d) w_j x_j \geq \sum_{j=1}^{n} H(|S_j|) w_j x_j \geq$$

(x: characteristic vector for C; d: largest size of sets.)

Y!

$$\geq \sum_{j=1}^{n} \sum_{i=1}^{m} (a_{ij} y_i) x_j \quad \text{— by (2)}$$

$$= \sum_{i=1}^{m} \underbrace{\sum_{j=1}^{n} (a_{ij} x_j)}_{\geq 1} y_i = \sum_{i=1}^{m} y_i$$

$\Rightarrow$ C is feasible

$$\Rightarrow \sum_{j \in C^G} w_j \quad \text{— (3)}$$

(4)

Now we prove (3) & then (2).

$S_j^k \triangleq$ S_j at the beginning of iteration k

$|S_j^k| \triangleq s_j^k$.

Let y_i be the average price paid for covering element i by the greedy algorithm. This is the average price of all elements $y_i = \frac{w_k}{s_k^k}$.

However $\frac{w_k}{s_k^k} \leq \frac{w_j}{s_j^k}$, $\forall j$ (as the algorithm picks the smallest price ~~element~~ set).

If we had t iterations, selected sets $S_1^1, S_2^2, \ldots, S_t^t$

$$\sum_{i=1}^{m} y_i = \sum_{k=1}^{t} \sum_{i \in S_k^k} y_i = \sum_{k=1}^{t} s_k^k \left(\frac{w_k}{s_k^k}\right) = \sum_{k=1}^{t} w_k = \sum_{j \in C^G} w_j$$

proves (3)

for (2) $\sum_{i=1}^{m} a_{ij} y_i = \sum_{k=1}^{t} \left(\sum_{i \in S_j \cap S_k^k} y_i \right) = \sum_{k=1}^{t} \sum_{i \in S_j^k \setminus S_j^{k+1}}$

$= \sum_{k=1}^{t} \underbrace{\left(s_j^k - s_j^{k+1}\right)}_{\text{so many}} \frac{w_k}{s_k^k}$

$\{a_{ij}\}_{i=1}^{m}$ characteristic vector for S_j

$S_j \cap S_k^k = S_j^k \setminus S_j^{k+1}$

(4)

Again $\sum_{i=1}^{m} a_{ij} y_i = \sum_{k=1}^{p} (s_j^k - s_j^{k+1}) w_k$ $\leq w_j \sum_{k=1}^{p} \left(\frac{s_j^k - s_j^{k+1}}{s_j^k}\right)$ (5)

Also terms of $H(s_j^k) - H(s_j^{k+1})$ add up to more than the difference of terms $s_j^k - s_j^{k+1}$ times $\frac{1}{s_j^k}$

$\left(\frac{1}{s_j^k} \geq \frac{1}{s_{j+1}^{k+1}}\right)$

e.g.:

$\rightarrow$ So $\sum_{i=1}^{m} a_{ij} y_i \leq w_j \sum_{k=1}^{p} \left(H(s_j^k) - H(s_j^{k+1})\right)$

$\leq w_j \, H(s_j^1)$ — (_)

which is (2)

□