CS60023 Approximation and Online Algorithms 2024

04.01.2024

Basic introduction to approximation and online algorithms;

Heuristics for vertex cover of $G(V, E)$

- ▷ Select (and delete) an arbitrary vertex $v \in V$ or a large degree vertex for inclusion in vertex cover C.
- ▷ Select an arbitrary edge $\{u, v\} \in E$, include both u & v in vertex cover C.

General set cover problem

- ▷ logarithmic bound of approximation ratio

General weighted set cover problem

- ▷ H_n or the n^{th} Harmonic number bound of approximation ratio

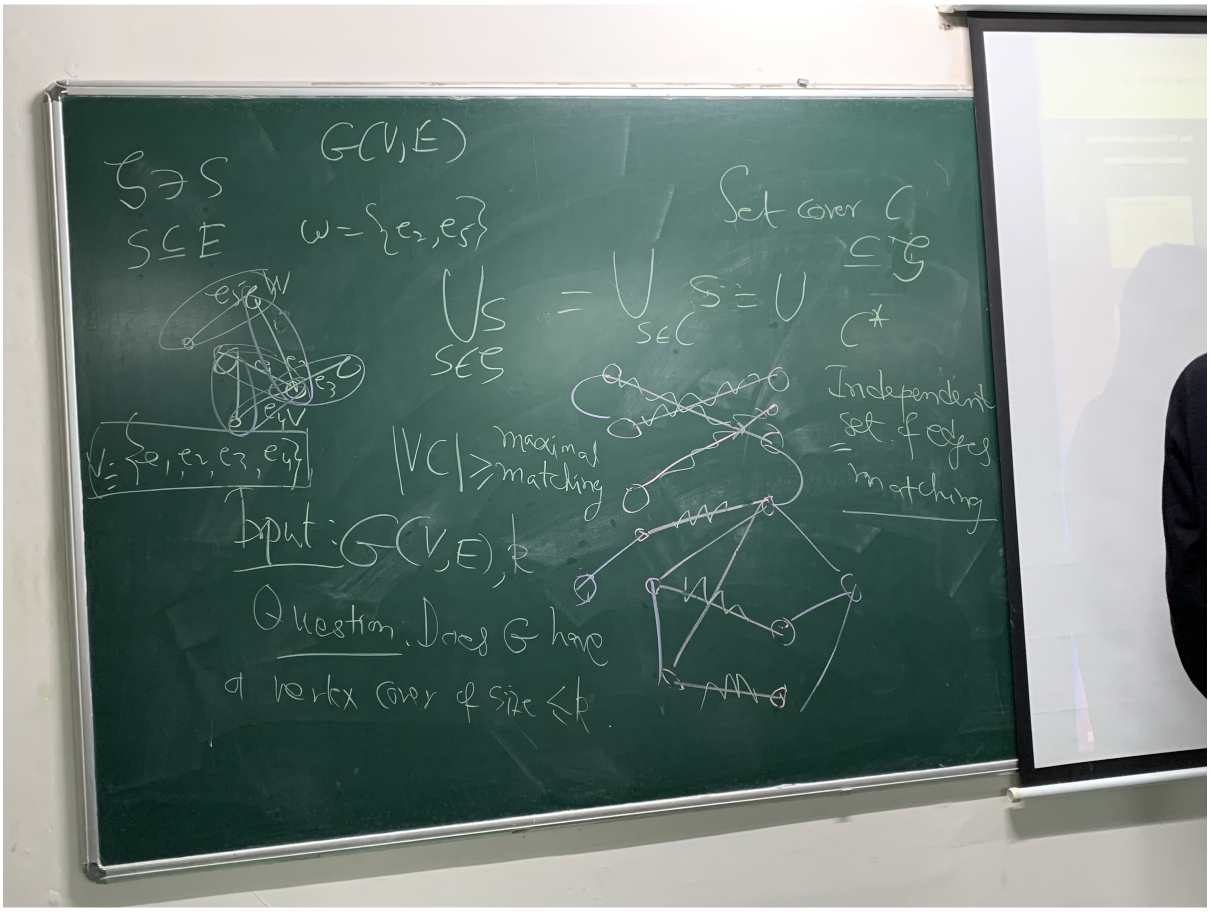
Dual fitting analysis technique for the greedy set cover;

05.01.2024

- ▷ Basic introduction and background to approximation and online algorithms; Minimization and maximization problems
- ▷ Large DAG subgraphs of directed graphs
  - Partitioning of edges into two sets so that each set induces a DAG, then we can choose the bigger one, ensuring at least $\frac{e}{2} \geqslant \frac{\text{OPT}}{2}$ edges are selected in the large DAG.
- ▷ Large cuts for undirected graphs
  - At least $\frac{e}{2}$ edges are across the cut which is at least $\frac{\text{OPT}}{2}$
- ▷ Vertex cover using DFS tree
  - Factor 2 ratio
- ▷ Vertex cover using Large Cut
  - Factor of $\log \Delta$ ratio
- ▷ General weighted set cover problem
  - H_n or the n^{th} Harmonic number bound of approximation ratio
- ▷ NP-Completeness reduction from 3-SAT to vertex cover

08.01.2024

- 2-approx vertex cover algorithm using maximal matching
- If an additional parameter k is input with graph $G = (V, E)$, does there exist a vertex cover of size $\leqslant k$?
- General weighted set cover problem
- linear programming duality
  - optimizing an objective function satisfying the given constraints
  - Decision version of the linear programming problem i.e. whether LP belongs to class P or NP



11.01.2024

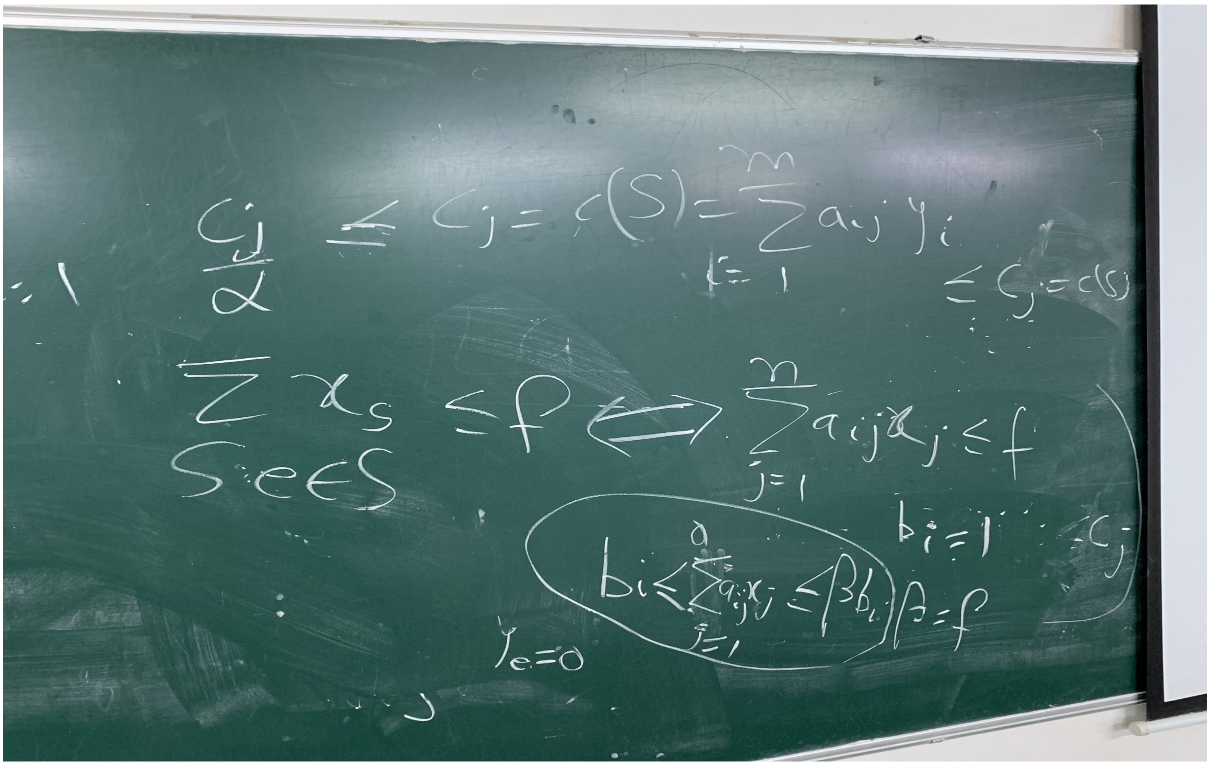
- Linear programming duality;
  - Minimization of an objective function and corresponding maximization of the dual version of the primal linear program
- Weak duality theorem

- Feasible solutions x and y to the primal and dual respectively; lower bounds of the constraints of the primal program define the objective function for the dual program; Symmetrically the upper bounds in the constraints of the dual program define the objective function in the primal program

- ▷ Primal-dual optimality and complementary slackness
- ▷ Membership of the linear program in the class co-NP
- ▷ Weighted set cover in terms of a primal-dual linear program

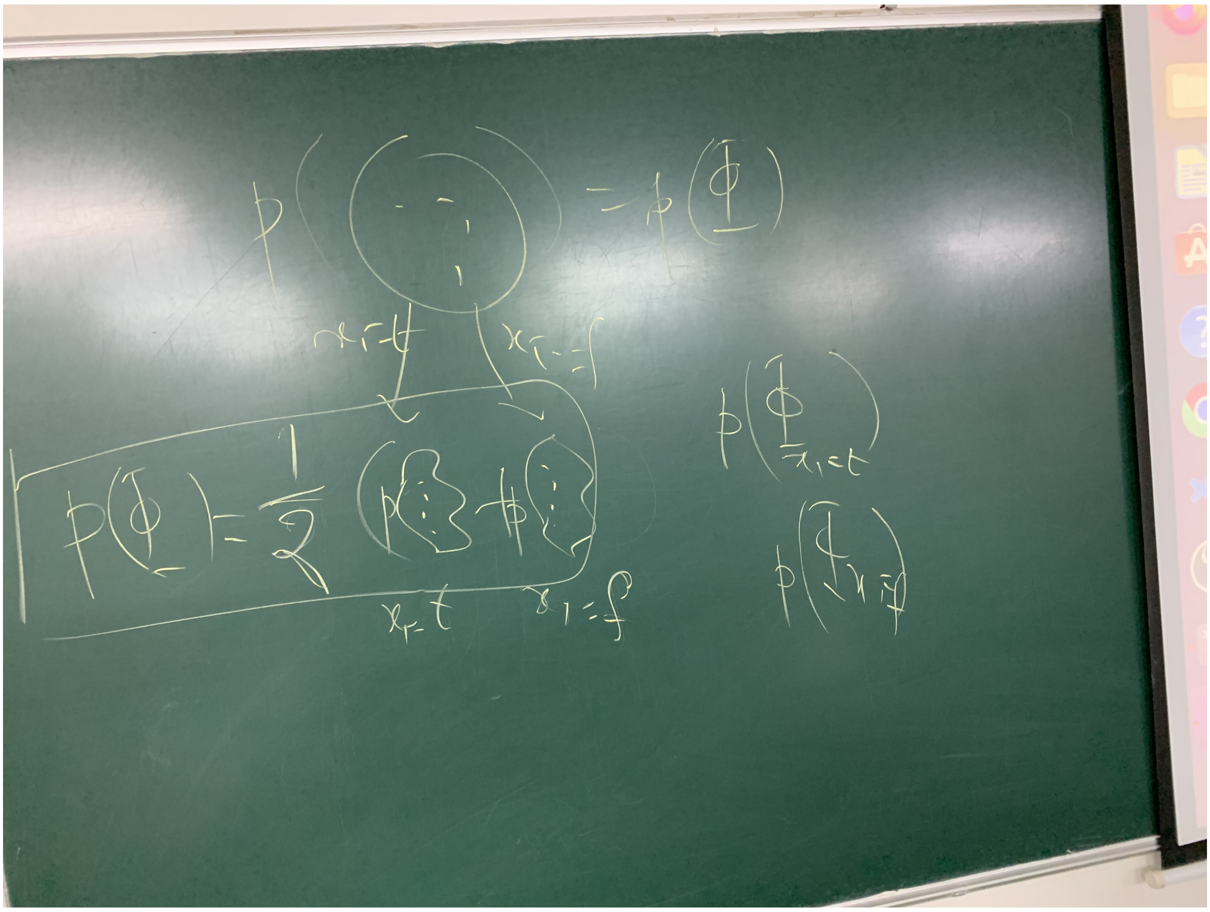
12.01.2024

- ▷ Use of indicator variables for vertex cover;
  - Discrete Integer Linear Program for vertex cover
  - NP-Hardness for ILP and weighted vertex cover
  - Linear programming relaxation for the ILP;
  - The lower bound and ensuing approximation cap
- ▷ The greedy set cover
  - The dual fitting analysis technique for the greedy set cover
  - The dual lower bound
  - The greedy set cover prices
  - The slackness condition and the weighted set cover problem where each element is in at most f sets i.e. there is an underlying guarantee that there exists an f-factor algorithm given that the algorithm satisfies the relaxed complementary slackness conditions.



15.01.2024

- ▷ The notion of prices in the primal integral solution of the linear programming relaxation of the integer program
- ▷ SAT
  - Random truth value assignment to literals. Q. What is the probability that at least 50% of the clauses will be satisfied ?
- ▷ MAXSAT
  - Find out the expected number of satisfied expressions



07.03.2024

- ▷ Online K-server problem
  - K servers need to be moved around to service requests appearing online at points of a metric space.
  - The total distance travelled by the K servers must be minimised, where any request arising at a point of the metric space must be serviced on site by moving a server to that site.
  - In every metric space with at least $K + 1$ points, no online algorithm for the K-server problem can have competitive ratio less than K.

08.03.2024

- ▷ The K-center problem
- ▷ Travelling salesman problem
- ▷ bin-packing problem
- ▷ L-reduction for the independent set problem
- ▷ Gap-preserving reduction

14.03.2024

- ▷ Online deterministic paging algorithms and their competitive ratio bounds
- ▷ Amortized bound for the competitive ratio for paging using a potential function
- ▷ Online coloring for complements of bipartite graphs
  - The formulation using maximal stable sets partitioning
  - The stable sets and the online competitive ratio bound
- ▷ Online coloring for complements of chordal graphs

21.03.2024

- ▷ Online Paging Problem
  - Results on deterministic paging algorithms
  - The algorithms LRU and FIFO are k-competitive
  - Let A be a deterministic online paging algorithm. If A is c-competitive, then $c \geqslant k$.
- ▷ Randomization in online algorithms
  - Randomized paging algorithms against oblivious adversaries
  - The MARKING algorithm is $2H_k$-competitive against any oblivious adversary where H_k is the k-th Harmonic number.
- ▷ Yao's minimax principle
- ▷ If R is a randomized online paging algorithm that is c-competitive against any oblivious adversary, then $c \geqslant H_k$.

01.04.2024

- ▷ Multiway Cut
  - Computational lower bounds on the sizes of cuts in the optimal solution
- ▷ The K-cut problem
- ▷ The Gomory-Hu tree and minimum weight cuts
- ▷ Properties of any optimal k-cut A and the approximation algorithm for computing a k-cut.
  - Establishing the novel lower bound

08.04.2024

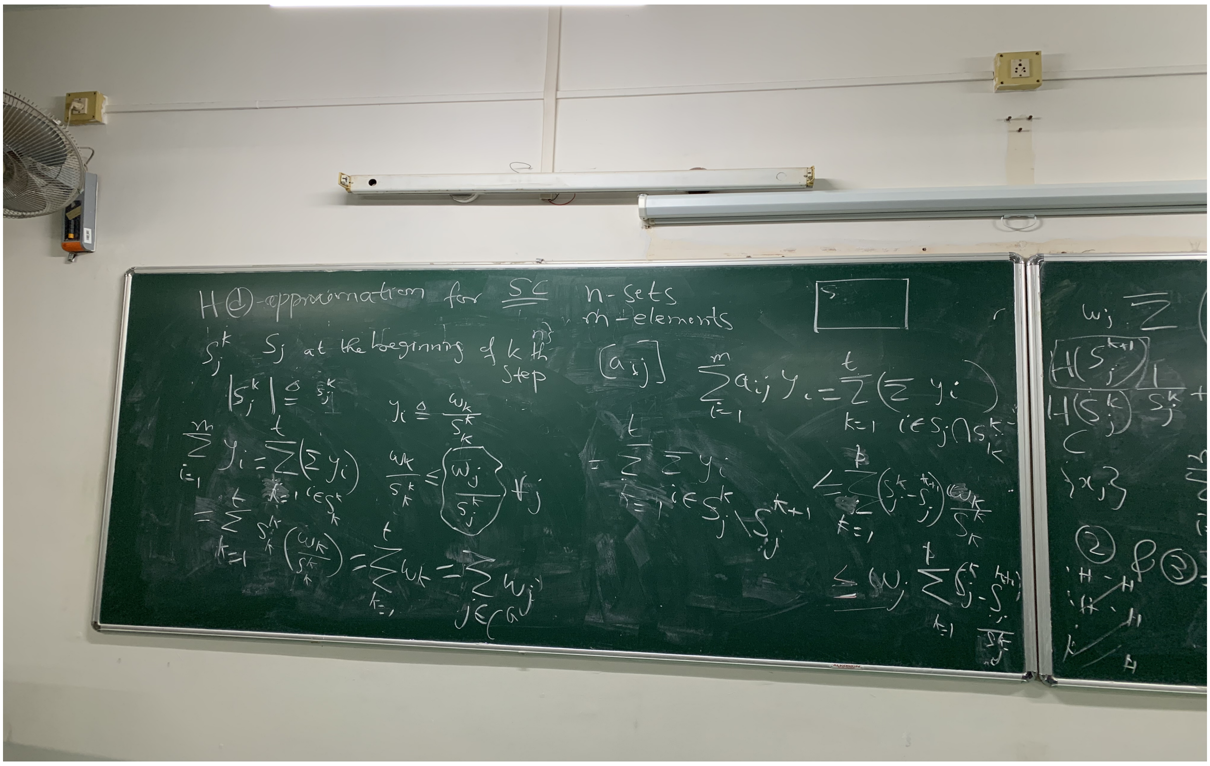
- ▷ Epsilon net theorem (Theorem 10.2.4 [Lectures on Discrete Geometry, Jiri Matousek])
- ▷ Epsilon nets and their applications
- ▷ Basic introduction to Hypergraph theory
  - Definition of shattered set
  - VC Dimension

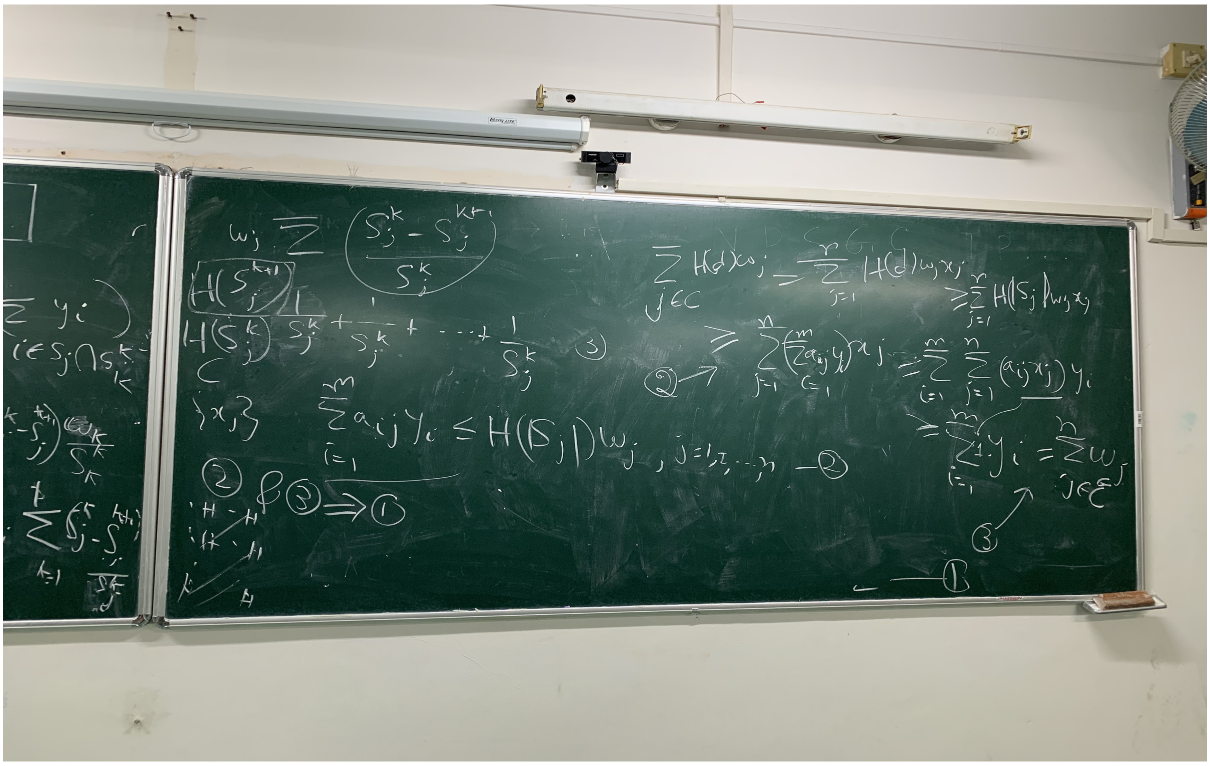
12.04.2024

- ▷ Randomized and deterministic construction of epsilon nets
  - Binomial random sampling of the set of vertices
  - Deterministic construction of ε-nets
- ▷ Epsilon nets and their applications recapitulation

18.01.2024

Board-work Coverage


H(d)-approximation for SC
n-sets
m-elements
Sj at the beginning of kth step
Step



18/01/24

$1+(d)$- approximation for s.c. (weighted)

$d \triangleq \max |S_i|, \; i \in J$ $\quad J = \{1, 2, \dots, n\}$ n-sets, m-elements

$\min \sum_{i=1}^{n} w_i x_i$ $\quad$ i/p : $[a_{ij}]$

MC $\sum_{j=1}^{n} a_{ij} x_j \geq 1, \; i \in I$ (if $b_i = 1$, then MC becomes SC)

$0 \leq x_j \leq 1 \quad j \in J$ $\qquad I = \{1, 2, \dots, m\}$

Sets S_j weight w_j

$a_{ij} = 1, \; i \in S_j$

$\quad\; = 0 \quad i \notin S_j$

$\left(\max \sum_{i=1}^{m} y_i\right)$ s.t. $\sum_{i=1}^{m} a_{ij} y_i \leq w_j, \; j \in J$

C - any set cover

C^* - optimal

C^G - greedy algorithmic answer

18/01/24

$$\sum_{j \in C} H(d)\, w_j = \sum_{j=1}^{n} H(d)\, w_j x_j \geq \sum_{j \in C^G} w_j \quad \text{— (1)}$$

Feasible solution :-

$$\sum_{i=1}^{m} y_i = \sum_{j \in C^G} w_j \quad \text{— (3)}$$

$$\sum_{i=1}^{m} a_{ij}\, y_i \leq H(|S_j|)\, w_j , \quad j = 1, 2, \dots n \quad \text{— (2)}$$

Need to show (2) & (3) $\Rightarrow$ (1)

$$\sum_{j \in C} H(d)\, w_j = \sum_{j=1}^{r} H(d)\, w_j x_j \geq \sum_{j=1}^{n} H(|S_j|)\, w_j x_j$$

$$\geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\, y_i \right) x_j$$

(by (2))

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \underline{(a_{ij} x_j)}\, y_i$$

$$\geq \sum_{i=1}^{m} 1 \cdot y_i = \sum_{\substack{j=1 \\ j \in C^G}}^{n} w_j$$

(by (3))

$\Rightarrow 1$

18/01/24

S_j^k at the beginning of the k-th steps $\quad S_j^0$

$|S_j^k| \triangleq s_j^k \qquad y_i \triangleq \frac{w_k}{s_k^k}$

$$\sum_{i=1}^{m} y_i = \sum_{k=1}^{t} \Big(\sum_{i \in S_k^k} y_i \Big)$$

There are t steps

$$= \sum_{k=1}^{t} s_k^k \left(\frac{w_k}{s_k^k} \right) = \sum_{k=1}^{t} w_k = \sum_{j \in C^G} w_j$$

$$\sum_{i=1}^{m} a_{ij} y_i = \sum_{k=1}^{t} \Big(\sum_{i \in S_j \cap S_k^k} y_i \Big)$$

$$= \sum_{k=1}^{t} \sum_{i \in S_j^k \setminus S_j^{k+1}} y_i$$

$$\le \sum_{k=1}^{t} \left(s_j^k - s_j^{k+1} \right) \frac{w_k}{s_k^k} \le w_j \sum_{k=1}^{p} \frac{\left(s_j^k - s_j^{k+1} \right)}{s_j^k}$$

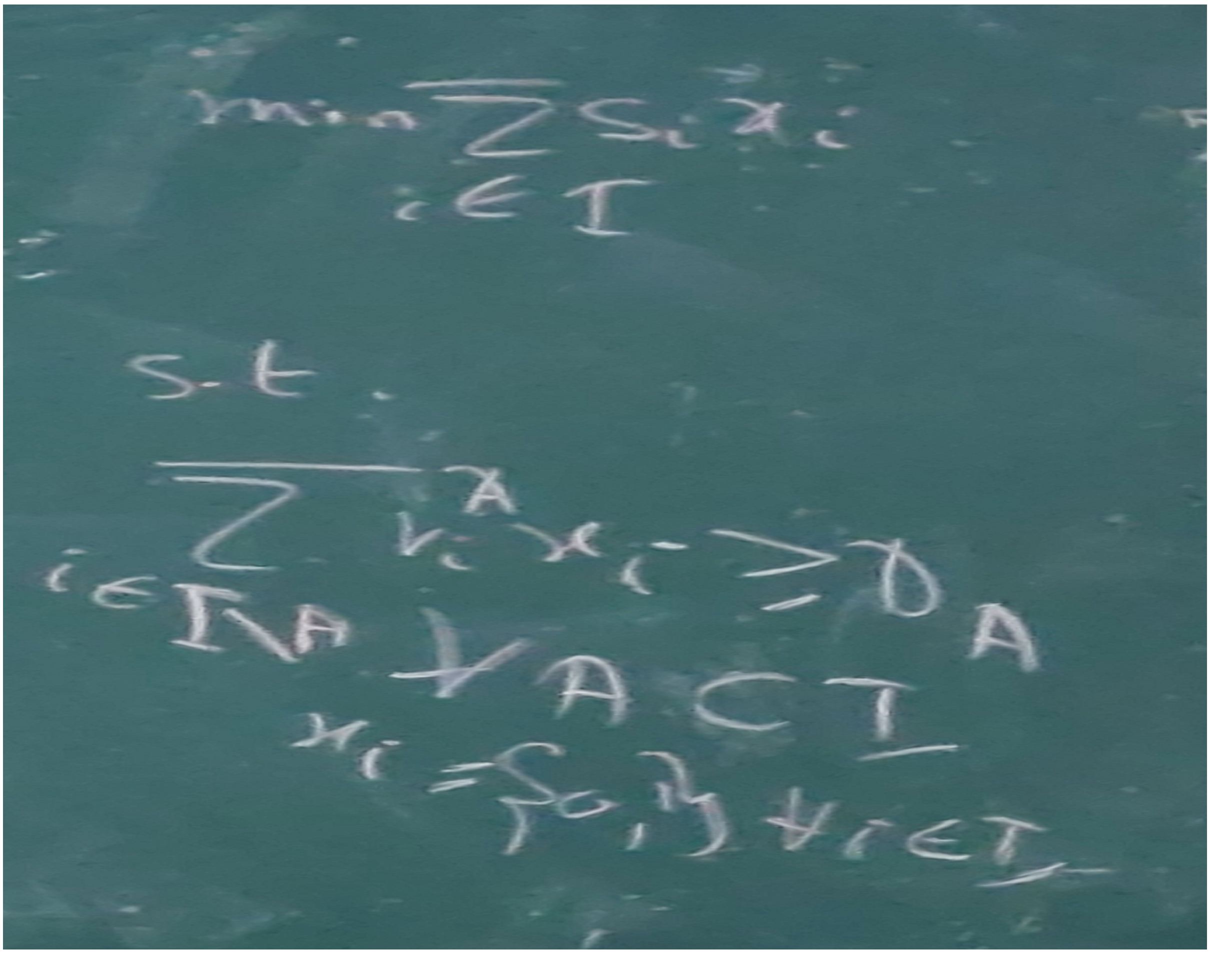


Figure 1: 01.02.2024

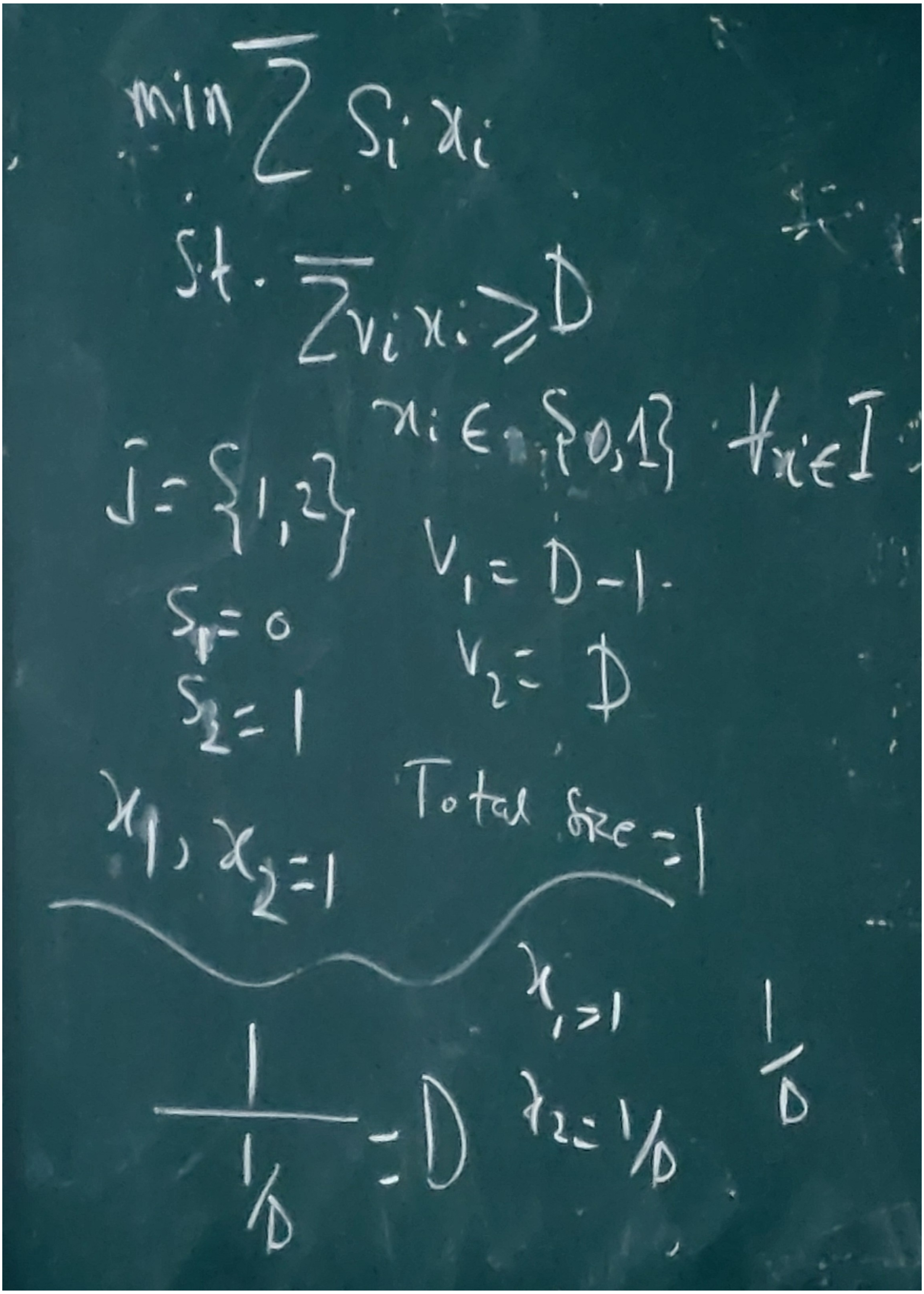
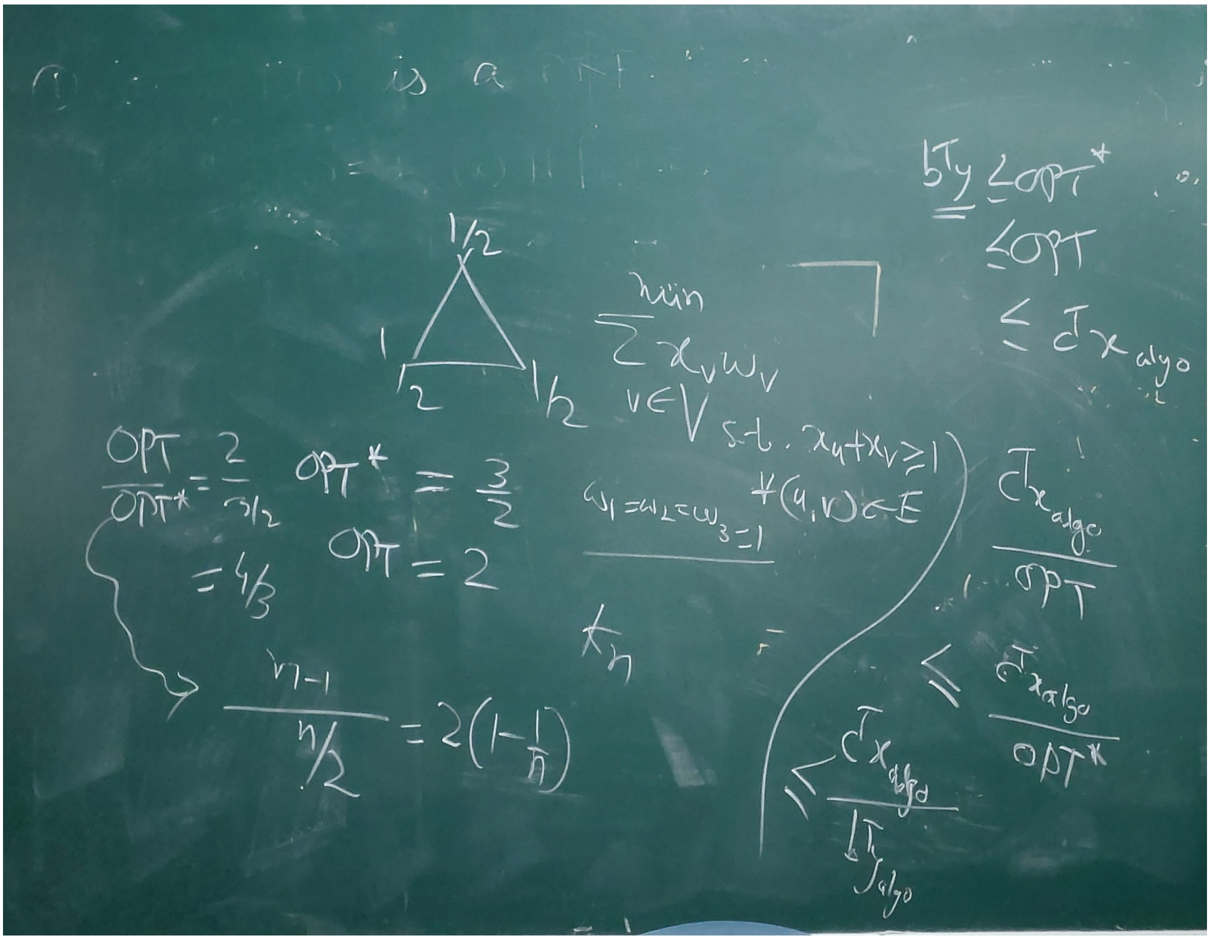


Figure 2: 01.02.2024



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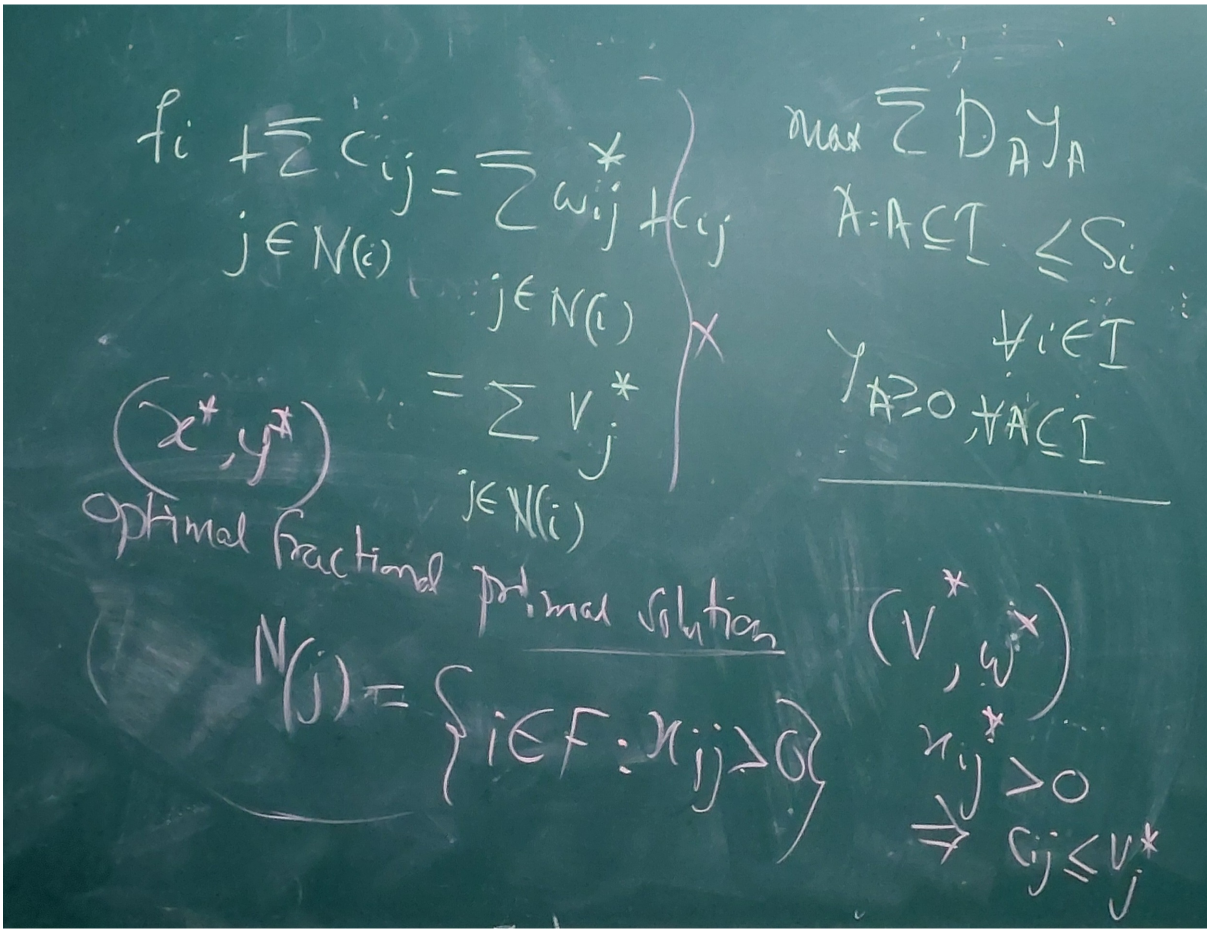

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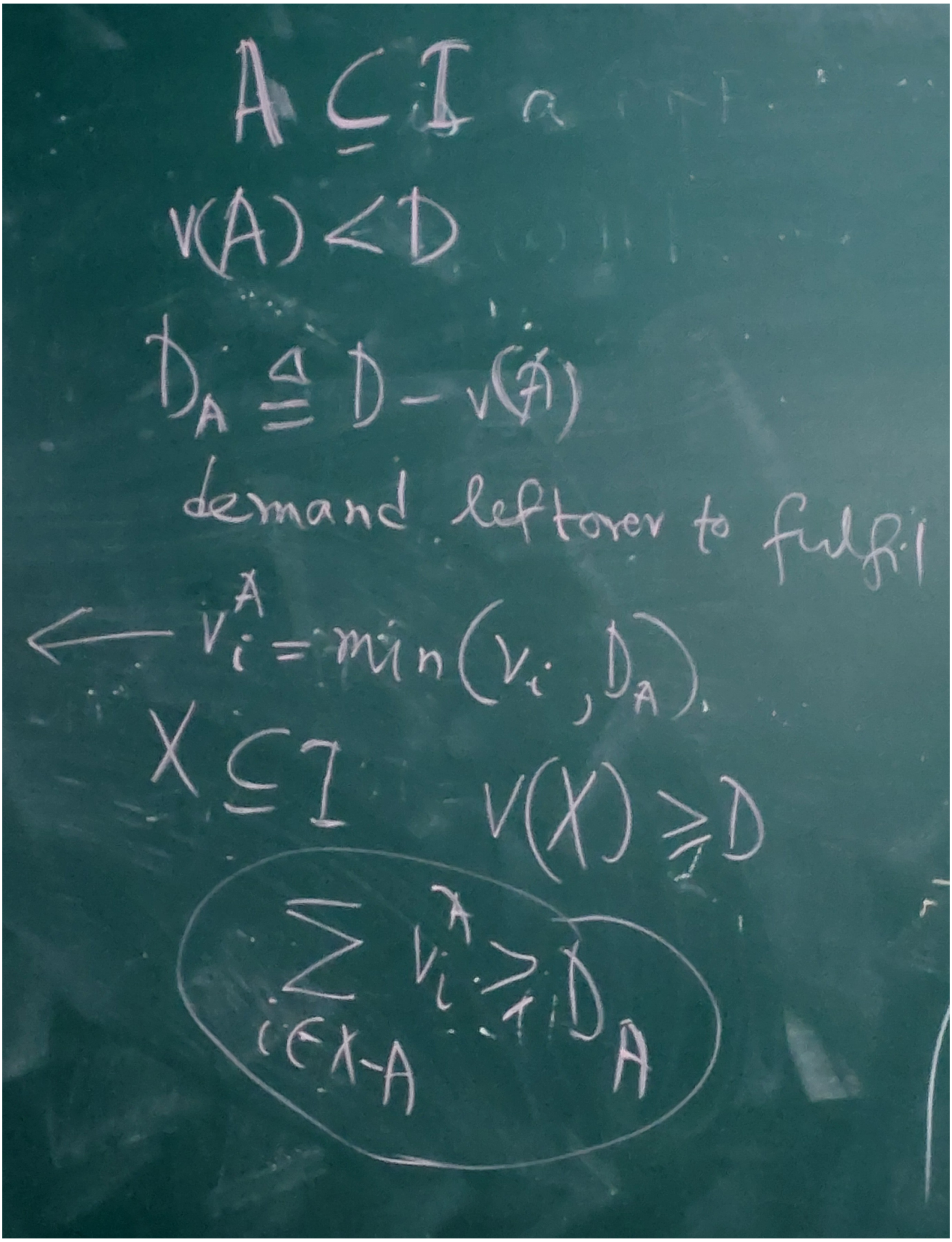

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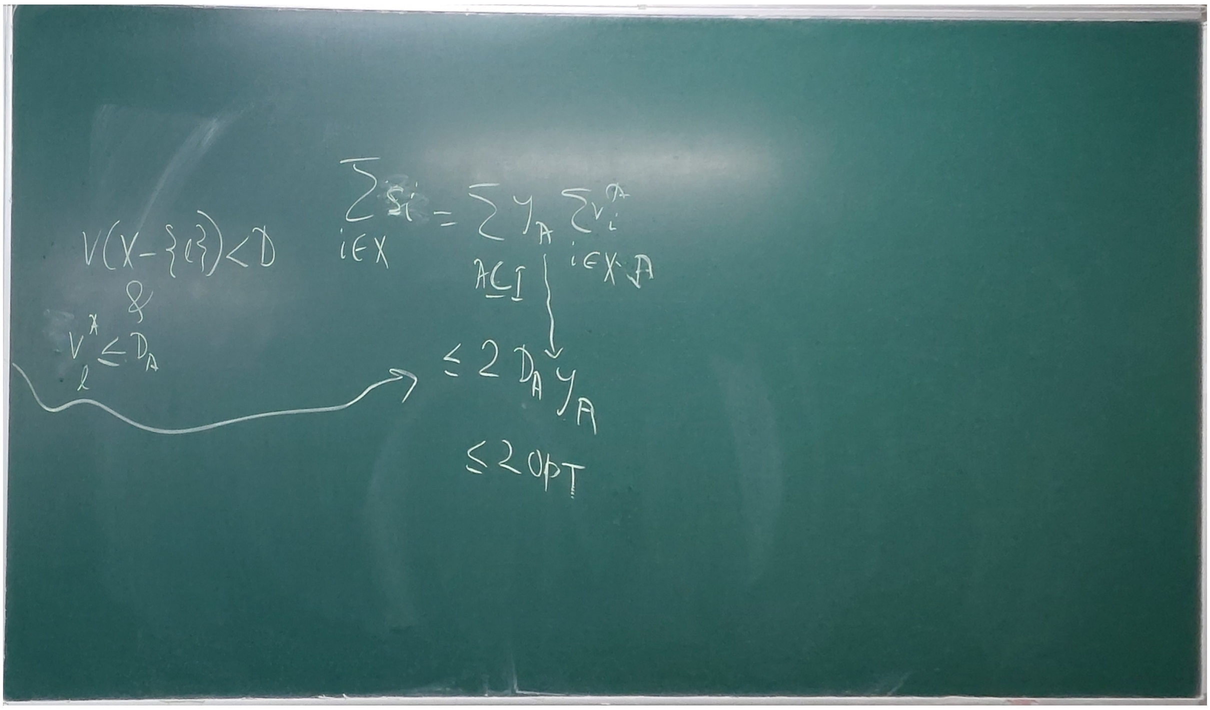
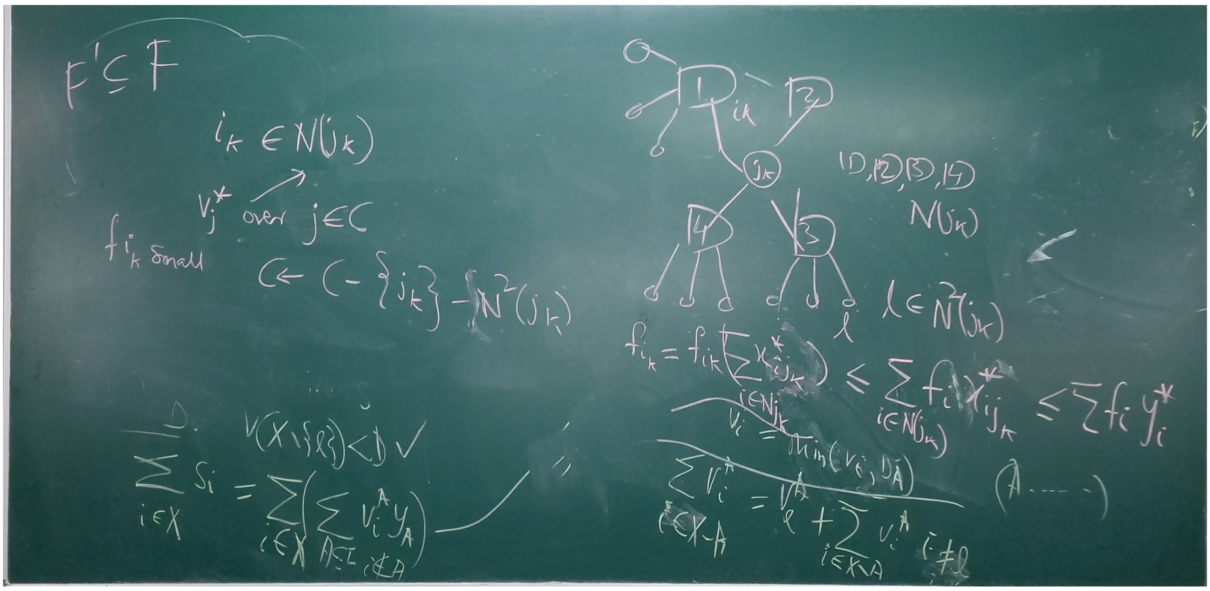


Figure 6: 02.02.2024



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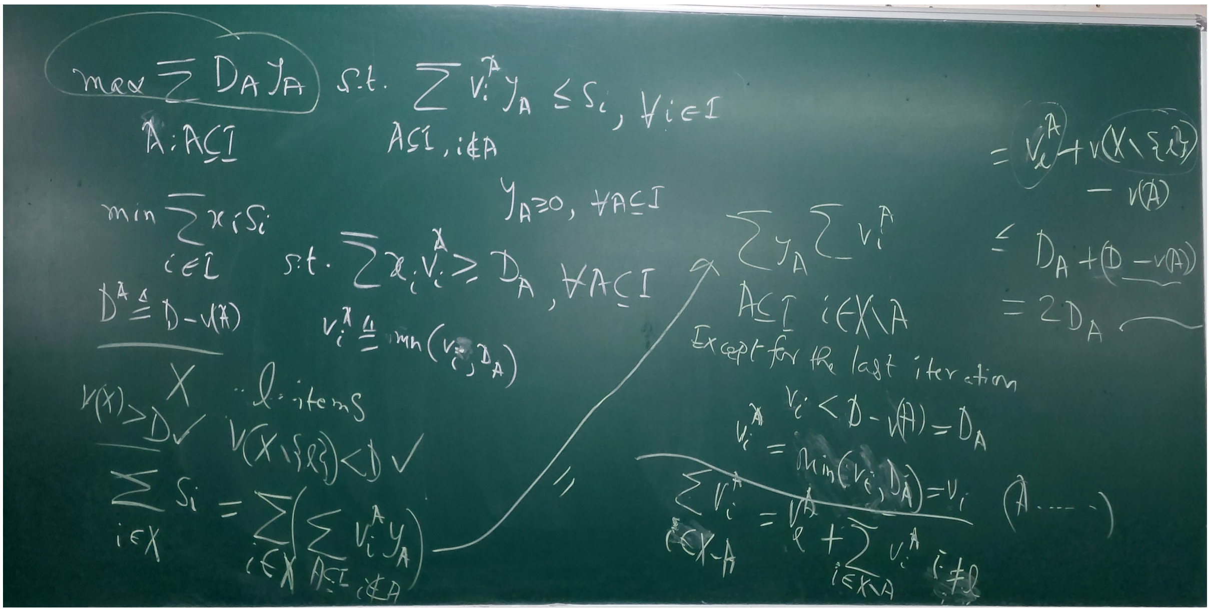

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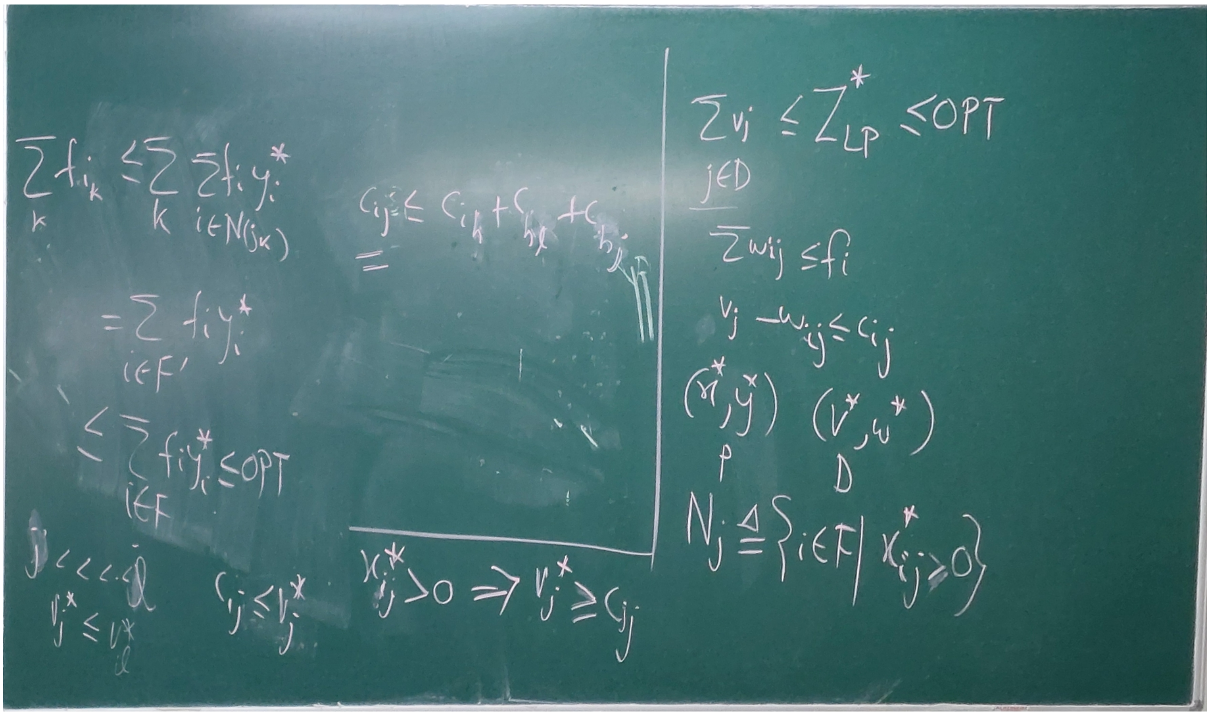

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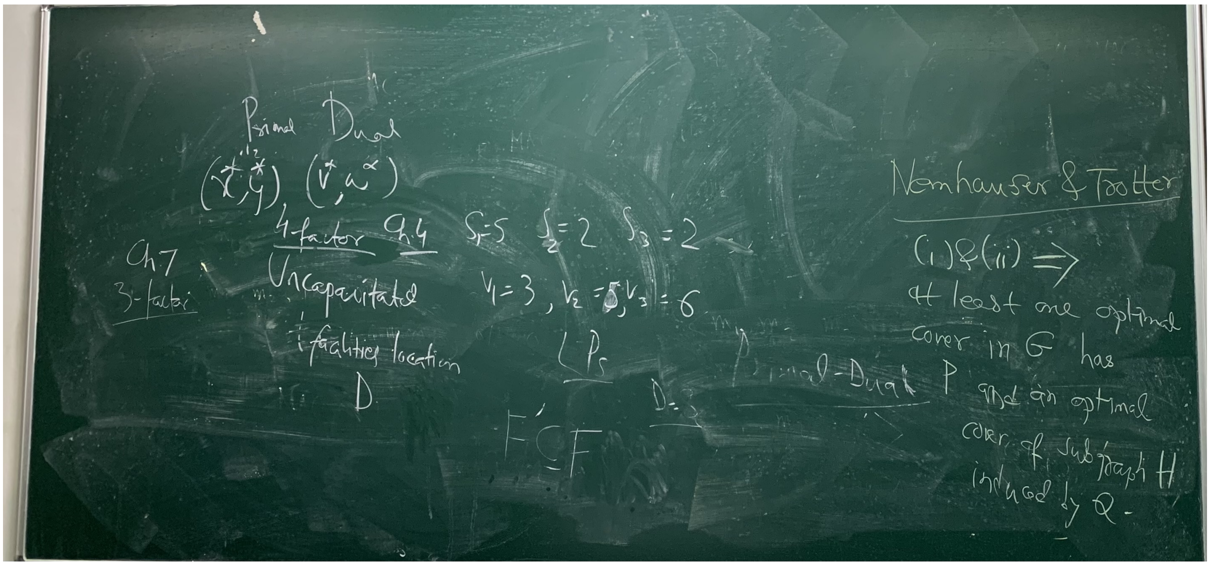
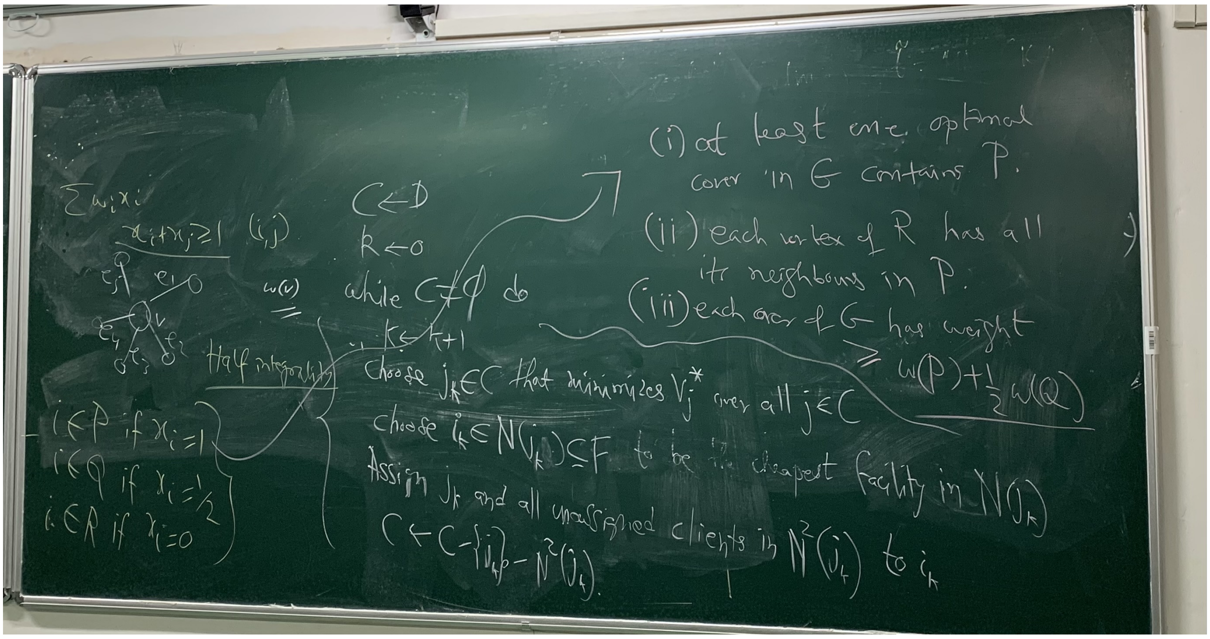


Figure 10: 09.02.2024



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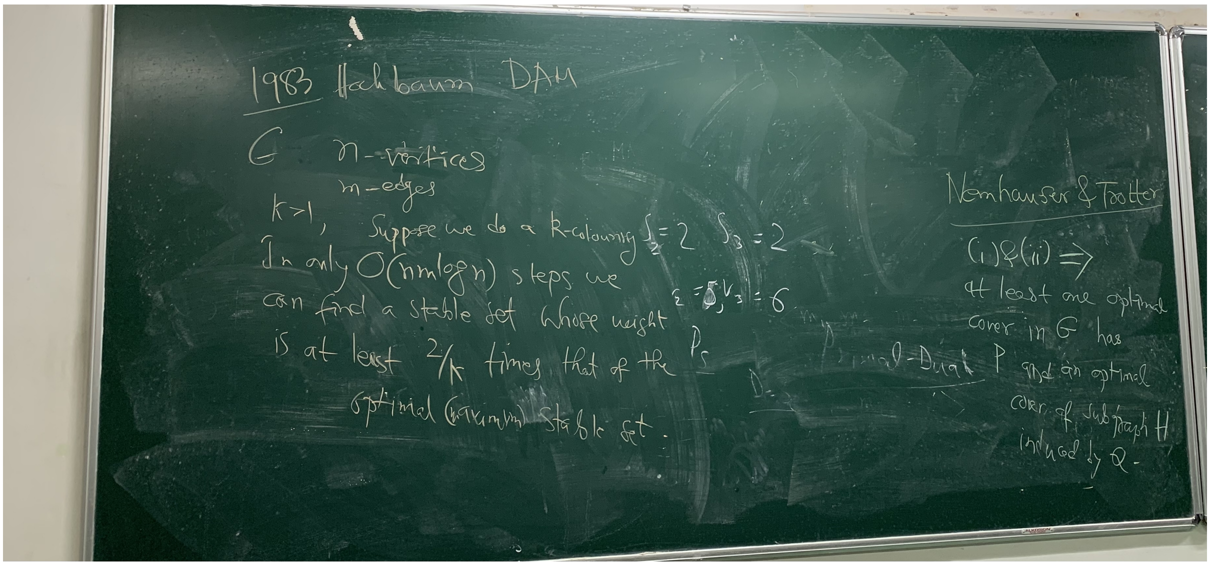

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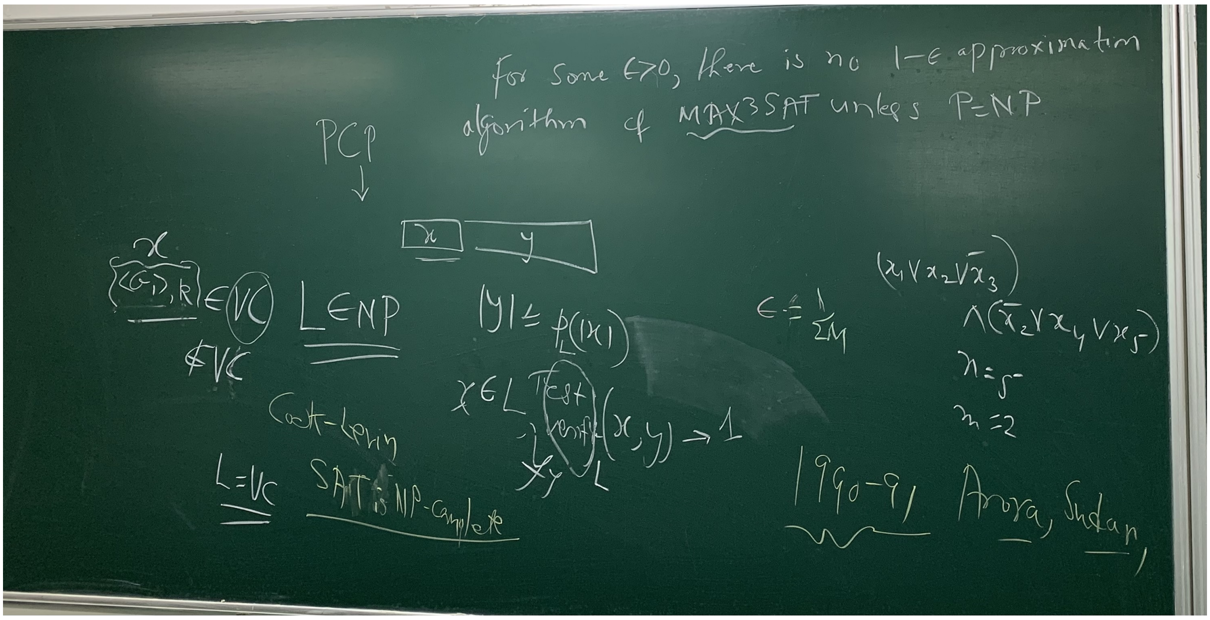

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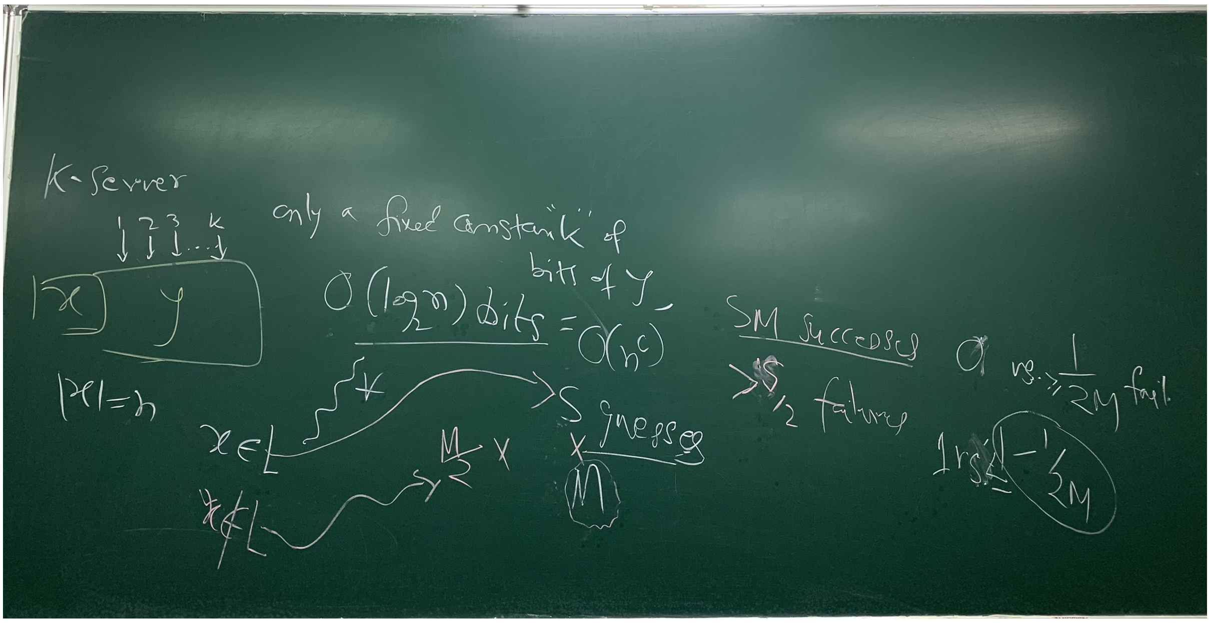

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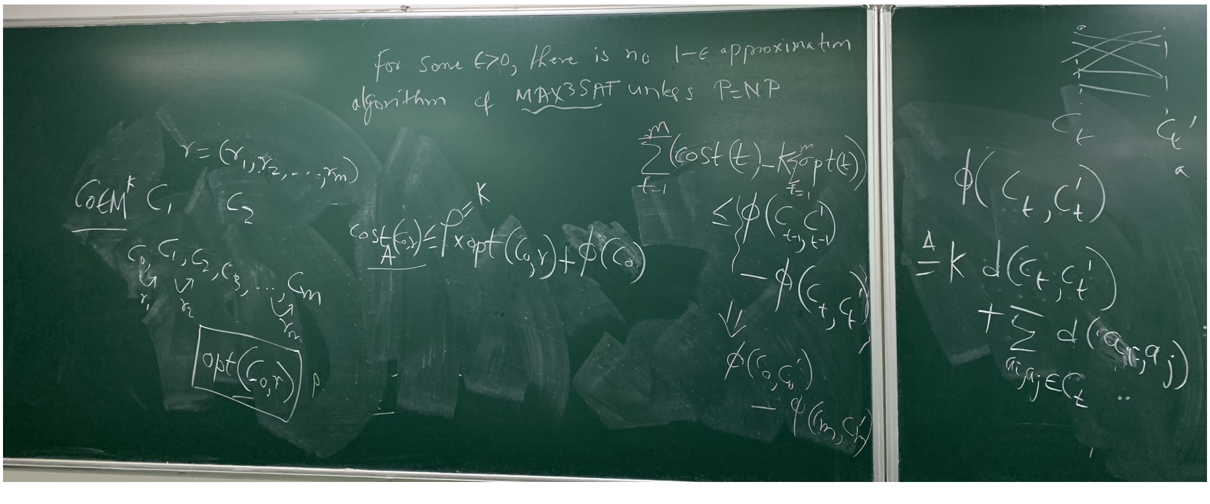


Figure 15: 26.02.2024

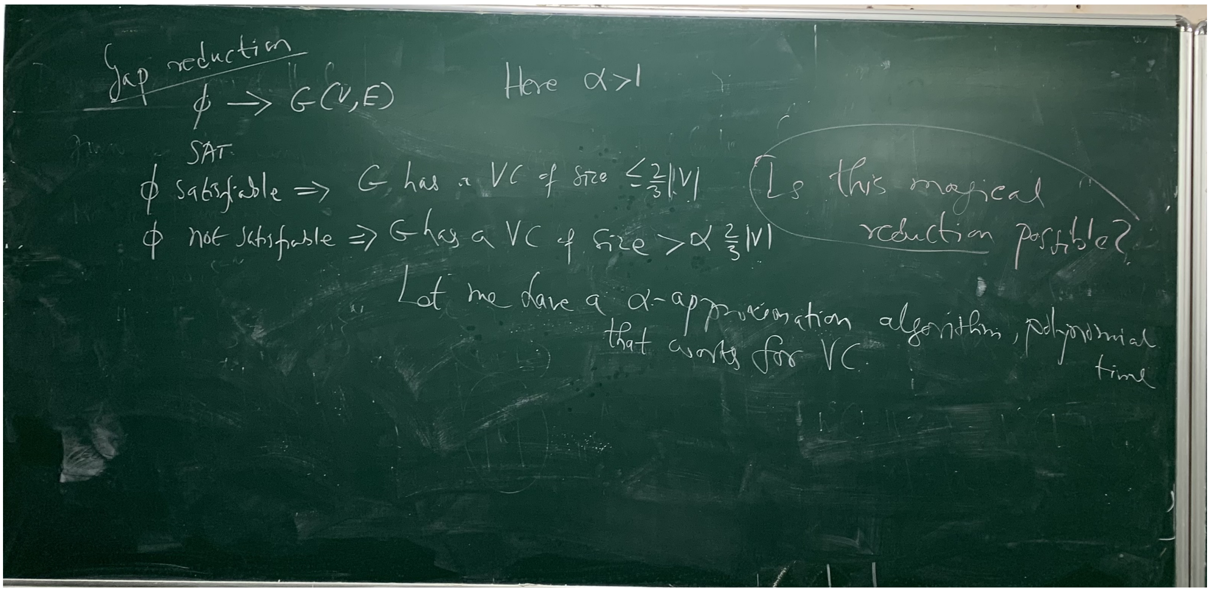
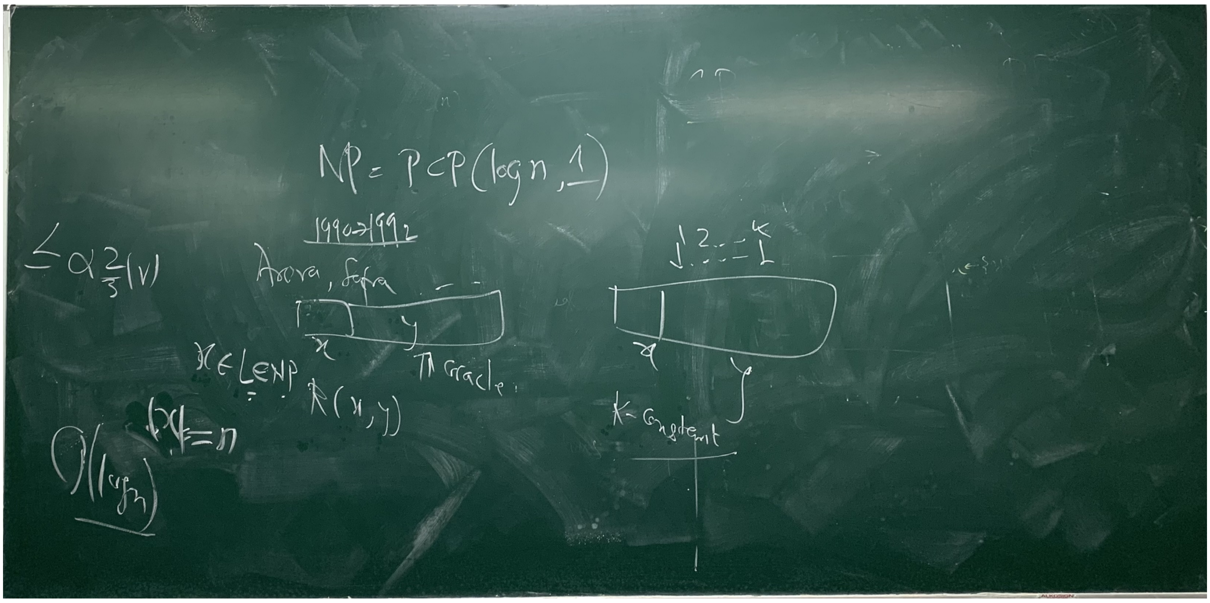


Figure 16: 29.02.2024



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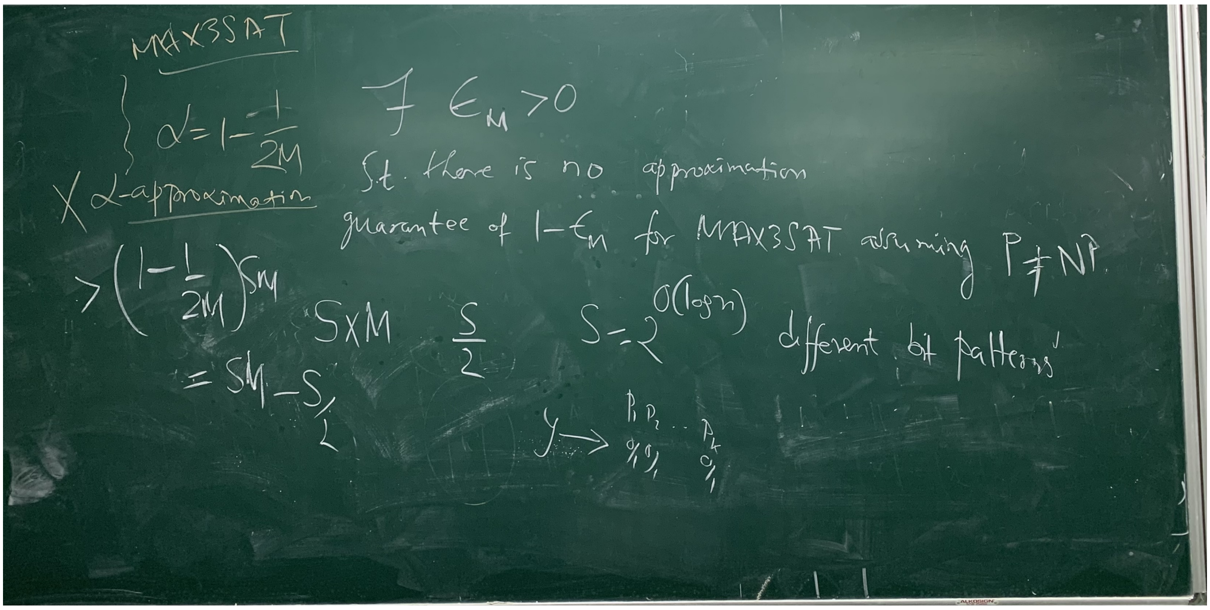

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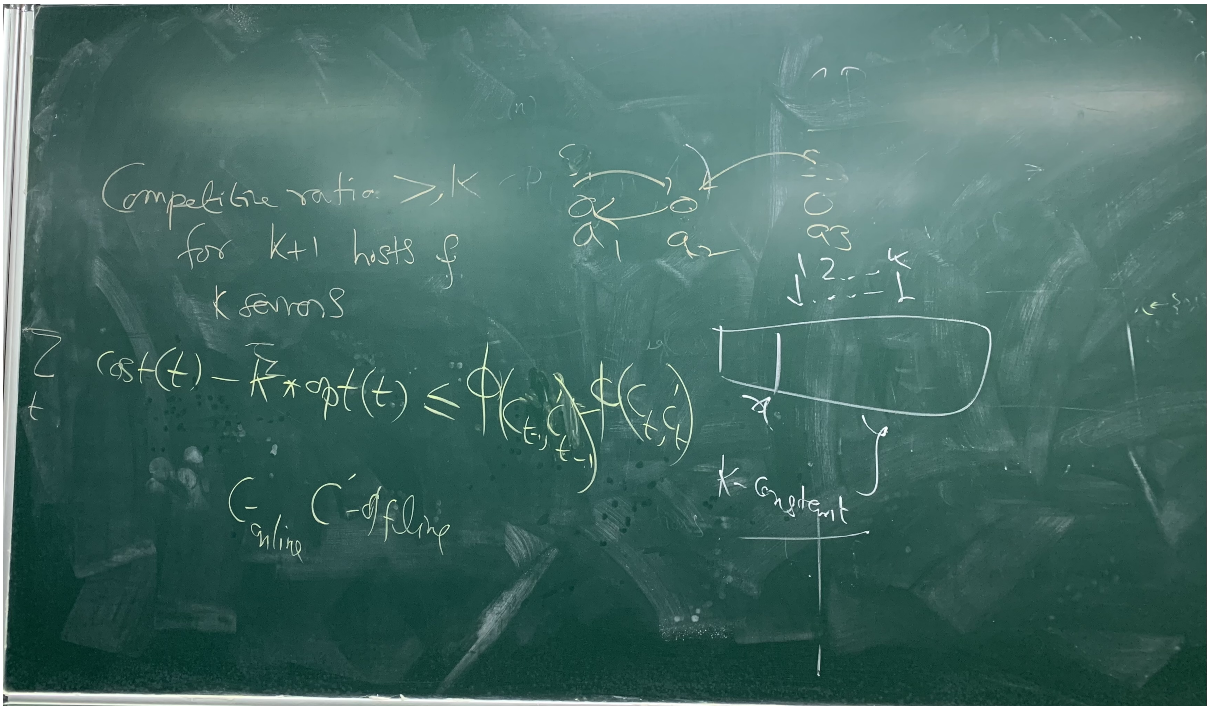

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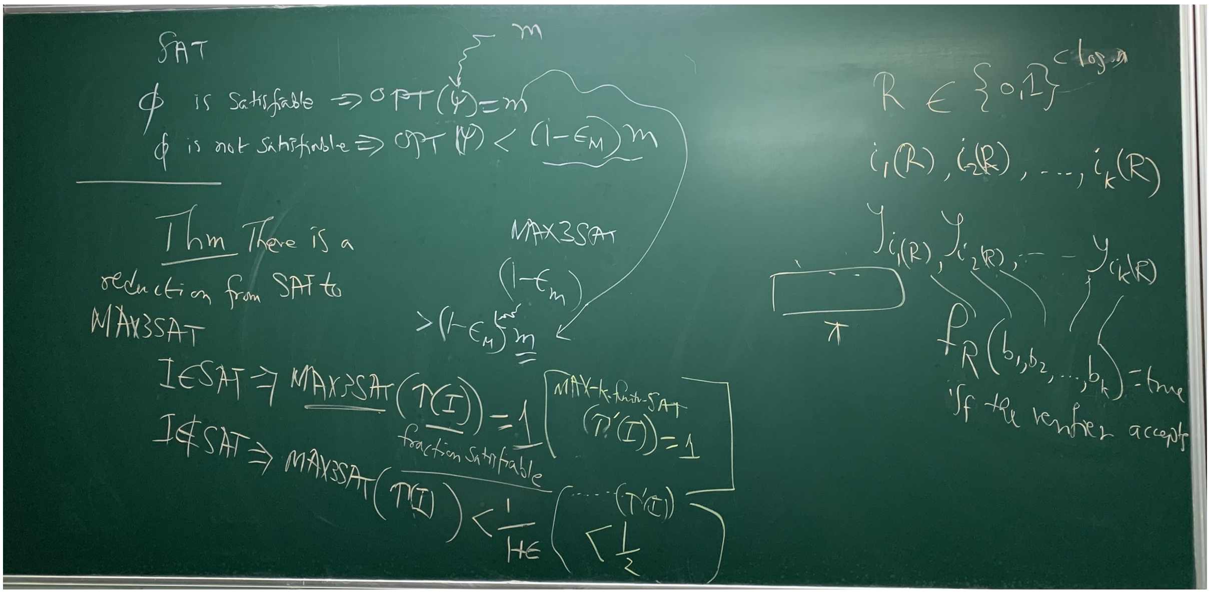
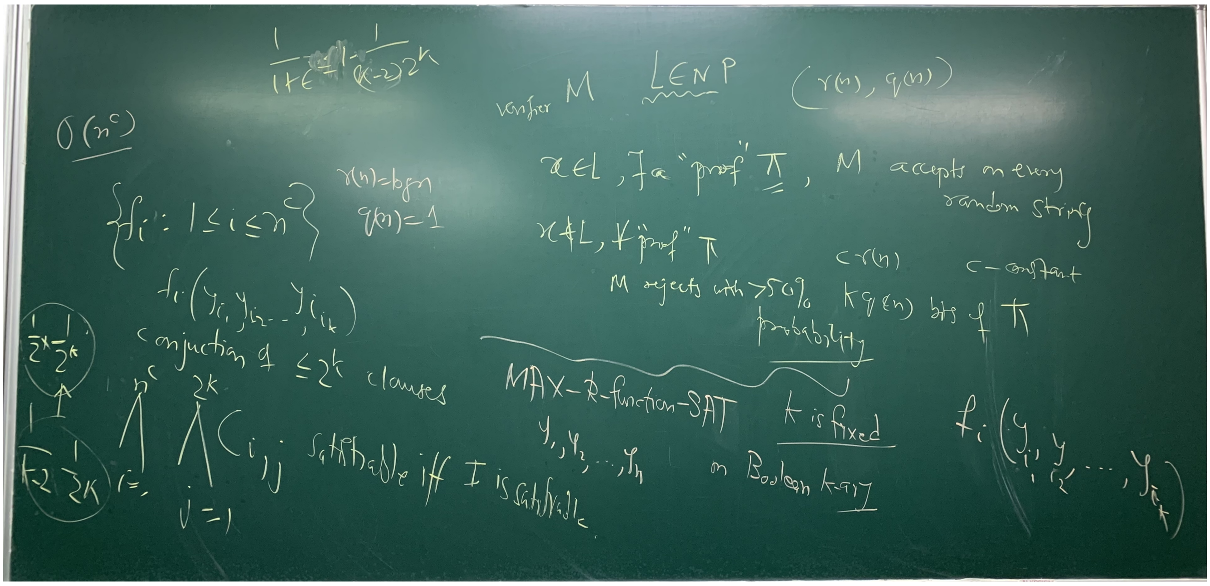


Figure 20: 01.03.2024



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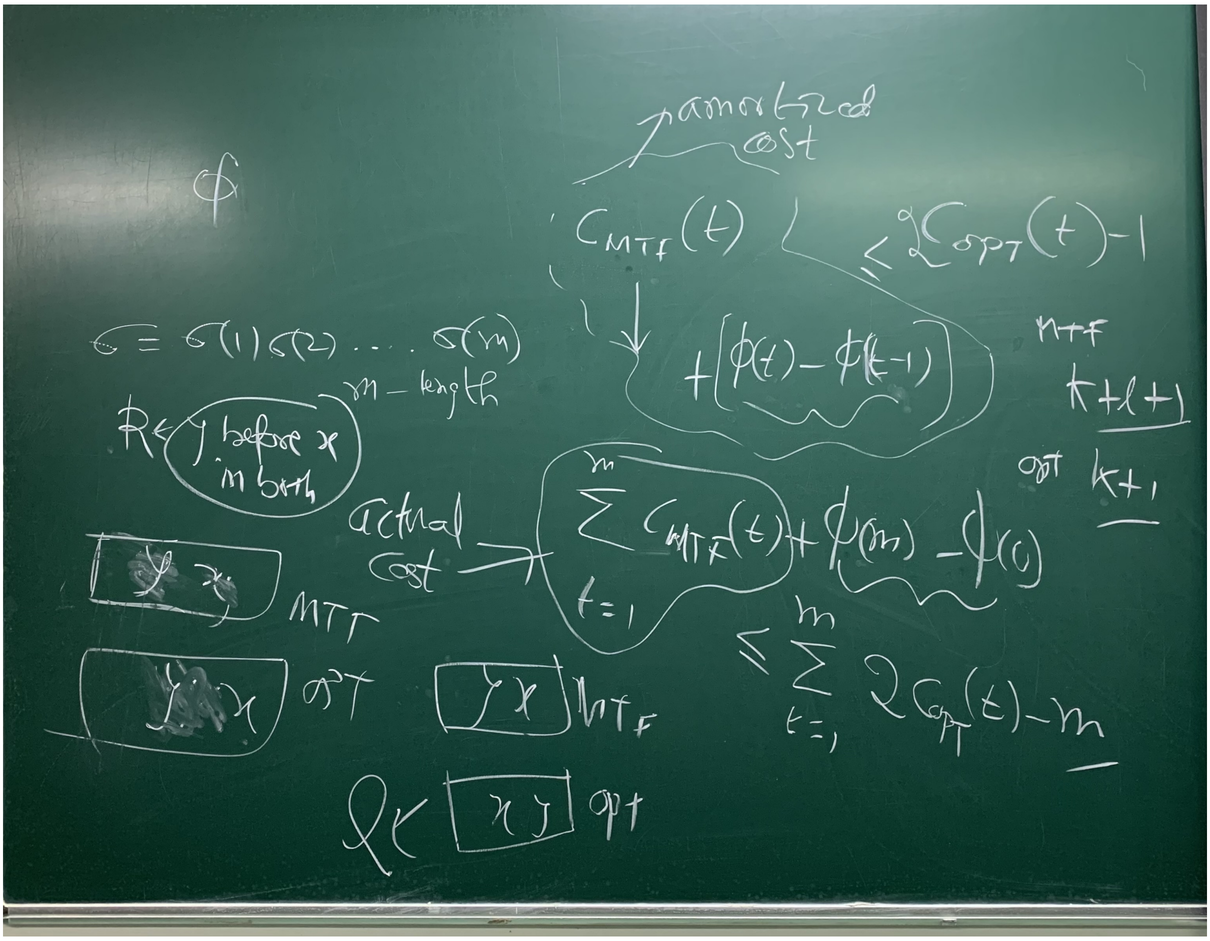

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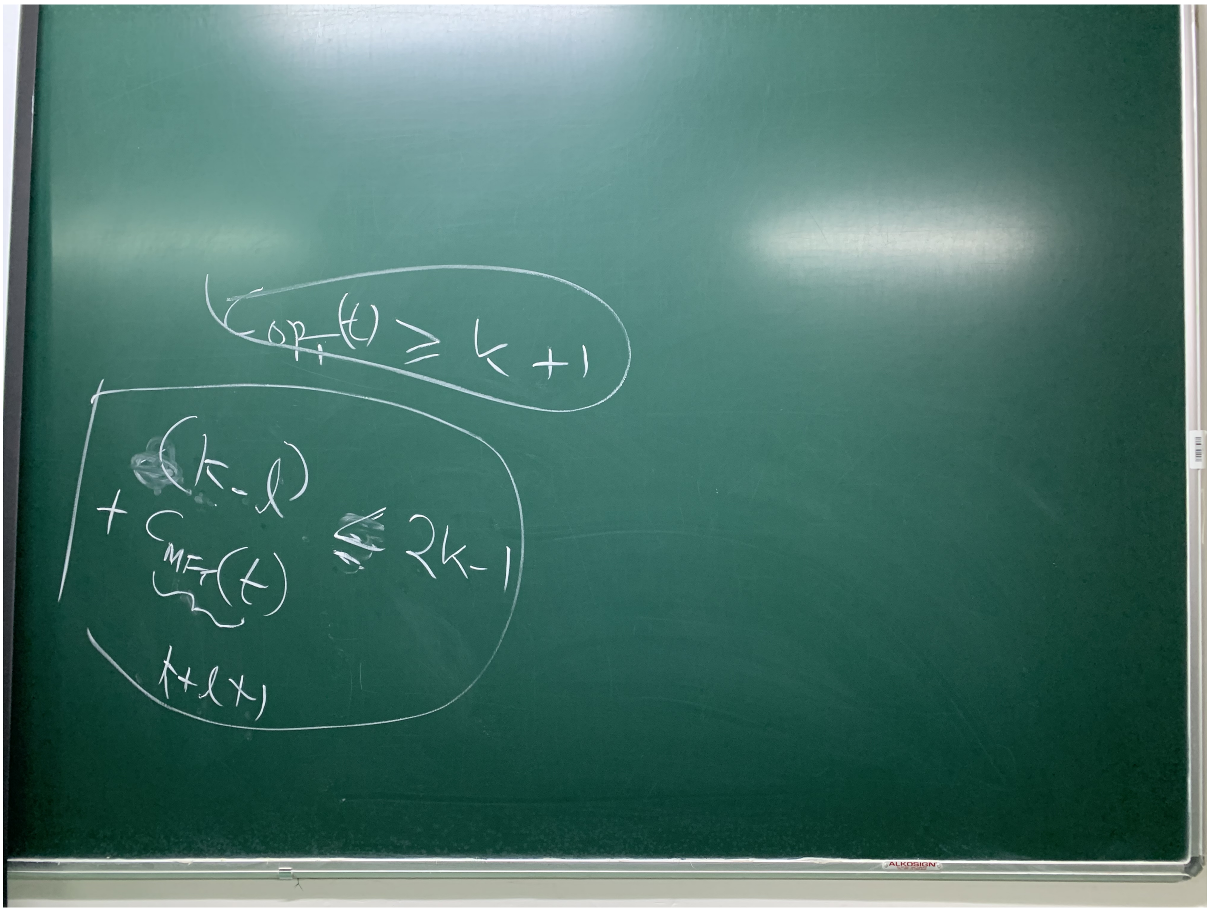

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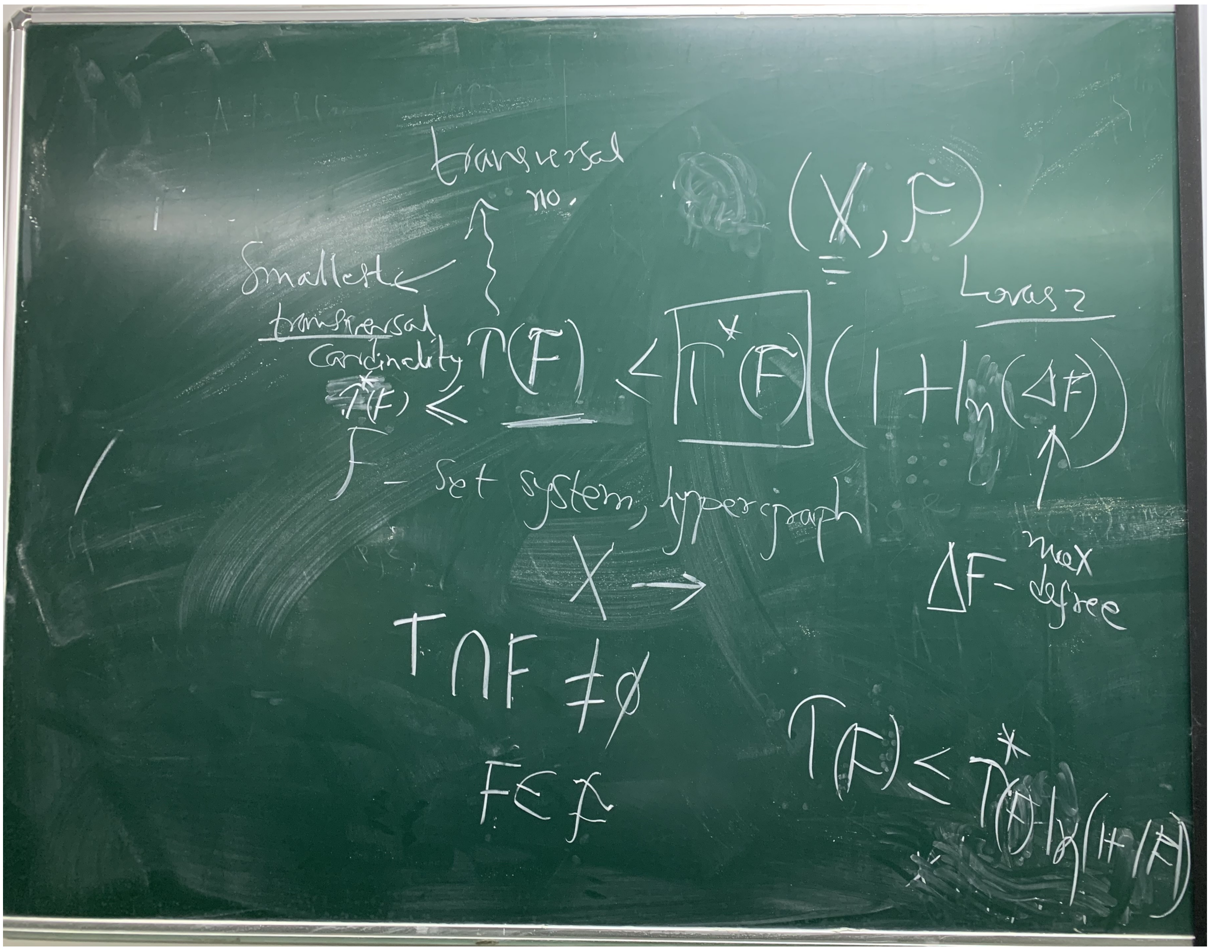

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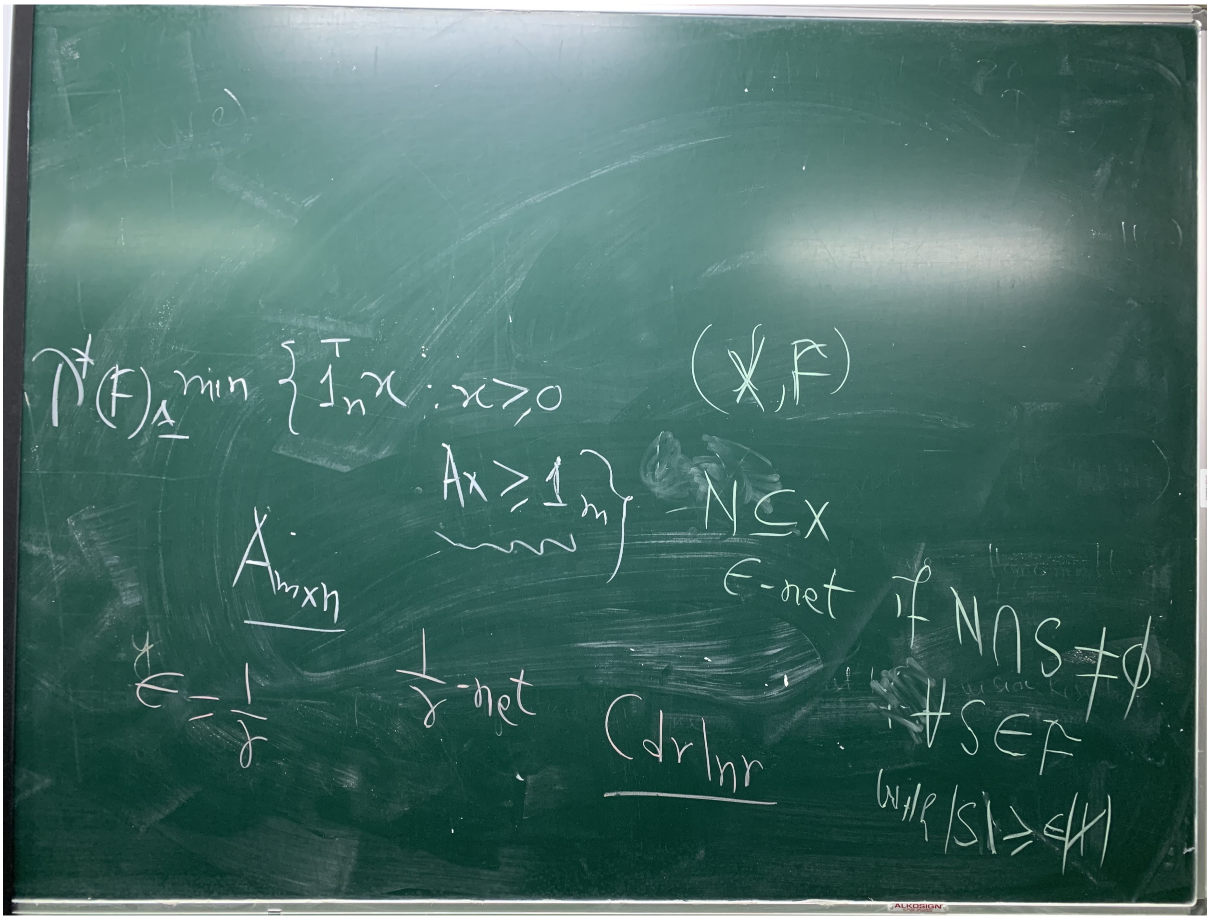

Figure 25: 05.04.2024