Quantum State Transformation

Quantum State Transition Postulate

It seems that nature does not allow arbitrary state transformation. Change of states (over time) of a closed[a] quantum mechanical system are caused by a specific class of transformations, mathematically known as unitary transformations.

[a] By a closed system we mean that no measurement is performed on the system.

Unitary Transformation

- A unitary transformation $U : H \rightarrow H$ is an isomorphism, where H an inner product space (Hilbert space).
- So U is a bijection that preserves inner product.
- In our notation $U : |x\rangle \mapsto |y\rangle$ and if $|x\rangle, |x'\rangle \in H$, then $< |x\rangle, |x'\rangle >=< |Ux\rangle, |Ux'\rangle >$.

Unitary Matrix

A unitary transformation will be represented by a unitary matrix. We call a complex matrix U as unitary if its conjugate transpose (Hermitian transpose) $U^\dagger$ (or U^*) is its inverse i.e $UU^\dagger = I$.

One Qubit Transformation

- We encode $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- So an 1-qubit unitary transformation or 1-qubit gate is a 2×2 unitary matrix U i.e. $U^\dagger U = I$.
- This means $U^\dagger = U^{-1}$ and $U^\dagger$ is also unitary.

Properties

It is clear that any quantum transformation is reversible, as $U|x\rangle = |y\rangle$ if and only if $U^\dagger|y\rangle = |x\rangle$.

Properties

- A unitary operator U is linear by definition.
- If a quantum state $|x\rangle$ is a superposition of $a_1|x_1\rangle + \cdots + a_n|x_n\rangle$, then $U|x\rangle = U(a_1|x_1\rangle + \cdots + a_n|x_n\rangle) = a_1U|x_1\rangle + \cdots + a_nU|x_n\rangle$.

Properties

- The inner-product of $U|x\rangle$ and $U|y\rangle$ is $\langle x|U^\dagger U|y\rangle = \langle x|I|y\rangle = \langle x|y\rangle$.
- A unitary operator preserves the norm or length of a state vector.
- It maps an orthonormal basis to another orthonormal basis e.g. $H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} : \{|0\rangle, |1\rangle\} \rightarrow \{|+\rangle, |-\rangle\}$.

Properties

We know that $U^\dagger U = I = UU^\dagger$ i.e.

$$\begin{bmatrix} \overline{u_{1i}} & \overline{u_{2i}} & \cdots & \overline{u_{ni}} \end{bmatrix} \cdot \begin{bmatrix} u_{1j} \\ u_{2j} \\ \cdots \\ u_{nj} \end{bmatrix} = I_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}.$$

So the columns of U are orthonormal.
$U^\dagger$ is also unitary, the columns of $U^\dagger$ are also orthonormal. So the rows of U are so.

Properties

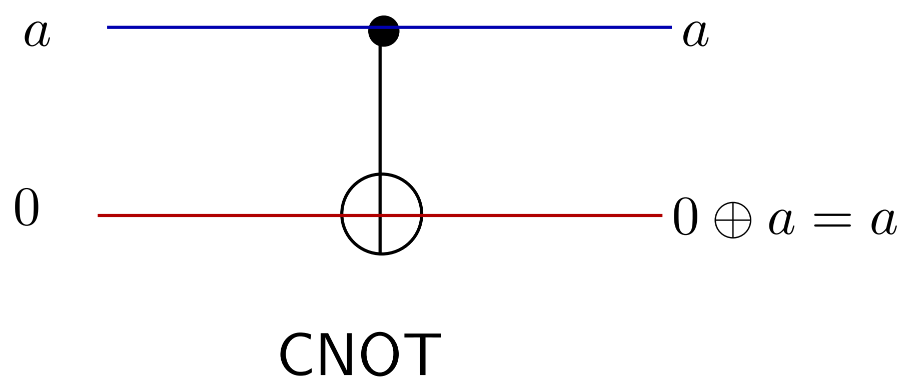
- If U_1 and U_2 are two unitary operators on the same space, their product U_1U_2 is also unitary:
$(U_1U_2)^\dagger U_1U_2 = U_2^\dagger U_1^\dagger U_1U_2 = U_2^\dagger IU_2 = I$.
- If U_1 and U_2 are unitary operators on spaces V_1 and V_2 respectively, then $U_1 \otimes U_2$ is an unitary operator on $V_1 \otimes V_2$:
$(U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) = (U_1^\dagger U_1) \otimes (U_2^\dagger U_2) = I_1 \otimes I_2$.

Properties

- If U_1 and U_2 are unitary operators on spaces V_1 and V_2 respectively, then $U_1 \oplus U_2$ is an unitary operator on $V_1 \oplus V_2$.
- $U_1 + U_2$ need not be unitary, even if U_1 and U_2 are.
- kU need not be unitary, even if U is.

Cloning of Classical Bit

We can use a CNOT gate to copy a classical bit.



No-Cloning Principle

There is no unitary operator that can create clone of an arbitrary qubit state. This is essentially an outcome of linearity.

- Assume that there is a 2-qubit unitary transformation U such that $U|a0\rangle = |aa\rangle$ for any qubit state $|a\rangle$.
- Let $|a\rangle$ and $|b\rangle$ be orthogonal states.

No-Cloning Principle

- We have $U|a0\rangle = |aa\rangle$ and $U|b0\rangle = |bb\rangle$.
- Consider the state $|c0\rangle$, where $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$. So $|c0\rangle = \frac{1}{\sqrt{2}}(|a0\rangle + |b0\rangle)$.

$$\begin{aligned} U|c0\rangle &= \frac{1}{\sqrt{2}}(U|a0\rangle + U|b0\rangle), \text{ by linearity} \\ &= \frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle) \text{ by cloning.} \end{aligned}$$

No-Cloning Principle

On the other hand,

$$\begin{aligned} U\left|c0\right\rangle &= \left|cc\right\rangle, \text{ by cloning} \\ &= \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle) \otimes \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle), \\ &= \frac{1}{2}(|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle). \end{aligned}$$

It is a contradiction as.
$\frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle) \neq \frac{1}{2}(|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle)$.

Outer Product

- Given a ket vector $|x\rangle$, its dual is a bra vector $\langle x|$.
- The inner product of $|x\rangle$ and $|y\rangle$ is $\langle x|y\rangle \in \mathbb{C}$.
- The outer product of $|x\rangle$ and $|y\rangle$ is $|x\rangle\langle y|$.

Outer Product

- If we represent $|x\rangle$ as a column vector $(x_1, \cdots, x_n)$, then $\langle x|$ is the row vector $[\overline{x_1} \ \cdots \ \overline{x_n}]$, where $\overline{x_i}$ is the conjugate of x_i.
- The outer product $|x\rangle\langle y|$, is the tensor product $|x\rangle \otimes \langle y|$.

Outer Products: 1-bit Base Vectors

$$
\begin{aligned}
|0\rangle\langle 0| &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
|0\rangle\langle 1| &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
|1\rangle\langle 0| &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
|1\rangle\langle 0| &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

1-Bit Transformation

Any 1-bit transformation is a linear combination of the outer products of vectors.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a(|0\rangle\langle 0|) + b(|0\rangle\langle 1|) + c(|1\rangle\langle 0|) + d(|1\rangle\langle 1|),$$

Property of Outer Product

Let $|x\rangle$, $|y\rangle$ be two states of the state-space of a quantum system, then

$$\begin{aligned} (|x\rangle\langle x|)|y\rangle &= |x\rangle\langle x|y\rangle, \\ &= \langle x|y\rangle|x\rangle. \end{aligned}$$

The outer product $|x\rangle\langle x|$ projects a vector $|y\rangle$ to the subspace spanned by $|x\rangle$.

Example

$$
\begin{aligned}
& (|0\rangle \langle 0|) \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) \\
= & \frac{1}{\sqrt{2}} ((|0\rangle \langle 0|) |0\rangle + (|0\rangle \langle 0|) |1\rangle) \\
= & \frac{1}{\sqrt{2}} (|0\rangle \langle 0|0\rangle + |0\rangle \langle 0|1\rangle) \\
= & \frac{1}{\sqrt{2}} |0\rangle .
\end{aligned}
$$

Note: $|0\rangle \langle 0|$ is not a unitary transformation.

1-qubit Unitary Transformations

The first 1-qubit unitary transformation is the identity map, the 2×2 identity matrix that keeps a qubit state unchanged.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1-qubit Unitary Transformations

- Next 1-qubit unitary transformation is the not gate. It should interchange the amplitudes of base vectors i.e. $\neg(a|0\rangle + b|1\rangle) = b|0\rangle + a|1\rangle$.
- The Boolean transformation matrix for NOT will do the job.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}, \text{ where } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

1-qubit Unitary Transformations

- 1-qubit not-gate is called X or σ_x or σ_1. It is one of the three Pauli matrices.
- In Boolean logic, identity and not are the only two possible 1-bit reversible gates.
- But the situation is different in case of quantum gates.

Unitary Transformations are Uncountable

- $U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is a unitary operator.
- The adjoint of U is $U^\dagger = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, satisfying
- $UU^\dagger = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- So for every value of θ we have a unitary gate.

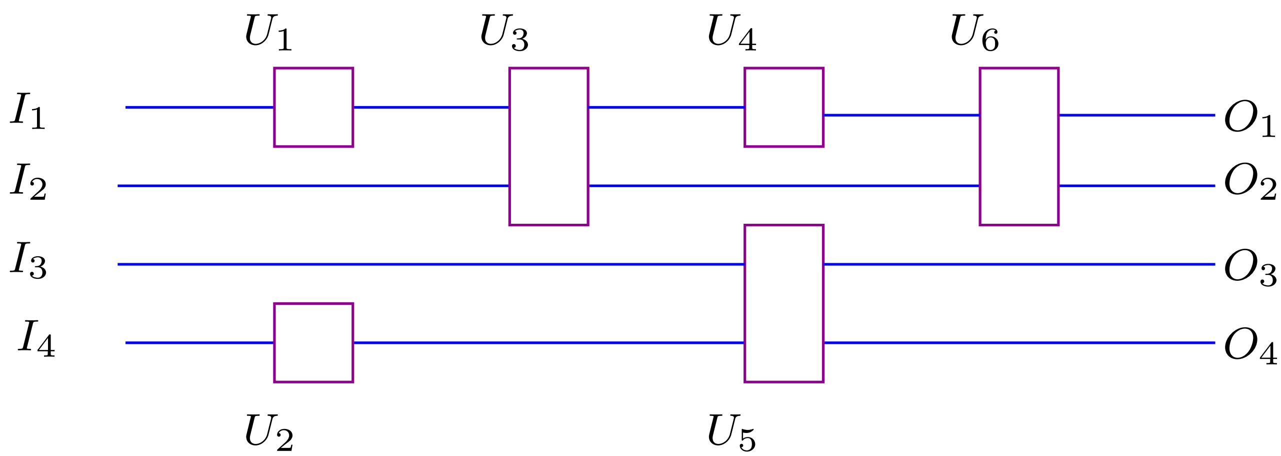
No Finite Set of Generators

- There are uncountably many unitary transformations. So it is impossible to get a finite set of generators or universal transformation gates.
- However there are finite set of transformations that can approximate any arbitrary transformation to any desired accuracy.

Tensor Product of Transformations

- Multi-qubit transformations can be expressed as linear combination of tensor products of 1-qubit or 2-qubit transformations.
- Let in an n-qubit system the transformations $U_1, \cdots, U_k$ are applied to qubits $(1, i_1), (i_1 + 1, i_2), \cdots, (i_{k-1}, n)$ respectively. The combined transformation on n-qubits is $U = U_1 \otimes \cdots \otimes U_k$.

Qubit Gate Array



Tensor Product of Transformations

- The leftmost transformation is $U_1 \otimes I \otimes I \otimes U_2$.
- Second one is $U_3 \otimes I \otimes I$.
- Third one is $U_4 \otimes I \otimes U_5$.
- Last transformation is $U_6 \otimes I \otimes I$.

General Form of an 1-Qubit Operator

Let U be an 1-qubit operator. We know

$$\begin{aligned} 1 &= \det(I) \\ &= \det(UU^\dagger) \\ &= |\det(U)||\det(U^\dagger)| \\ &= |\det(U)||\det(\overline{U})| \\ &= |\det(U)|^2 = |\det(U)|. \end{aligned}$$

General Form of 1-Qubit Operator

- Let $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We know that $|det(U)| = 1$ and $U^{-1} = U^\dagger$.
- Ignoring the global phase, we have $\begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $\overline{a}$ is the complex conjugate of a.
- So, $U = e^{i\phi}\begin{bmatrix} a & -\overline{c} \\ c & \overline{a} \end{bmatrix}$, where $|a|^2 + |c|^2 = 1$.

General Form of 1-Qubit Operator

- We can rewrite
$$U = e^{i\phi}\begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix} = e^{i\phi}\begin{bmatrix} e^{i\delta}\cos\theta & -e^{-i\gamma}\sin\theta \\ e^{i\gamma}\sin\theta & e^{-i\delta}\cos\theta \end{bmatrix}.$$
- If we substitute $\delta = -(\alpha+\beta)$ and $\gamma = (\alpha-\beta)$, then

$$\begin{aligned} U &= e^{i\phi}\begin{bmatrix} e^{-i(\alpha+\beta)}\cos\theta & -e^{-i(\alpha-\beta)}\sin\theta \\ e^{i(\alpha-\beta)}\sin\theta & e^{i(\alpha+\beta)}\cos\theta \end{bmatrix} \\ &= e^{i\phi}\begin{bmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} e^{-i\beta} & 0 \\ 0 & e^{i\beta} \end{bmatrix} \end{aligned}$$

A Few important 1-Qubit Gates

We have already talked about the Pauli matrix X. Two other Pauli matrices are Y and Z.

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -ib \\ ia \end{bmatrix} \Rightarrow$$

$a\,|0\rangle + b\,|1\rangle \mapsto -ib\,|0\rangle + ia\,|1\rangle$, where

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\,|1\rangle\,\langle 0| - i\,|0\rangle\,\langle 1|.$$

This is also known as σ_y or σ_2.

A Few important 1-Qubit Gates

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix} \Rightarrow$$

$a|0\rangle + b|1\rangle \mapsto a|0\rangle - b|1\rangle$, where

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

Also known as σ_z or σ_3.

A Few important 1-Qubit Gates

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} a+b \\ a-b \end{bmatrix}$$

$H : a|0\rangle + b|1\rangle \mapsto \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$.

$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} =$ $\frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$ is called the Hadamard gate.

A Few important 1-Qubit Gates

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ is known as a phase gate and $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{i\pi/8} \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}$ is known as $\pi/8$ gate.

Pauli Matrices

The general form of three Pauli matrices is

$$\sigma_a = \begin{bmatrix} \delta_{az} & \delta_{ax} - i\delta_{ay} \\ \delta_{ax} + i\delta_{ay} & -\delta_a z \end{bmatrix},$$

where $a \in \{x, y, z\}$ and δ_{ab} is the Dirac's delta-function.

Properties of Pauli Matrices

Four Pauli matrices are I, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Some of its properties are the following:

1. $X^2 = Y^2 = Z^2 = -iXYZ = I$.
2. $Z = IZ = (-iXYZ)Z = -iXY(ZZ) = -iXYI = -iXY$.

Properties of Pauli Matrices

3. $X = XI = X(-iXYZ) = -i(XX)YZ = -iIYZ = -iYZ.$

4. $Y = YI = Y(ZZ) = (YZ)Z = i(-iYZ)Z = iXZ.$

5. $ZYX = ZY(-iYZ) = -iZ(YY)Z = -iZIZ = -iZZ = -iI = -XYZ.$

6. $YZ = iX = iIX = i(iZYX)X = -ZY$, etc.

Visualization on Bloch Sphere

We have already shown that a single qubit state, ignoring the global phase, corresponds to a point on Bloch Sphere by the following mapping:. $|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$, where $0 \leq \theta \leq \pi$ and $-\pi \leq \phi \leq \pi$.

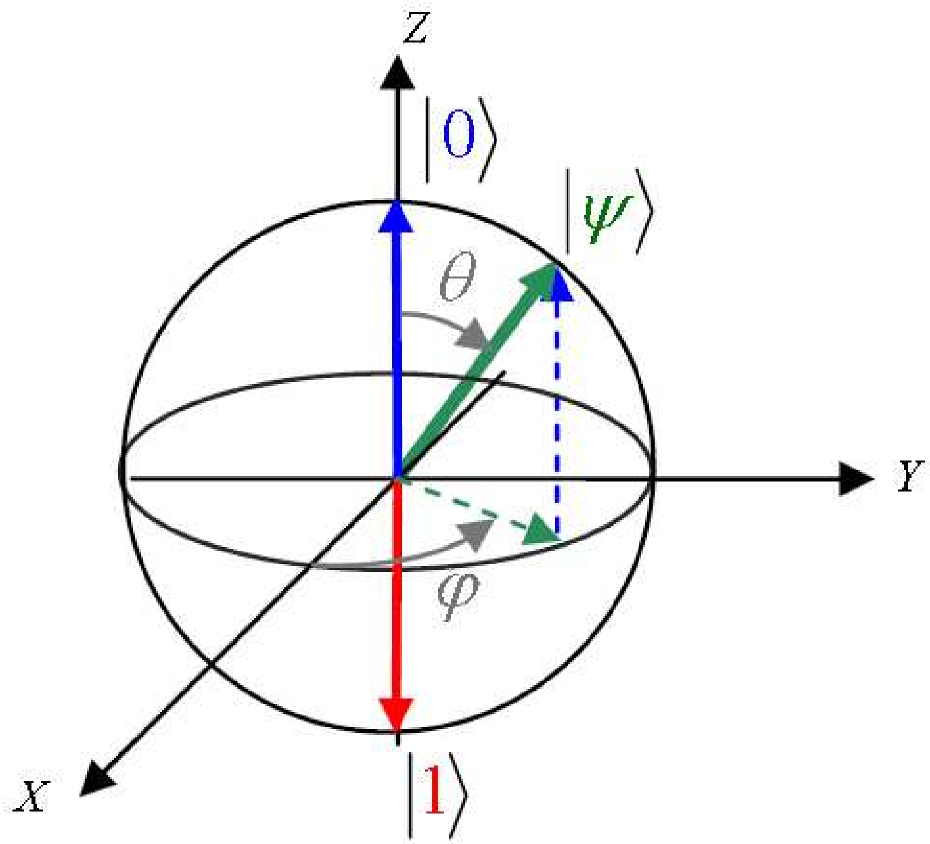


Image Source:
`https://commons.wikimedia.org/wiki/File:Sphere_bloch.jpg`

Note

The position of $|0\rangle$ is the north pole of the sphere (where the z_+-axis meets the sphere). So $\theta = 0$. And $|1\rangle$ is at the south pole.
Other axial points on the sphere are the following :

Other Axial Points

- x_+: $\theta = \pi/2, \phi = 0$, state: $\frac{|0\rangle+|1\rangle}{\sqrt{2}} = |+\rangle$
- x_-: $\theta = \pi/2, \phi = \pi$, state: $\frac{|0\rangle-|1\rangle}{\sqrt{2}} = |-\rangle$
- y_+: $\theta = \pi/2, \phi = \pi/2$, state: $\frac{|0\rangle+i|1\rangle}{\sqrt{2}} = |i\rangle$
- y_-: $\theta = \pi/2, \phi = -\pi/2$, state: $\frac{|0\rangle-i|1\rangle}{\sqrt{2}} = |-i\rangle$

Bloch Vector

The point corresponding to the qubit state $\cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle$ has the Cartesian coordinates $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on the Bloch sphere. This is the Bloch vector corresponding to the given qubit state.

We wish to see the effect of transformation on Bloch vectors.

Transformation as Rotation

An unitary transformation may be viewed as a rotation of quantum state vector in the state space. So it is important to look at the eigenvalues and eigenvectors of some of the important transformations.

Eigenvalues and Eigenvectors

Given a square matrix A, if there is a vector $|x\rangle$ satisfying $A|x\rangle = \lambda|x\rangle$, then $|x\rangle$ is called an eigenvector of A and λ is the corresponding eigenvalue.

We shall use the known fact that $\det(A - \lambda I) = 0$ to compute eigenvectors and eigenvalues of Pauli matrices.

Eigenvalues of Y

We have

$$\begin{aligned} \det\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \lambda I\right) &= 0 \\ \Rightarrow \det(\begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix}) &= 0 \\ \Rightarrow \lambda^2 - 1 &= 0 \\ \Rightarrow \lambda &= \pm 1. \end{aligned}$$

In fact all Pauli matrices have $\lambda = \pm 1$.

Eigenvectors of Y

So we have

$$\begin{aligned}
\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \pm \begin{bmatrix} a \\ b \end{bmatrix} \\
\Rightarrow \begin{bmatrix} -ib \\ ia \end{bmatrix} &= \pm \begin{bmatrix} a \\ b \end{bmatrix}
\end{aligned}$$

$$\Rightarrow \ -ib = \pm a \ \text{ and } \ ia = \pm b$$

$$\begin{aligned}
\Rightarrow \ a^2 &= -b^2 \\
\Rightarrow \ a &= \pm ib.
\end{aligned}$$

Eigenvectors of Y

We also have $|a|^2 + |b|^2 = 1$. So the eigenvectors of Y are

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \pm i \frac{1}{\sqrt{2}} \end{bmatrix} = |i\rangle, |-i\rangle.$$

Corresponding points on the Bloch sphere are $(0, \pm 1, 0)$ (Bloch vector), where $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Points where the y-axis meets the sphere.

Note

- The Pauli matrix Y or σ_y rotates a qubit state in its state-space.
- The vectors that do not change "directions" are $|i\rangle$ and $|-i\rangle$.
- These are not to be confused with the 3-dimensional Block vectors - $(0, \pm 1, 0)$.
- We shall see the connection of Y with the Bloch vectors.