Bottom-UP Parsing

The Process

- The parse tree is built starting from the leaf nodes labeled by the terminals (tokens).
- It tries to build the leftmost internal node (labeled by non-terminal) whose children (their subtrees) have been constructed.
- In other words it tries to discover the rightmost derivations in reverse order and use the corresponding reductions.

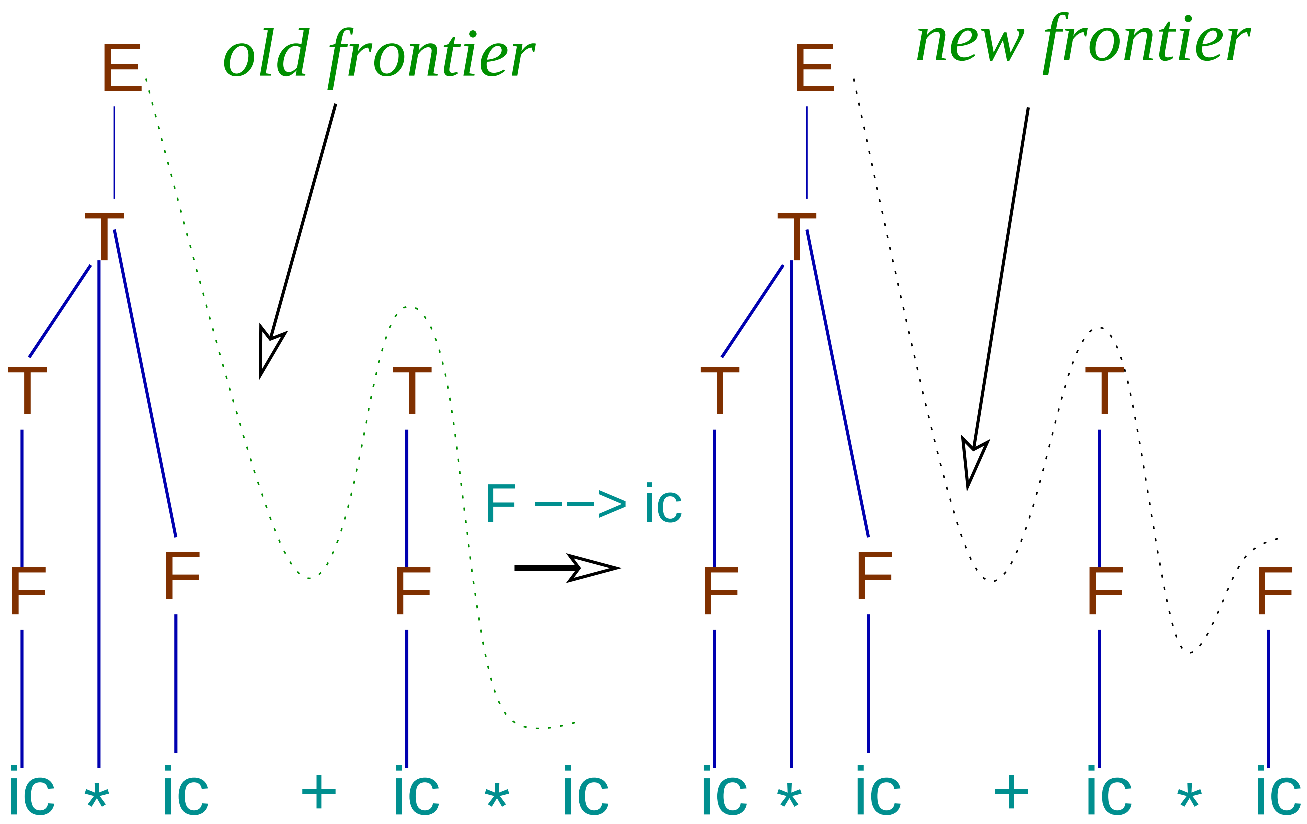
The Process

- The process ends at the root of the tree labeled by the start symbol, or with an error condition.
- At any intermediate point there is a sequence of roots of sub-trees. This sequence may be called the frontier of the parse tree.

The Process

- At every step the parser tries to find an appropriate β in the frontier, which can be reduced by a rule $A \to \beta$, forming a bigger subtree of the parse tree.
- If no such β is available, the parser either calls the scanner to get a new token, creates a leaf node and extend the frontier, or reports an error.

Growing Frontier



The Process

- As a parser reads input from left-to-right, the first reduction is the last step of derivation at the left-most end.
- Input further away from the left-end were produced by earlier steps of derivation.
- The reduction takes place following the sequence of rightmost derivations in reverse order.

Rightmost Derivation of $id_1 + id_2 * id_3$

$$
\begin{aligned}
E &\rightarrow E + \underline{T} \\
&\rightarrow E + T * \underline{F} \\
&\rightarrow E + \underline{T} * id_3 \\
&\rightarrow E + \underline{F} * id_3 \\
&\rightarrow \underline{E} + id_2 * id_3 \\
&\rightarrow \underline{T} + id_2 * id_3 \\
&\rightarrow \underline{F} + id_2 * id_3 \\
&\rightarrow id_1 + id_2 * id_3
\end{aligned}
$$

Reduction of $id_1 + id_2 * id_3$

$$
\begin{aligned}
& \underline{id_1} + id_2 * id_3 \\
\rightarrow\ & \underline{F} + id_2 * id_3 \\
\rightarrow\ & \underline{T} + id_2 * id_3 \\
\rightarrow\ & E + \underline{id_2} * id_3 \\
\rightarrow\ & E + \underline{F} * id_3 \\
\rightarrow\ & E + T * \underline{id_3} \\
\rightarrow\ & E + \underline{T * F} \\
\rightarrow\ & \underline{E + T} \\
\rightarrow\ & E
\end{aligned}
$$

Frontiers of $id_1 + id_2 * id_3$

	$\underline{id_1}$	one new token
$\rightarrow$	$\underline{F}$	
$\rightarrow$	$\underline{T}$	
$\rightarrow$	$E + \underline{id_2}$	two new tokens
$\rightarrow$	$E + \underline{F}$	
$\rightarrow$	$E + T * \underline{id_3}$	two new tokens
$\rightarrow$	$E + \underline{T * F}$	
$\rightarrow$	$\underline{E + T}$	
$\rightarrow$	E	

Handle

- Let $\alpha\beta x$ and $\alpha A x$ be the $(i+1)^{th}$ and i^{th} right sentential forms, and $A \rightarrow \beta$ be a production rule ($x \in \Sigma^*$).
- If k is the position of β in $\alpha\beta x$, the doublet $(A \rightarrow \beta, k)$ is called a handle of the frontier $\alpha\beta$ or the right sentential form $\alpha\beta x$.

Example

- In the first example $(ic + ic * ic \cdots)$, after the reduction of $E + F$ to $E + T$, the parser does not find any other handle in the frontier and invokes the scanner. It supplies the token for '$*$'.
- The parser forms the corresponding leaf node and includes it in the frontier $(E + T*)$.

Example

- Still there is no handle and the scanner is invoked again to get the next token '`ic`'.
- The parser detects the handle $(F \to \texttt{ic}, E + T * \underline{\texttt{ic}})$ and reduces it to F.

Handle

- In an unambiguous grammar the rightmost derivation is unique, so a handle of a right sentential form is unique.
- But that is not be true for an ambiguous grammar.

Example

Let the input be $ic_1 + ic_2 * ic_3$. The ambiguous expression grammar is $E \rightarrow E + E \mid E * E \mid ic$.

Handle	I	II	Reduction
1^{st}	$\underline{ic_1}$	$\underline{ic_1}$	$E \rightarrow ic$
2^{nd}	$E + \underline{ic_2}$	$E + \underline{ic_2}$	$E \rightarrow ic$
3^{rd}	$E + E * \underline{ic_3}$	$\underline{E + E}$	$E \rightarrow ic$, $E \rightarrow E + E$

Example

Let the input be $ic_1 + ic_2 * ic_3$. The ambiguous expression grammar is $E \rightarrow E + E \mid E * E \mid ic$.

Handle	I	II	Reduction
4^{th}	$E + \underline{E * E}$	$E * \underline{ic_3}$	$E \rightarrow E * E$, $E \rightarrow ic$
5^{th}	$\underline{E + E}$	$\underline{E * E}$	$E \rightarrow E + E$, $E \rightarrow E * E$
accept	E	E	

Shift-Reduce Parsing

A bottom-up parser essentially takes two types of actions,

- if it detects a handle in the frontier, that is reduced to get a new frontier, or
- if the handle is not present, it calls the scanner, gets a new token and extends (shifts) the frontier.

Note

The parser may fail to detect a handle and may report an error. But if discovered, the handle is always present at the right end of the frontier.

Shift-Reduce Parsing

- A shift-reduce parser uses a stack to hold the frontier (left end at the bottom of the stack).
- A frontier is a prefix of a right-sentential form at most up to the right-most handle[a].
- A prefix of the frontier is also called a viable prefix of the right sentential form.

[a] In the previous example of the ambiguous grammar, the right sentential form $E + E * ic$ has two handles $E + E$ or ic.

Accept

If the parser can successfully reduce the whole input to the start symbol of the grammar. It reports acceptance of the input i.e. the input string is syntactically (grammatically) correct.

Example

Consider our old grammar:

1	`P`	$\rightarrow$	`main DL SL end`
2	`DL`	$\rightarrow$	`D DL | D`
4	`D`	$\rightarrow$	`T VL ;`
5	`VL`	$\rightarrow$	`id VL | id`
7	`T`	$\rightarrow$	`int | float`
9	`SL`	$\rightarrow$	`S SL |` ε

Production Rules

11 `S → ES | IS | WS | IOS`

15 `ES → id := E ;`

16 `IS → if be then SL end |`

`if be then SL else SL end`

18 `WS → while be do SL end`

19 `IOS → scan id ; | print e ;`

[a]

---

[a]We are considering `BE` and `E` as terminals.

Input

Let the input be

```
main
    int id ;
    id := E ;
    print E ;
end$
```

The end of input is marked by eof ($) and the bottom-of-stack is also marked by $.

Parsing

Value Stack	Next Input	Handle	Action
$	main	nil	shift
$ main	int	nil	shift
$ main int	id	$(T \rightarrow$ int$)$	reduce
$ main T	id	nil	shift
$ main T id	;	(VL $\rightarrow$ id)	reduce
$ main T VL	;	nil	shift
$ main T VL ;	id	$(D \rightarrow$ T VL ;$)$	reduce
$ main D	id	(DL $\rightarrow$ D)	reduce

Note

- The position of the handle is always on the top-of-stack. But the problem is the detection of it.
- When to ask for a new token from the scanner and push it in the stack; and when to reduce the handle from the top-of-stack using a grammar rule.

Automaton of Viable Prefixes

- It is known that the viable prefixes of any CFG is a regular language.
- For some class of context-free grammar it is possible to design a DFA that can be used (along with some heuristic information) to take the shift-reduce decision of a parser on the basis of the DFA state and a fixed number of token look-ahead.

$LR(k)$ Parsing

$LR(k)$ is an important class of CFG where a bottom-up parsing technique can be used efficiently[a].
The 'L' is for left-to-right scanning of input, and 'R' is for discovering the rightmost derivation in reverse order (reduction) by looking ahead at most k input tokens.

[a] Operator precedence parsing is another bottom-up technique that we shall not discuss. The time complexity of $LR(k)$ is $O(n)$ where n is the length of the input.

Note

We shall consider the cases where $k = 0$ and $k = 1$. We shall also consider two other special cases, simple $LR(1)$ or SLR and look-ahead LR or LALR. An $LR(0)$ parser does not look-ahead to decide its shift or reduce actions[a].

[a] It may look-ahead for early detection of error.

Note

- An LR parser decides about shift or reduce actions depending on the state of the automaton accepting the viable prefixes and examining a fixed number of current input tokens (look-ahead).
- The states of the deterministic automaton are subsets of items defined as follows.

$LR(0)$ Items

- Given a context-free grammar G, an $LR(0)$ item corresponding to a production rule $A \rightarrow \alpha$ is $A \rightarrow \beta \bullet \gamma$ where $\alpha = \beta\gamma$.
- $LR(0)$ items corresponding to the rule $E \rightarrow E + T$ are $E \rightarrow \bullet E + T, \cdots,$ $E \rightarrow E + T\bullet$.
- The $LR(0)$ item of $A \rightarrow \varepsilon$ is $A \rightarrow \bullet$.

Viable Prefix and Valid Item

An $LR(0)$ item $A \to \alpha_1 \bullet \alpha_2$ is said to be valid for a viable prefix $\alpha\alpha_1$ if there is a right-most sentential form $\alpha\alpha_1\alpha_2 x$, where $x \in \Sigma^*$. It essentially means that during parsing the viable prefix $\alpha\alpha_1$ may be extended to a handle $\alpha_1\alpha_2$,

$$S \Rightarrow_{rm}^{*} \alpha A x \Rightarrow_{rm} \alpha\alpha_1\alpha_2 x.$$

Note

- Given a viable prefix there may be more than one valid items.
- As an example, in the expression grammar, the valid items corresponding to the viable prefix $E+T$ are $E \rightarrow E+T\bullet$ and $T \rightarrow T \bullet *F$.

Note

- Using the first one the prefix can be extended to right sentential form as $E + T\varepsilon = E + T,\ E + T + \ ic, \cdots.$
- Using the second one the prefix can be extended as $E + T * \ ic, \cdots.$

Main Theorem

The main theorem of LR parsing claims that, the set of valid items of a viable prefix α forms the state of a deterministic finite automaton that can be reached from the start state by a path labeled by α.

Note

- An item $A \rightarrow \alpha \bullet \beta$ in the state of the automaton indicates that the parser has already seen the string of terminals x derived from α ($\alpha \rightarrow x$) and it expects to see a string of terminals derivable from β.
- If $\beta = B\mu$ i.e. $A \rightarrow \alpha \bullet B\mu$, where B is a non-terminal; then the parser also expects to see any string generated by 'B'.

Note

- So all the items of the form $B \rightarrow \bullet\gamma$ are included in the state of $A \rightarrow \alpha \bullet B\beta$.
- In terms of finite automaton, it is equivalent to ε-transition from the state of $A \rightarrow \alpha \bullet B\mu$. So $B \rightarrow \bullet\gamma$ is included in the DFA state of $A \rightarrow \alpha \bullet B\mu$ (ε-closure).

Canonical $LR(0)$ Collection

The set of states of the the DFA of the viable prefix automaton is a collection of the set of $LR(0)$ items and is known as the canonical $LR(0)$ collection[a].

[a] It is a set of sets.

Example

Consider the following grammar:

$$
\begin{array}{rrcl}
1: & P & \rightarrow & m\,L\,s\,e \\
2: & L & \rightarrow & D\,L \\
3: & L & \rightarrow & D \\
4: & D & \rightarrow & T\,V\,; \\
5: & V & \rightarrow & d\,V \\
6: & V & \rightarrow & d \\
7: & T & \rightarrow & i \\
8: & T & \rightarrow & f
\end{array}
$$

Closure()

If i is an $LR(0)$ item, then Closure(i) is defined as follows:

- $i \in$ Closure(i) - basis,
- If $A \rightarrow \alpha \bullet B\beta \in$ Closure(i) and $B \rightarrow \gamma$ is a production rule, then $B \rightarrow \bullet\gamma \in$ Closure(i).

The closure of I, a set of $LR(0)$ items, is defined as Closure(I) $= \bigcup_{i \in I}$ Closure(i).

Example

Let $i = P \rightarrow m \bullet L\ s\ e$,

$$
\begin{aligned}
\text{Closure}(i) \quad = \quad & \{ \\
& P \rightarrow m \bullet L\ s\ e \\
& L \rightarrow \bullet D\ L \\
& L \rightarrow \bullet D \\
& D \rightarrow \bullet T\ V\ ; \\
& T \rightarrow \bullet i \\
& T \rightarrow \bullet f \\
& \}
\end{aligned}
$$

Goto(I, X)

Let I be a set of $LR(0)$ items and $X \in \Sigma \cup N$. The set of $LR(0)$ items, Goto(I, X) is

$$\text{Closure}\left(\{A \to \alpha\, X \bullet \beta :\; A \to \alpha \bullet X\, \beta \in I\}\right).$$

Goto() is the state transition function δ of the DFA.

Example

From our previous example
Goto(Closure($P \rightarrow m \bullet L\ s\ e$), D) is

$$\{L \rightarrow D \bullet L$$

$$L \rightarrow D\bullet$$

$$L \rightarrow \bullet DL$$

$$L \rightarrow \bullet D$$

$$D \rightarrow \bullet TV;$$

$$T \rightarrow \bullet i$$

$$T \rightarrow \bullet f\}$$

Augmented Grammar

We augment the original grammar with a new start symbol, say S', that has only one production rule $S' \rightarrow S\$$, where S is the start symbol of the original grammar. When we come to a state corresponding to $(S' \rightarrow S\$, S)$ or with the $LR(0)$ item $S' \rightarrow S \bullet \$$, we know that the input string is well-formed and the parser accepts it.

$LR(0)$ Automaton

- The alphabet of the automaton is $\Sigma \cup N$.
- The start state is $s_0 = \text{Closure}(S' \rightarrow \bullet S\$)$, the automaton expects to see the string generated by S followed by $\$$.
- All constructed states are final states[a] of the automaton as it accepts a prefix language.

[a]The constructed automaton is incompletely specified and all unspecified transitions lead to the only non-final state.

$LR(0)$ Automaton

- For every $X \in \Sigma \cup N$ and for all states s already constructed, we compute Goto(s, X)[a] to build the automaton.

[a] This nothing but $\delta(s, X)$.

Example: States

s_0 :	$S' \rightarrow \bullet P\$$	$P \rightarrow \bullet m\ L\ s\ e$	
s_1 :	$S' \rightarrow P \bullet \$$		
s_2 :	$P \rightarrow m \bullet L\ s\ e$	$L \rightarrow \bullet D\ L$	$L \rightarrow \bullet D$
	$D \rightarrow \bullet T\ V\ ;$	$T \rightarrow \bullet i$	$T \rightarrow \bullet f$
s_3 :	$P \rightarrow m\ L \bullet s\ e$		
s_4 :	$L \rightarrow D \bullet L$	$L \rightarrow D \bullet$	$L \rightarrow \bullet D\ L$
	$L \rightarrow \bullet D$	$D \rightarrow \bullet T\ V\ ;$	$T \rightarrow \bullet i$
	$T \rightarrow \bullet f$		

States

s_5 :	$D \rightarrow T \bullet V\ ;$ $\quad V \rightarrow \bullet d\ V$ $\quad V \rightarrow \bullet d$
s_6 :	$T \rightarrow i \bullet$
s_7 :	$T \rightarrow f \bullet$
s_8 :	$P \rightarrow m\ L\ s \bullet e$
s_9 :	$L \rightarrow D\ L \bullet$
s_{10} :	$D \rightarrow T\ V \bullet ;$
s_{11} :	$V \rightarrow d \bullet V$ $\quad V \rightarrow d \bullet$ $\quad V \rightarrow \bullet d\ V$ $\quad V \rightarrow \bullet d$

States

s_{12} :	$P \rightarrow m\ L\ s\ e\bullet$
s_{13} :	$D \rightarrow T\ V\ ;\bullet$
s_{14} :	$V \rightarrow d\ V\bullet$

State Transitions

CS	NS (Input)											
	m	s	e	$;$	d	i	f	P	L	D	V	T
0	2							1				
2						6	7		3	4		5
3		8										
4						6	7		9	4		5
5					11						10	
8			12									
10				13								
11					11						14	

Items

- Kernel item:
  $\{S' \rightarrow \bullet S\$\} \cup \{A \rightarrow \alpha \bullet \beta : \alpha \neq \varepsilon\}$.
- Non-kernel item: $\{A \rightarrow \bullet\alpha\} \setminus \{S' \rightarrow \bullet S\$\}$.

Every non-kernel item in a state comes from the closure operation and can be generated from the kernel items. So it is not necessary to store them explicitly.

Complete Item

- An item of the form $A \rightarrow \alpha \bullet$ is known as a complete item.
- If a state has a complete item $A \rightarrow \alpha \bullet$, it indicates that the parser has possibly seen a handle and it may be reduced.
- But there may be other complications that we shall discuss afterward.

Structure of LR Parser

- Every LR-parser has a similar structure with a core parsing program.
- A stack to store the states of the DFA of viable prefixes and a parsing table.
- The content of the table is different for different types of LR parsers[a].

[a] Depends on the type of DFA and other information.

Structure of LR Parsing Table

- The parsing table has two parts, action and goto.
- The action(i, a) is a function of two parameters, i is the current state of the DFA[a] and 'a' is the current token.
- The table is indexed by 'i' and 'a'. The action stored in the table, are of four different types.

[a] The current state is available at the top of the stack.

Action-1

- action$(i, a) = s_j$, push the state j in the stack[a]. In the automaton $\delta(i, a) = j$.
- The parser has not yet found the handle and augments the frontier by including a new token (forms a leaf node).

[a] In fact the input token and the related attributes are also pushed in the same or a different stack (value stack) for semantic actions. But that is not required for acceptance of input.

Action-2

- $\text{action}(i, a) = r_j$, reduce the handle by the rule number $j :\ A \rightarrow \alpha$.
- If $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$, then the top k states on the stack $\$ \cdots qq_{i_1}q_{i_2} \cdots q_{i_k}$, corresponding to this α[a], are popped out and $\delta(q, A) = \text{goto}(q, A) = p$ is pushed.
- Old stack: $\$ \cdots qq_{i_1}q_{i_2} \cdots q_{i_k}$

  New stack: $\$ \cdots qp$

[a] $\text{Action}(q, \alpha_1) = q_{i_1}, \cdots, \text{Action}(q_{i_{k-1}}, \alpha_k) = q_{i_k}$.

Goto in the Table

- After a reduction (action 2) by the rule $A \rightarrow \alpha$, the top-of-stack has the state q.
- The parser driver needs to find $\delta(q, A) = \text{goto}(q, A) = p$ and push it on the stack.
- This information is stored in the goto portion of the table. This is the state-transition function restricted to the non-terminals.

Action-3 & 4

- An LR-parser accepts the input at the accept state when the eof ($) is reached.
- A parser rejects the input at a state where the table entry is undefined on the current token.

Configuration

- A configuration of an LR-parser is specified by the content of the stack and the remaining input.
- An LR-parser starts with the initial state at the top of the stack and the input. This is the initial configuration: $(\$q_0, a_1 \cdots a_j a_{j+1} \cdots a_n\$)$.

Configuration

- At any point of computation, the top-of-stack contains the current state of the DFA. A configuration is $(\$q_0 q_{i_1} \cdots q_{i_k}, a_j a_{j+1} \cdots a_n \$)$.
- In terms of the sentential form it is $\alpha_1 \alpha_2 \cdots \alpha_k a_j \cdots a_n \$$.

Final Configurations

- A final configuration is $(\$q_0 q_f, \$)$, where $\text{Goto}(q_0, S) = q_f$, and the token stream is empty.
- An error configuration.: $(\$q_0 \cdots q, a_j a_{j+1} \cdots a_n \$)$, where $\text{Action}(q, a_j)$ is not defined.