Fuzzy Relations, Rules and Inferences

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Fuzzy Relations
Crisp relations

To understand the fuzzy relations, it is better to discuss first crisp relation.

Suppose, $A$ and $B$ are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Note :

(1) $A \times B \neq B \times A$

(2) $|A \times B| = |A| \times |B|$

(3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

The mapping so mentioned is called a relation.
Crisp relations

Example 1:
Consider the two crisp sets $A$ and $B$ as given below. $A = \{1, 2, 3, 4\}$
$B = \{3, 5, 7\}$.

Then, $A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7)\}$

Let us define a relation $R$ as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2, 3), (4, 5)\}$ in this case.

We can represent the relation $R$ in a matrix form as follows.

$$R = \begin{bmatrix}
3 & 5 & 7 \\
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 \\
4 & 0 & 1 & 0 
\end{bmatrix}$$
Operations on crisp relations

Suppose, \( R(x, y) \) and \( S(x, y) \) are the two relations define over two crisp sets \( x \in A \) and \( y \in B \)

**Union:**

\[
R(x, y) \cup S(x, y) = \max(R(x, y), S(x, y));
\]

**Intersection:**

\[
R(x, y) \cap S(x, y) = \min(R(x, y), S(x, y));
\]

**Complement:**

\[
\overline{R(x, y)} = 1 - R(x, y)
\]
Example: Operations on crisp relations

Example:
Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

\[
R = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
S = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix};
\]

Find the following:

1. $R \cup S$
2. $R \cap S$
3. $\overline{R}$
Composition of two crisp relations

Given $R$ is a relation on $X$, $Y$ and $S$ is another relation on $Y$, $Z$. Then $R \circ S$ is called a composition of relation on $X$ and $Z$ which is defined as follows.

$$R \circ S = \{ (x, z) | (x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y \}$$

Max-Min Composition

Given the two relation matrices $R$ and $S$, the max-min composition is defined as $T = R \circ S$;

$$T(x, z) = \max \{ \min \{ R(x, y), S(y, z) \} \text{ and } \forall y \in Y \}$$
Example:
Given
\[X = \{1, 3, 5\}; \ Y = \{1, 3, 5\}; \ R = \{(x, y)|y = x + 2\}; \ S = \{(x, y)|x < y\}\]
Here, \(R\) and \(S\) is on \(X \times Y\).

Thus, we have
\[R = \{(1, 3), (3, 5)\}\]
\[S = \{(1, 3), (1, 5), (3, 5)\}\]

\[
R = \begin{bmatrix}
1 & 0 & 1 & 0 \\
3 & 0 & 0 & 1 \\
5 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
S = \begin{bmatrix}
1 & 0 & 1 & 1 \\
3 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 \\
5 & 0 & 0 & 0
\end{bmatrix}
\]

Using max-min composition \(R \circ S = \)
Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, ..., X_n$
- Here, n-tuples $(x_1, x_2, ..., x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Example:

$X = \{ \text{typhoid, viral, cold} \}$ and $Y = \{ \text{running nose, high temp, shivering} \}$

The fuzzy relation $R$ is defined as

\[
\begin{bmatrix}
\text{running\ nose} & \text{high\ temperature} & \text{shivering} \\
\text{typhoid} & 0.1 & 0.9 & 0.8 \\
\text{viral} & 0.2 & 0.9 & 0.7 \\
\text{cold} & 0.9 & 0.4 & 0.6
\end{bmatrix}
\]
Fuzzy Cartesian product

Suppose

A is a fuzzy set on the universe of discourse X with \( \mu_A(x) \) \( x \in X \)

B is a fuzzy set on the universe of discourse Y with \( \mu_B(y) \) \( y \in Y \)

Then \( R = A \times B \subset X \times Y \); where R has its membership function given by

\[ \mu_R(x, y) = \mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\} \]

Example :

\[ A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\} \] and \( B = \{(b_1, 0.5), (b_2, 0.6)\} \)

\[
R = A \times B =
\begin{bmatrix}
a_1 & b_1 & b_2 \\
0.2 & 0.2 \\
a_2 & 0.5 & 0.6 \\
a_3 & 0.4 & 0.4
\end{bmatrix}
\]
Operations on Fuzzy relations

Let \( R \) and \( S \) be two fuzzy relations on \( A \times B \).

**Union:**

\[
\mu_{R \cup S}(a, b) = \max\{\mu_R(a, b), \mu_S(a, b)\}
\]

**Intersection:**

\[
\mu_{R \cap S}(a, b) = \min\{\mu_R(a, b), \mu_S(a, b)\}
\]

**Complement:**

\[
\mu_R^-(a, b) = 1 - \mu_R(a, b)
\]

**Composition**

\[
T = R \circ S \\
\mu_{R \circ S} = \max_{y \in Y}\{\min(\mu_R(x, y), \mu_S(y, z))\}
\]
Operations on Fuzzy relations: Examples

Example:

\[ X = (x_1, x_2, x_3); \ Y = (y_1, y_2); \ Z = (z_1, z_2, z_3); \]

\[ R = \begin{bmatrix}
  y_1 & y_2 \\
  x_1 & 0.5 & 0.1 \\
  x_2 & 0.2 & 0.9 \\
  x_3 & 0.8 & 0.6 \\
\end{bmatrix} \]

\[ S = \begin{bmatrix}
  z_1 & z_2 & z_3 \\
  y_1 & 0.6 & 0.4 & 0.7 \\
  y_2 & 0.5 & 0.8 & 0.9 \\
\end{bmatrix} \]

\[ R \circ S = \begin{bmatrix}
  z_1 & z_2 & z_3 \\
  x_1 & 0.5 & 0.4 & 0.5 \\
  x_2 & 0.5 & 0.8 & 0.9 \\
  x_3 & 0.6 & 0.6 & 0.7 \\
\end{bmatrix} \]

\[ \mu_{R \circ S}(x_1, y_1) = \max\{ \min(x_1, y_1), \min(y_1, z_1), \min(x_1, y_2), \min(y_2, z_1) \} \]

\[ = \max\{ \min(0.5, 0.6), \min(0.1, 0.5) \} = \max\{0.5, 0.1\} = 0.5 \text{ and so on.} \]
Fuzzy relation: An example

Consider the following two sets $P$ and $D$, which represent a set of paddy plants and a set of plant diseases. More precisely:

$P = \{ P_1, P_2, P_3, P_4 \}$ a set of four varieties of paddy plants

$D = \{ D_1, D_2, D_3, D_4 \}$ of the four various diseases affecting the plants.

In addition to these, also consider another set $S = \{ S_1, S_2, S_3, S_4 \}$ be the common symptoms of the diseases.

Let, $R$ be a relation on $P \times D$, representing which plant is susceptible to which diseases, then $R$ can be stated as

\[
R = \begin{bmatrix}
P_1 & 0.6 & 0.6 & 0.9 & 0.8 \\
P_2 & 0.1 & 0.2 & 0.9 & 0.8 \\
P_3 & 0.9 & 0.3 & 0.4 & 0.8 \\
P_4 & 0.9 & 0.8 & 0.4 & 0.2 \\
\end{bmatrix}
\]
Fuzzy relation : An example

Also, consider $T$ be the another relation on $D \times S$, which is given by

$$S = \begin{bmatrix}
S_1 & S_2 & S_3 & S_4 \\
D_1 & 0.1 & 0.2 & 0.7 & 0.9 \\
D_2 & 1.0 & 1.0 & 0.4 & 0.6 \\
D_3 & 0.0 & 0.0 & 0.5 & 0.9 \\
D_4 & 0.9 & 1.0 & 0.8 & 0.2
\end{bmatrix}$$

Obtain the association of plants with the different symptoms of the disease using max-min composition.

**Hint:** Find $R \circ T$, and verify that

$$R \circ S = \begin{bmatrix}
P_1 & S_1 & S_2 & S_3 & S_4 \\
P_2 & 0.8 & 0.8 & 0.8 & 0.9 \\
P_3 & 0.8 & 0.8 & 0.8 & 0.9 \\
P_4 & 0.8 & 0.8 & 0.7 & 0.9
\end{bmatrix}$$
Fuzzy relation: Another example

Let, \( R = x \) is relevant to \( y \) and \( S = y \) is relevant to \( z \) be two fuzzy relations defined on \( X \times Y \) and \( Y \times Z \), respectively, where \( X = \{1, 2, 3\} \), \( Y = \{\alpha, \beta, \gamma, \delta\} \) and \( Z = \{a, b\} \).

Assume that \( R \) and \( S \) can be expressed with the following relation matrices:

\[
R = \begin{bmatrix}
1 & 0.1 & 0.3 & 0.5 & 0.7 \\
2 & 0.4 & 0.2 & 0.8 & 0.9 \\
3 & 0.6 & 0.8 & 0.3 & 0.2
\end{bmatrix}
\]

and

\[
S = \begin{bmatrix}
\alpha & a \\
\beta & b \\
\gamma & \alpha \\
\delta & \beta
\end{bmatrix}
= \begin{bmatrix}
0.9 & 0.1 \\
0.2 & 0.3 \\
0.5 & 0.6 \\
0.7 & 0.2
\end{bmatrix}
\]
Fuzzy relation : Another example

Now, we want to find $R \circ S$, which can be interpreted as a derived fuzzy relation $x \text{ is relevant to } z$.

Suppose, we are only interested in the degree of relevance between $2 \in X$ and $a \in Z$. Then, using max-min composition,

$$\mu_{R \circ S}(2, a) = \max\{(0.4 \land 0.9), (0.2 \land 0.2), (0.8 \land 0.5), (0.9 \land 0.7)\}$$

$$= \max\{0.4, 0.2, 0.5, 0.7\} = 0.7$$
(Binary) fuzzy relations are fuzzy sets $A \times B$ which map each element in $A \times B$ to a membership grade between 0 and 1 (both inclusive). Note that a membership function of a binary fuzzy relation can be depicted with a 3D plot.

Important: Binary fuzzy relations are fuzzy sets with two dimensional MFs and so on.
Let, $X = R^+ = y$ (the positive real line)
and $R = X \times Y = "y \text{ is much greater than } x"

The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x, y) = \begin{cases} 
\frac{(y-x)}{4} & \text{if } y > x \\
0 & \text{if } y \leq x 
\end{cases}$$

Suppose, $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then

$$R = \begin{bmatrix}
3 & 0 & 0.25 & 0.5 & 0.75 & 1.0 \\
4 & 0 & 0 & 0.25 & 0.5 & 0.75 \\
5 & 0 & 0 & 0 & 0.25 & 0.5
\end{bmatrix}$$
Problems to ponder:

How you can derive the following?

If x is A or y is B then z is C;

Given that

1. $R_1$: If x is A then z is c [$R_1 \in A \times C$]
2. $R_2$: If y is B then z is C [$R_2 \in B \times C$]

Hint:

- You have given two relations $R_1$ and $R_2$.
- Then, the required can be derived using the union operation of $R_1$ and $R_2$
Two-valued logic vs. Multi-valued logic

- The basic assumption upon which crisp logic is based - that every proposition is either TRUE or FALSE.
- The classical two-valued logic can be extended to multi-valued logic.
- As an example, three valued logic to denote true(1), false(0) and indeterminacy ($\frac{1}{2}$).
Two-valued logic vs. Multi-valued logic

Different operations with three-valued logic can be extended as shown in the following truth table:

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>∧</th>
<th>∨</th>
<th>¬a</th>
<th>→</th>
<th>=</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Fuzzy connectives used in the above table are:

AND (∧), OR (∨), NOT (¬), IMPLICATION (⇒) and EQUAL (=).
Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Connective</th>
<th>Usage</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>¬</td>
<td>NOT</td>
<td>¬P</td>
<td>(1 - T(P))</td>
</tr>
<tr>
<td>(\vee)</td>
<td>OR</td>
<td>(P \vee Q)</td>
<td>(\text{max}{T(P), T(Q)})</td>
</tr>
<tr>
<td>(\wedge)</td>
<td>AND</td>
<td>(P \wedge Q)</td>
<td>(\text{min}{T(P), T(Q)})</td>
</tr>
<tr>
<td>(\implies)</td>
<td>IMPLICATION</td>
<td>((P \implies Q)) or ((\neg P \vee Q))</td>
<td>(\text{max}{(1 - T(P)), T(Q)})</td>
</tr>
<tr>
<td>=</td>
<td>EQUALITY</td>
<td>((P = Q)) or ([(P \implies Q) \wedge (Q \implies P)])</td>
<td>(1 -</td>
</tr>
</tbody>
</table>
Example 1:

P : Ram is honest

1. $T(P) = 0.0$ : Absolutely false
2. $T(P) = 0.2$ : Partially false
3. $T(P) = 0.4$ : May be false or not false
4. $T(P) = 0.6$ : May be true or not true
5. $T(P) = 0.8$ : Partially true
6. $T(P) = 1.0$ : Absolutely true.
Example 2: Fuzzy proposition

P: Mary is efficient; T(P) = 0.8;
Q: Ram is efficient; T(Q) = 0.6

1. Mary is not efficient.
   \[ T(\neg P) = 1 - T(P) = 0.2 \]

2. Mary is efficient and so is Ram.
   \[ T(P \land Q) = \min\{ T(P), T(Q) \} = 0.6 \]

3. Either Mary or Ram is efficient
   \[ T(P \lor Q) = \max T(P), T(Q) = 0.8 \]

4. If Mary is efficient then so is Ram
   \[ T(P \implies Q) = \max\{1 - T(P), T(Q)\} = 0.6 \]
The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.

While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.

The degree of truth of each fuzzy proposition is expressed by a value in the interval [0,1] both inclusive.
Suppose, $X$ is a universe of discourse of five persons. Intelligent of $x \in X$ is a fuzzy set as defined below.

Intelligent: $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

We define a fuzzy proposition as follows:

$P : x$ is intelligent

The canonical form of fuzzy proposition of this type, $P$ is expressed by the sentence $P : v$ is $F$.

Predicate in terms of fuzzy set.

$P : v$ is $F$ ; where $v$ is an element that takes values $v$ from some universal set $V$ and $F$ is a fuzzy set on $V$ that represents a fuzzy predicate.

In other words, given, a particular element $v$, this element belongs to $F$ with membership grade $\mu_F(v)$. 
For a given value $v$ of variable $V$ in proposition $P$, $T(P)$ denotes the degree of truth of proposition $P$. 

$T(P) = \mu_F(v)$ for $v \in V$
Fuzzy Implications
A fuzzy implication (also known as fuzzy If-Then rule, fuzzy rule, or fuzzy conditional statement) assumes the form:

**If** $x$ **is** $A$ **then** $y$ **is** $B$

where, $A$ and $B$ are two linguistic variables defined by fuzzy sets $A$ and $B$ on the universe of discourses $X$ and $Y$, respectively.

Often, $x$ **is** $A$ is called the **antecedent** or premise, while $y$ **is** $B$ is called the **consequence** or conclusion.
Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High

The fuzzy implication is denoted as $R : A \rightarrow B$

In essence, it represents a binary fuzzy relation $R$ on the (Cartesian) product of $A \times B$
Fuzzy implication: Example 2

Suppose, $P$ and $T$ are two universes of discourses representing pressure and temperature, respectively as follows.

$P = \{1, 2, 3, 4\}$ and $T = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$

Let the linguistic variable **High temperature** and **Low pressure** are given as

$T_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$

$P_{LOW} = (1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)$
Fuzzy implications : Example 2

Then the fuzzy implication **If temperature is High then pressure is Low** can be defined as

\[
R : T_{\text{HIGH}} \rightarrow P_{\text{LOW}}
\]

where, \( R = \)

\[
\begin{bmatrix}
20 & 0.2 & 0.2 & 0.2 & 0.2 \\
25 & 0.4 & 0.4 & 0.4 & 0.4 \\
30 & 0.6 & 0.6 & 0.6 & 0.4 \\
35 & 0.6 & 0.6 & 0.6 & 0.4 \\
40 & 0.7 & 0.7 & 0.6 & 0.4 \\
45 & 0.8 & 0.8 & 0.6 & 0.4 \\
50 & 0.8 & 0.8 & 0.6 & 0.4
\end{bmatrix}
\]

**Note** : If temperature is 40 then what about low pressure?
Interpretation of fuzzy rules

In general, there are two ways to interpret the fuzzy rule $A \rightarrow B$ as

- $A$ coupled with $B$
- $A$ entails $B$
Interpretation as \( A \) coupled with \( B \)

\[
R : A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) \ast \mu_B(y) \big|_{(x,y)} ; \text{where } \ast \text{ is called a T-norm operator.}
\]

**T-norm operator**

The most frequently used T-norm operators are:

- **Minimum** : \( T_{\text{min}}(a, b) = \min(a, b) = a \land b \)

- **Algebric product** : \( T_{\text{ap}}(a, b) = ab \)

- **Bounded product** : \( T_{\text{bp}}(a, b) = 0 \lor (a + b - 1) \)

- **Draastic product** : \( T_{\text{dp}} = \begin{cases} 
  a & \text{if } b = 1 \\
  b & \text{if } a = 1 \\
  0 & \text{if } a, b < 1
\end{cases} \)

Here, \( a = \mu_A(x) \) and \( b = \mu_B(y) \). \( T_* \) is called the function of T-norm operator.
Based on the T-norm operator as defined above, we can automatically define the fuzzy rule \( R : A \rightarrow B \) as a fuzzy set with two-dimensional MF:

\[ \mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b) \]

with \( a=\mu_A(x) \), \( b=\mu_B(y) \), and \( f \) is the fuzzy implication function.
Interpretation as \( A \) coupled with \( B \)

In the following, few implications of \( R : A \to B \)

**Min operator:**

\[
R_m = A \times B = \int_{X \times Y} \mu_A(x) \land \mu_B(y) |_{(x,y)} \text{ or } f_{min}(a, b) = a \land b
\]

[Mamdani rule]

**Algebraic product operator**

\[
R_{ap} = A \times B = \int_{X \times Y} \mu_A(x) \cdot \mu_B(y) |_{(x,y)} \text{ or } f_{ap}(a, b) = ab
\]

[Larsen rule]
Product Operators

Bounded product operator

\[ R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y) |_{(x,y)} = \int_{X \times Y} 0 \lor (\mu_A(x) + \mu_B(y) - 1) |_{(x,y)} \]

or \( f_{bp} = 0 \lor (a + b - 1) \)

Drastic product operator

\[ R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\odot} \mu_B(y) |_{(x,y)} \]

or \( f_{dp}(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \)
Interpretation of A entails B

There are three main ways to interpret such implication:

Material implication:
\[ R : A \rightarrow B = \bar{A} \cup B \]

Propositional calculus:
\[ R : A \rightarrow B = \bar{A} \cup (A \cap B) \]

Extended propositional calculus:
\[ R : A \rightarrow B = (\bar{A} \cap \bar{B}) \cup B \]
Interpretation of A entails B

With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

Zadeh’s arithmetic rule:

\[ R_{za} = \bar{A} \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y)) \big|_{(x,y)} \]

or

\[ f_{za}(a, b) = 1 \wedge (1 - a + b) \]

Zadeh’s max-min rule:

\[ R_{mm} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \lor (\mu_A(x) \wedge \mu_B(y)) \big|_{(x,y)} \]

or

\[ f_{mm}(a, b) = (1 - a) \lor (a \wedge b) \]
Interpretation of A entails B

Boolean fuzzy rule

\[ R_{bf} = \overline{A} \cup B = \int_{X \times Y} (1 - \mu_A(x)) \lor \mu_B(x)|_{(x,y)} \]

or

\[ f_{bf}(a, b) = (1 - a) \lor b; \]

Goguen’s fuzzy rule:

\[ R_{gf} = \int_{X \times Y} \mu_A(x) * \mu_B(y)|_{(x,y)} \]

where \( a * b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{if } a > b \end{cases} \)
Example 3: Zadeh’s Max-Min rule

If \( x \) is \( A \) then \( y \) is \( B \) with the implication of Zadeh’s max-min rule can be written equivalently as:

\[
R_{mm} = (A \times B) \cup (\bar{A} \times Y)
\]

Here, \( Y \) is the universe of discourse with membership values for all \( y \in Y \) is 1, that is, \( \mu_Y(y) = 1 \forall y \in Y \).

Suppose \( X = \{a, b, c, d\} \) and \( Y = \{1, 2, 3, 4\} \)
and \( A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\} \)
\( B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\} \) are two fuzzy sets.

We are to determine \( R_{mm} = (A \times B) \cup (\bar{A} \times Y) \)
Example 3: Zadeh’s min-max rule:

The computation of $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$ is as follows:

\[
A \times B = \begin{bmatrix}
1 & 2 & 3 & 4 \\
a & 0 & 0 & 0 & 0 \\
b & 0.2 & 0.8 & 0.8 & 0 \\
c & 0.2 & 0.6 & 0.6 & 0 \\
d & 0.2 & 1.0 & 0.8 & 0
\end{bmatrix}
\]

\[
\bar{A} \times Y = \begin{bmatrix}
1 & 2 & 3 & 4 \\
a & 1 & 1 & 1 & 1 \\
b & 0.2 & 0.2 & 0.2 & 0.2 \\
c & 0.4 & 0.4 & 0.4 & 0.4 \\
d & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Example 3: Zadeh’s min-max rule:

Therefore,

\[
R_{mm} = (A \times B) \cup (\tilde{A} \times Y) =
\]

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 \\
0.2 & 0.8 & 0.8 & 0.2 \\
0.4 & 0.6 & 0.6 & 0.4 \\
0.2 & 1.0 & 0.8 & 0
\end{bmatrix}
\]
Example 3:

\[ X = \{ a, b, c, d \} \]
\[ Y = \{ 1, 2, 3, 4 \} \]

Let, \[ A = \{ (a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0) \} \]
\[ B = \{ (1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0) \} \]

Determine the implication relation:

**If** \( x \) **is A then** \( y \) **is B**

Here, \( A \times B = \)

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0.2 & 0.8 & 0.8 & 0 \\
c & 0.2 & 0.6 & 0.6 & 0 \\
d & 0.2 & 1.0 & 0.8 & 0 \\
\end{array}
\]
Example 3:

and $\bar{A} \times Y =$

$$
\begin{bmatrix}
 1 & 2 & 3 & 4 \\
a & 1 & 1 & 1 \\
b & 0.2 & 0.2 & 0.2 \\
c & 0.4 & 0.4 & 0.4 \\
d & 0 & 0 & 0
\end{bmatrix}
$$

$R_{mm} = (A \times B) \cup (\bar{A} \times Y) =$

$$
\begin{bmatrix}
 1 & 2 & 3 & 4 \\
a & 1 & 1 & 1 \\
b & 0.2 & 0.8 & 0.8 & 0.2 \\
c & 0.4 & 0.6 & 0.6 & 0.4 \\
d & 0.2 & 1.0 & 0.8 & 0
\end{bmatrix}
$$

This $R$ represents If $x$ is $A$ then $y$ is $B$. 

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Example 3:

IF $x$ is A THEN $y$ is B ELSE $y$ is C.

The relation $R$ is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of $R$ is given by

$$\mu_R(x, y) = \max[\min\{\mu_A(x), \mu_B(y)\}, \min\{\mu_{\bar{A}}(x), \mu_C(y)\}]$$
Example 4:

\[ X = \{ a, b, c, d \} \]
\[ Y = \{ 1, 2, 3, 4 \} \]
\[ A = \{ (a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0) \} \]
\[ B = \{ (1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0) \} \]
\[ C = \{ (1, 0), (2, 0.4), (3, 1.0), (4, 0.8) \} \]

Determine the implication relation:

If \( x \) is \( A \) then \( y \) is \( B \) else \( y \) is \( C \)

Here, \( A \times B = \)

\[
\begin{bmatrix}
  a & 0 & 0 & 0 & 0 \\
  b & 0.2 & 0.8 & 0.8 & 0 \\
  c & 0.2 & 0.6 & 0.6 & 0 \\
  d & 0.2 & 1.0 & 0.8 & 0 
\end{bmatrix}
\]
Example 4:

and $\bar{A} \times C =$

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
a & 0 & 0.4 & 1.0 & 0.8 \\
b & 0 & 0.2 & 0.2 & 0.2 \\
c & 0 & 0.4 & 0.4 & 0.4 \\
d & 0 & 0 & 0 & 0
\end{bmatrix}
\]

$R =$

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
a & 0 & 0.4 & 1.0 & 0.8 \\
b & 0.2 & 0.8 & 0.8 & 0.2 \\
c & 0.2 & 0.6 & 0.6 & 0.4 \\
d & 0.2 & 1.0 & 0.8 & 0
\end{bmatrix}
\]
Interpretation of fuzzy implication

If $x$ is $A$ then $y$ is $B$

If $x$ is $A$ then $y$ is $B$ else $y$ is $C$
Fuzzy Inferences
Let’s start with propositional logic. We know the following in propositional logic.

1. **Modus Ponens** : $P, P \implies Q, \iff Q$

2. **Modus Tollens** : $P \implies Q, \neg Q \iff \neg P$

3. **Chain rule** : $P \implies Q, Q \implies R \iff, P \implies R$
An example from propositional logic

Given

1. $C \lor D$
2. $\neg H \implies (A \land \neg B)$
3. $C \lor D \implies \neg H$
4. $(A \land \neg B) \implies (R \lor S)$

From the above can we infer $R \lor S$?

Similar concept is also followed in fuzzy logic to infer a fuzzy rule from a set of given fuzzy rules (also called fuzzy rule base).
Inferring procedures in Fuzzy logic

Two important inferring procedures are used in fuzzy systems:

- **Generalized Modus Ponens (GMP)**
  
  If $x$ is $A$ Then $y$ is $B$

  $x$ is $A'$

  ________________

  $y$ is $B'$

- **Generalized Modus Tollens (GMT)**

  If $x$ is $A$ Then $y$ is $B$

  $y$ is $B'$

  ________________

  $x$ is $A'$
Fuzzy inferring procedures

Here, \( A, B, A' \) and \( B' \) are fuzzy sets.

To compute the membership function \( A' \) and \( B' \) the max-min composition of fuzzy sets \( B' \) and \( A' \), respectively with \( R(x, y) \) (which is the known implication relation) is to be used.

Thus,

\[
B' = A' \circ R(x, y) \quad \mu_B(y) = \max\left[\min(\mu_{A'}(x), \mu_R(x, y))\right]
\]

\[
A' = B' \circ R(x, y) \quad \mu_A(x) = \max\left[\min(\mu_{B'}(y), \mu_R(x, y))\right]
\]
Generalized Modus Ponens (GMP)

\[ P : \text{If } x \text{ is } A \text{ then } y \text{ is } B \]

Let us consider two sets of variables \( x \) and \( y \) be
\[ X = \{x_1, x_2, x_3\} \text{ and } Y = \{y_1, y_2\}, \text{ respectively.} \]

Also, let us consider the following.
\[ A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\} \]
\[ B = \{(y_1, 1), (y_2, 0.4)\} \]

Then, given a fact expressed by the proposition \( x \text{ is } A' \),
where \( A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\} \)
derive a conclusion in the form \( y \text{ is } B' \) (using generalized modus ponens (GMP)).
Example: Generalized Modus Ponens

If $x$ is $A$ Then $y$ is $B$

$x$ is $A' \\ 
\hline \\
\vdots \\
$y$ is $B'$

We are to find $B' = A' \circ R(x, y)$ where $R(x, y) = \max\{A \times B, \overline{A} \times Y\}$

$$
\begin{align*}
A \times B &= \begin{bmatrix}
    x_1 & y_1 & y_2 \\
    x_2 & 1 & 0.4 \\
    x_3 & 0.6 & 0.4 \\
\end{bmatrix}, \\
\overline{A} \times Y &= \begin{bmatrix}
    x_1 & 0.5 & 0.5 \\
    x_2 & 0 & 0 \\
    x_3 & 0.4 & 0.4 \\
\end{bmatrix}
\end{align*}
$$

Note: For $A \times B$, $\mu_{A \times B}(x, y) = \min(\mu_A x, \mu_B(y))$
Example: Generalized Modus Ponens

\[ R(x, y) = (A \times B) \cup (\overline{A} \times y) = \begin{bmatrix} x_1 & 0.5 & 0.5 \\ x_2 & 1 & 0.4 \\ x_3 & 0.6 & 0.4 \end{bmatrix} \]

Now, \( A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\} \)

Therefore, \( B' = A' \circ R(x, y) = \begin{bmatrix} 0.6 & 0.9 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix} \)

Thus we derive that \( y \) is \( B' \) where \( B' = \{(y_1, 0.9), (y_2, 0.5)\} \)
Example: Generalized Modus Tollens

Generalized Modus Tollens (GMT)

P: If \( x \) is \( A \) Then \( y \) is \( B \)

Q: \( y \) is \( B' \)

\[
\begin{align*}
\text{If } x \text{ is } A \text{ Then } y \text{ is } B \\
y \text{ is } B' \times \\
\hline \\
x \text{ is } A'
\end{align*}
\]
Example: Generalized Modus Tollens

Let sets of variables $x$ and $y$ be $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2\}$, respectively.

Assume that a proposition If $x$ is $A$ Then $y$ is $B$ given where $A = \{(x_1, 0.5), (x_2, 1.0), (x_3, 0.6)\}$ and $B = \{(y_1, 0.6), (y_2, 0.4)\}$

Assume now that a fact expressed by a proposition $y$ is $B$ is given where $B' = \{(y_1, 0.9), (y_2, 0.7)\}$.

From the above, we are to conclude that $x$ is $A'$. That is, we are to determine $A'$.
Example: Generalized Modus Tollens

- We first calculate $R(x, y) = (A \times B) \cup (\overline{A} \times y)$

$$
R(x, y) =
\begin{bmatrix}
0.5 & 0.5 \\
1 & 0.4 \\
0.6 & 0.4 \\
\end{bmatrix}
$$

- Next, we calculate $A' = B' \circ R(x, y)$

$$
A' =
\begin{bmatrix}
0.9 & 0.7 \\
0.5 & 0.5 \\
1 & 0.4 \\
0.6 & 0.4 \\
\end{bmatrix}
=\begin{bmatrix}
0.5 & 0.9 & 0.6 \\
\end{bmatrix}
$$

- Hence, we calculate that $x$ is $A'$ where $A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$
Apply the fuzzy GMP rule to deduce \textbf{Rotation is quite slow}

Given that:

- If temperature is High then rotation is Slow.
- temperature is Very High

Let,

\[ X = \{30, 40, 50, 60, 70, 80, 90, 100\} \] be the set of temperatures.

\[ Y = \{10, 20, 30, 40, 50, 60\} \] be the set of rotations per minute.
The fuzzy set High (H), Very High (VH), Slow (S) and Quite Slow (QS) are given below.

\[
H = \{(70, 1), (80, 1), (90, 0.3)\}
\]

\[
VH = \{(90, 0.9), (100, 1)\}
\]

\[
S = \{(30, 0.8), (40, 1.0), (50, 0.6)\}
\]

\[
QS = \{(10, 1), (20, 0.8)\}
\]

1. If temperature is High then the rotation is Slow.

\[
R = (H \times S) \cup (H \times Y)
\]

2. temperature is Very High

Thus, to deduce "rotation is Quite Slow", we make use the composition rule

\[
QS = VH \circ R(x, y)
\]
Any questions??