Quality and Quantity, 13(1979) 161-173 161

CLIQUES, CLUBS AND CLANS *

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1. Introduction

In the analysis of social networks adequate concepts are necessary to indicate various types and configurations of social groups such as peer groups, *coteries,* acquaintance groups, etc. The problem has theoretically been argued convincingly by e.g. Kadushin (1968), who introduced the general concept of "social circle". In the actual empirical study of social networks there is therefore a need for adequate operational and analytically useful concepts to represent such more or less closely knit groups.

Many of these can be developed with the help of the theory of graphs and networks. A well-known concept, more or less corresponding to that of the peer group is the clique: a group all members of which are in contact with each other or are friends, know each other, etc. However, similar concepts will be necessary to denote less closely knit, yet significantly homogeneous social groups, such as "acquaintance groups", where every pair of members, if they are not in mutual contact, have mutual acquaintances, or common third contacts, etc. In this latter type of social group an important aspect is brought out by the question of whether the homogeneity of a social group is due to its position in a larger social network in which it is embedded, or whether it is a property of the group itself as a more or less autarchic unit, independent of the surrounding social network. In the first case, for instance, a group may be as closely knit as an "acquaintance network",

* Paper presented at the QUANTUM conference on Historical Social Research, Cologne, Germany, August 9-13, 1971.

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just because there are "mutual acquaintances" outside the group in the surrounding network. Changes in the environment, i.e. the outside network, may change the character of the group as an acquaintance group, In the latter case, however, a group is an acquaintance group, because mutual acquaintances linking members are themselves members of that group. Therefore changes in the outside social network will not affect the nature and structure of the group itself as an acquaintance group. Such concepts can be worked out in terms of graph theoretic cluster-concepts. For instance, the familiar concept of n-clique deals with the tightness of a group as a global property, due to the interrelationships or interactions of all members of a larger social network. The concepts of clubs and clans, to be introduced here denote a local property of structural autarchy in the sense that the interrelationships within the particular social group are sufficient for its homogeneity, and independent of those interrelationships involving members or parts of the surrounding larger social network.

In this paper we shall introduce three different cluster concepts of graphs, cliques, clubs and clans, and investigate their interrelationships. The graphs treated here will be simple graphs: finite, non-empty, and having no loops or multiple lines. We shall mainly follow the notation and concepts given by Harary (1969), to which we may refer the reader for further reference.

We shall suffice here with a cursory introduction of the concepts and notation used.

A graph G is a set of points together with a set of lines. To simplify notation here, we shall use the same symbol G to denote the set of points of G. Any line of G connects some pair of points, $u, v \in G_t$ which then are said to be adjacent to each other in G. We shall also consider subgraphs of G, indicated by their pointset. If $H \subset G$ is a subset of G, the subgraph H of G consists of all points of H together with all lines of G, which connect points, $u, v \in H$ in G. A path, connecting two points u, v of a subgraph H, in that same subgraph H, consists of points $u, w_1, w_2, \ldots w_{l-1}, v \in$ H, such that u is adjacent to w_1, w_i is adjacent to w_{i+1}, consecutively and w_{i-1} is adjacent to v. The length l of a path is given by the number of its lines. A cycle C_l of length l is a path of length l, where $u = v$. A subgraph H is connected in G, if each pair of points $u, v \in H$ is connected by a path in H. A complete graph K_p is a graph of p points, where each pair of points is adjacent to each other.

We shall also consider maximal subgraphs with respect to a given property. They are subgraphs of G satisfying that property, such that no larger subgraphs with that property exist in G, which contain them. A well known example is given by the cliques of a graph G: maximal complete subgraphs K_p of G.

The distance of a pair of points u, v in a certain subgraph H, denoted by

$$d_H(u, v)$$

is given by the length of a shortest path connecting u and v in H. If u, v are not connected in H, this distance is infinity. We shall frequently make use of the well-known relation that, if H is a subgraph of G, than for every pair of points $u, v \in H$

$$d_G(u,v) \leq d_H(u,v) \quad (1)$$

The distance between any two points in a subgraph of G cannot be smaller than their distance in G itself.

The diameter of a subgraph H is given by the largest distance between a pair of points in that subgraph. If we extend the pointset $H \subset G$ with a point $w \in G - H$ or a subset $S \subset G - H$, we may consider the distances in the larger subgraphs corresponding to $H \cup \{w\}$ or $H \cup S$, denoting for simplicity distances as $d_{H,w}$ or $d_{H,S}$ [1]. The degree of a point u in G is the number of points (neighbors) adjacent to u in G. The set of those points is called the 1-neighborhood of u in G, to be denoted by $V_1(u)$. We may restrict the set of neighbors to those in a subgraph H of G only, to be denoted as $V_1^H(u)$. Similar extensions may be made to n-neighborhoods of u: points at distance n from u.

2. Cluster concepts

In standard graph theory a familiar cluster concept is given by the cliques of a graph G. As mentioned above, they are given by the set of maximal complete subgraphs of G. Another cluster type definition of subgraphs of graphs are the "n-cliques", introduced by Luce (1950; see also Luce and Perry, 1949) as given by the following definition [2],

DEFINITION 1: An n-clique L of a graph G is a maximal subgraph of G such that for all pairs of points u, v of L the distance in G

$$d_G(u, v) \leq n \, . \quad (2)$$

The reader may note that, due to the maximality of L, for every point $w \in G - L$, there is a point $v \in L$ for which

$$d_G(w,v) > n. \quad (3)$$

From this definition, it can be seen that the n-clique is a global concept, based on the total structure of the network, as based on the graph and reflected in its distance matrix. The distances between points in a certain subset of points can be based on shortest paths, involving other points from the network not belonging to that group.

It is well-known, therefore, that in the subgraph formed by the points of an *n*-clique *L,* the distances between points can be larger than *n*. This follows from the familiar property, referred to above in condition (1), that for any two points *u, v* of the subgraph *L* of *G* we must have

$$d_G(u, v) \leq d_L\{u, v).$$

The condition (2) therefore does not imply that for each $u, v \in L$

$$d_L(u, v) \leq n .$$

Consequently, the diameter of *L* may be larger than *n.*

In a recent article Alba (1973) has illustrated this phenomenon with the example, given here in Fig. 1.

If we restrict our attention to 2-cliques *L (n* = 2) and designate 2-cliques by their pointsets, it can be seen that *L* = {1, 2, 3, 4, 5} is a 2-clique. However, its diameter is 3, i.e. the largest distance in *L* is 3, in the case of the pair of points 4 and 5. In fact, an *n*-clique can be disconnected (diameter infinity) as we shall illustrate further in this paper.

The concept of *n*-clique therefore does not embody the idea of particular tightness or even connectedness of the particular group concerned as an essential feature of the corresponding cluster of points in a graph. Yet in many, if not most, problems in social network analysis, leading to a graphtheoretic formulation, this idea of interconnectedness is a basic feature of the "tightness" of sets of points, underlying the definition of a cluster. Putting up a similar argument for the connectedness of clusters as subgraphs, as well as for their tightness as measured by their diameter, Alba introduced "sociometric cliques", as a more satisfactory subclass of n-cliques. They are n-cliques with diameter *n* and consequently connected. As the sociometric context is not essential, we suggest as a more appropriate name "*n*-clan".

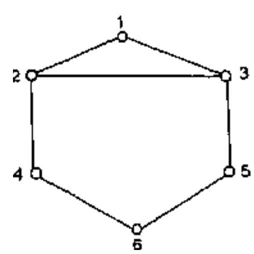

Fig. 1. Graph G.

DEFINITION 2. An n-clan M of a graph G is an n-clique of G such that for all pairs of points u, v of M the distance in M

$$d_M(u, v) \leq n . \tag{4}$$

Consequently, for an n-clan M of G the following relations hold:

(1) for all points $u, v \in M$:

$$d_M(u, v) \leq n ; \tag{5}$$

(2) for all points $w \in G - M$ there is a $u \in M$ for which:

$$d_G(u, w) > n . \tag{6}$$

The relations (5) and (6) imply that M is an n-clique, as from relations (1) and (5) we have

$$d_G(u, v) \leq d_M(u, v) \leq n , \tag{7}$$

whereas relation (6) ensures the maximality of M as an n-clique.

Instead of the restriction of cliques to clans we could also have looked immediately at clusters corresponding to subgraphs of diameter n. These we shall call "n-clubs".

DEFINITION 3. An n-club N of a graph G is a maximal subgraph of G of diameter n.

For all points u, v of an n-club N we have for the subgraph N *of* G

$$d_N(u, v) \leq n . \tag{8}$$

Fig. 2. N of diameter 2.

The maximality of N as an n-club of G implies that for all points $w \in G - N$, there is a point $u \in N$ such that

$$d_{N,w}(u, w) > n .$$

This condition, however, is not sufficient for the maximality of N, as illustrated by the graph of Fig. 2. The points on the cycle C_4: {1, 2, 3, 4}

form a subgraph of diameter 2. Neither point 5 nor point 6 can be

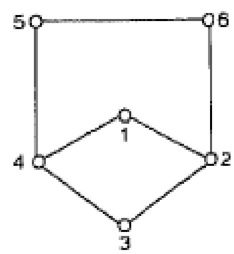

Fig. 2. N of diameter 2.

added without enlarging the diameter of the resulting subgraph to 3, Yet *N* as a whole is a graph of diameter 2 which contains the C_4.

3. Interrelationship of cliques, clubs and clans

By definition *n*-clans are *n*-cliques of diameter *n.* But how are they related to *n*-clubs? According to Alba (1973) all *n*-clubs are *n*-clans, as formulated in his theorem 2.1, which, in our terminology, states that a subgraph of *G* is an *n*-clan if and only if (iff) it is an *n*-club. This theorem is incorrect, as the "if"-part is deficient. This can be shown with the following proposition.

Proposition 1: Every *n*-club *N* of a graph *G* is contained in some *n*-clique *L* of *G.*

Proof: An *n*-club *N* of *G* satisfies relation (8) and, as a subgraph also relation (1). Therefore, we have for all points $u, v \in N$

$$d_G(u, v) \leq d_N(u, v) \leq n$$

Hence *N* is contained in some *n*-clique *L* of *G.* However, *N* can be properly contained in such an *n*-clique *L.* For instance, there may be a point *w* of *L,* not in *N,* such that for all points $u \in N$ we have

$$d_G(u, w) \leq n \ ,$$

whereas there is a point $v \in N$ such that

$$d_{N,w}(v, w) > n \ .$$

For instance, in the example provided by Alba (1973), as given in Fig. 1 in this paper, the set {1, 2, 3, 4} is obviously a 2-club, which is not a 2-clique and therefore not a 2-clan, as it is properly contained in the 2-clique {1,2,3,4,5}.

Proposition 2: Every *n*-clan *M* of *G* is an *n*-club of *G.*

Proof: Let *M* be an *n*-clan of *G.* It therefore satisfies relations (5), (6) and (7). Now assume *M* to be contained in a larger subgraph of diameter *n* formed by $M \cup$ S, where $S \subset G - M$: *S* is a subset of points in the other points of *G.* However, then we must have, for all $u, v \in M; s, w \in S$:

$$d_G\{u, w) \leq d_{M,S}(u, w) \leq n \ ;$$

$$d_G(s, w) \leq d_{M,S}(s, w) \leq n \ ;$$

and, obviously,

$$d_G(u, v) \le d_{M,S}(u, v) \le d_M(u, v) \le n.$$

This violates relation (6) and contradicts the maximality of M as an n-clique of G: $M \cup S$ is contained in some n-clique L of G. Consequently, there can be no such set S in G and M is a maximal subset of G with diameter n. That is, M is also an n-club of G,

In our example of Fig. 1, the only 2-clique, which is also a 2-clan and hence a 2-club is formed by the set of points {2, 3, 4, 5, 6}.

An obvious corollary of proposition 2 is:

Corollary 1: If an n-club N of G is contained in an n-clan M, then $N = M$.

Our results can be summarily illustrated with the aid of Fig. 1, where we restrict ourselves to distance 2 or diameter 2, ($n = 2$).

(a) 2-cliques of G: *a*1 {1,2, 3,4,5};*a*2: {2,3,4,5,6}.

(b)2-clubs of G: *b*1: {1,2, 3,4}; *b*2: {1,2, 3, 5}; *b*3: {2,3,4,5,6}.

(c)2-clans of G: *c*l: {2,3,4,5,6}.

The 2-clubs *b*1 and *b*2 are not 2-clans. They are properly contained in the 2-clique *a*1, which is not a 2-clan, as it has diameter 3. The 2-clique *a2* is a 2-clan (*c*l) and hence also a 2-club (*b*3).

4. The systems of cliques, clubs and clans of a graph

From the foregoing discussion it will be clear that for any graph G we can distinguish:

(1) the system of n-cliques of G: the class $\mathscr{L}_n(G) = \mathscr{L}_n$, the elements of which are indicated by the pointsets L of the different n-cliques L of G;

(2) the system of n-clubs of G: the class $\mathscr{N}_n(\mathrm{G}) = \mathscr{N}_n$, containing the pointsets N of the different n-clubs N of G;

(3) the system of n-clans of G: the class $\mathscr{M}_n(G) = \mathscr{M}_n$, containing the pointsets M of the different n-clans of G.

In this section we consider more closely the possible interrelationships of these classes $\mathscr{L}_n$, $\mathscr{N}_n$, and $\mathscr{M}_n$ of a graph G. Consider the symmetric difference of $\mathscr{L}_n$ *and* $\mathscr{N}_n$:

$$\mathscr{L}_n \,\Delta\, \mathscr{N}_n = (\mathscr{L}_n \cup \mathscr{N}_n) - (\mathscr{L}_n \cap \mathscr{N}_n),$$

containing only those n-cliques or n-clubs which are not common to both. Define

$$\sim\mathscr{N}_n =_{\mathrm{def}} \mathscr{N}_n \cap (\mathscr{L}_n \,\Delta\, \mathscr{N}_n),$$

which contains only n-clubs which are not n-clans, and

$$\sim\mathcal{L}_n =_{\text{def}} \mathcal{L}_n \cap (\mathcal{L}_n \,\Delta\, \mathcal{N}_n),$$

the subclass of *n*-cliques, which are not *n*-clans. The foregoing results of propositions 1 and 2 can be collected in the following lemma.

Lemma 1: For each *n*-club $N \in \mathcal{N}_n$, there is an *n*-clique $L \in \mathcal{L}_n$ such that $N \subseteq L$; (a) N is an *n*-clan ($N \in \mathcal{M}_n$) iff for every $v \in G - N$ there is a $u \in N$ such that

$$d_G(u, v) > n;$$

(b) N is not an *n*-clan ($N \in \sim\mathcal{N}_n$) iff there is a $v \in G - N$, such that for all $u \in N$

$$d_G(u, v) \leq n\ .$$

According to this lemma every *n*-club $N \in \mathcal{N}_n$ is either equal to an *n*-clique $L \in \mathcal{L}_n$, $(N = L)$, and then an n-clan ($N \in \mathcal{M}_n$) or N is properly contained in some *n*-clique $L \in \mathcal{L}_n$,), $(N \subset L)$. The concept of *n*-clique ($L \in \mathcal{L}_n$,) defines a class of clusters or subgraphs based on "close" reachability of points, through paths including points in G external to L.

On the other hand n-clubs ($N \in \mathcal{N}_n$) are based on the condition of "close" reachability of points, involving internal points of N only. In that sense *n*-clubs N have a property of local autarchy: the closeness or tightness of their communication structure is independent of the relations of its points with the surrounding network. Obviously, this latter, more stringent, condition leads to "smaller" clusters: *n*-clubs cannot be larger than *n*-cliques, as they are included in them. In fact, as we can see from the examples mentioned in this paper, an *n*-clique L can contain more than one *n*-club N and, conversely, an *n*-club can be contained in more than one *n*-clique L.

The n-clans ($M \in \mathcal{M}_n$) belong to both $\mathcal{L}_n$ and $\mathcal{N}_n$ they are *n*-cliques as well as *n*-clubs. As *n*-clubs they share the property of connectedness with sufficiently narrow diameter. As *n*-cliques they have the advantage of "size": they are as "large" as *n*-cliques.

We therefore can subsume the interrelationship of these classes in the following three, mutually exclusive subclasses:

(a) $\mathcal{M}_n = \mathcal{L}_n \cap \mathcal{N}_n$: the class of *n*-clans as the intersection of the class of *n*-cliques and the class of *n*-clubs;

(b) $\sim\mathcal{L}_n$: the subclass of *n*-cliques which are not *n*-clans;

(c) $\sim\mathcal{N}_n$: the subclass of *n*-clubs, which are properly contained in *n*-cliques.

For *n* - 1 we trivially have $\mathcal{L}_n = \mathcal{N}_n = \mathcal{M}_n$, all systems reducing to the system of cliques of G; the class of maximal complete subgraphs of G. A similar trivial reduction can be seen for the case of G itself being an n-club. Excluding these trivial cases, we may note that, except for the nullgraph, $\mathfrak{N}_n$ and $\mathcal{L}_n$ are never empty.

The following three cases deserve some interest:

(I) $\mathcal{M}_n = \varnothing$: there are no n-clans;

(II) $\sim\mathfrak{N}_n = \varnothing$: all n-clubs are n-clans; (Alba's case);

(III) $\sim \mathcal{L}_n = \varnothing$: all n-cliques are n-clans.

Case I. $= \mathcal{M}_n = \varnothing$ *(No clans)*

In this case $\sim\mathfrak{N}_n$ and $\mathcal{L}_n$ are disjoint classes. Every $N \in \sim\mathfrak{N}_n$, so condition (b) of Lemma 1 holds for every $N \in \mathfrak{N}_n$. We shall illustrate this case for $n = 2$, (distances and diameter 2). Extensions to general n, if necessary, are left to the reader.

We can define for all $N \in \sim\mathfrak{N}_2$ (= $\mathfrak{N}_2$ in this case) and for every $v \in G - N$ ($\neq \varnothing$) the set

$$N_v =_{\text{def}} \{u \in N;\ d_{N,V}(u, v) > 2\} \tag{9}$$

Note that N_v is never empty, as $G - N$ never is, under the present assumptions. Consequently, for every $v \in G - N$ and for all corresponding $u \in N_v$ we have

$$V_1^{(N)}(u) \cap V_1^{(N)}(v) = \varnothing. \tag{10}$$

Their 1-neighborhoods in N are disjoint in G. Obviously, $v \in G - N$ and $u \in N_v$ cannot be adjacent in G either, nor in any subgraph of G. Given relation (10) such a pair of points u, v can therefore have

$$d_G(u, v) = 2$$

if and only if

$$V_1^{G-N}(u) \cap V_1^{G-N}(v) \neq \varnothing, \tag{11}$$

that is when their 1-neighborhoods in $G - N$ are not disjoint in G. These considerations establish the validity of the following proposition:

Proposition 3. $\mathcal{M}_2 = \varnothing$, iff for all $N \in \mathfrak{N}_2$, there is a point $v \in G - N$, such that for all $u \in N_v$

$$V_1^{(G-N)}(u) \cap V_1^{(G-N)}(v) \neq \varnothing$$

In this case as $\mathcal{M}_2 = \varnothing$, there are no 2-clans, all 2-clubs $N \in \mathfrak{N}_2$ are properly contained in larger 2-cliques L. This situation is illustrated with the graph G of Fig. 3, which has no 2-clans as can be deduced from the enumeration of its classes $\mathfrak{N}_2$ and $\mathcal{L}_n$ given in Table I. In this

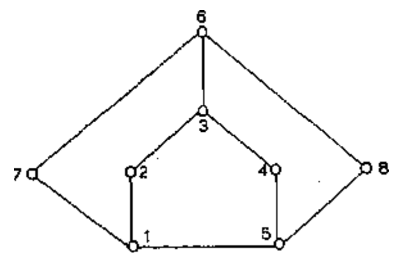

Fig. 3. A graph G without 2-clans.

example all 2-cliques contain 2-clubs. More than one 2-club, a star and a cycle C_s, is contained in each 2-clique.

Case II. $\mathcal{N}_n = \varnothing$. *Alba's case: all clubs are clans*

We have $\mathcal{N}_n \subseteq \mathcal{L}_n$. This case is therefore equivalent to $\mathcal{M}_n = \mathcal{N}_n$. All $N \in \mathcal{N}_n$ satisfy condition (a) of Lemma 1. For the case of diameter 2 $(n = 2)$ one can see easily that: $\mathcal{M}_2 = \mathcal{N}_2$ iff for every $N \in \mathcal{N}_n$ all $v \in G - N$, there is a $u \in N_v$ satisfying

(a) 1-neighborhoods *in G - N* disjoint:

$$V_1^{(G-N)}(u) \cap V_1^{(G-N)}(v) \neq \varnothing \tag{12}$$

or, equivalently,

(b) 1-neighborhoods in G disjoint:

$$V_1(u) \cap V_1(v) \neq \varnothing \tag{13}$$

An example for $n = 2$ is given in Fig. 4, where the sets $\{s_i\}$, $\{u_j\}$ and $\{v_k\}$ indicate points of degree 2, each adjacent solely to the points 1, 3 or, respectively 1, 2 or 2, 3. All the 2-clubs are 2-clans and therefore

TABLE I

$\mathcal{N}_2$		$\mathcal{L}_2$
Star: {2,3,4,6}	$\subset$	{1,2,3,4,5,6}
C_5: {1,2,3,4,5}	$\subset$	{1,2,3,4,5,6}
Star: {1,2,5,7}	$\subset$	{1,2,3,5,6,7}
C_5: {1,2,3,6,7}	$\subset$	{1,2,3,5,6,7}
Star: {1,4,5,8}	$\subset$	{1,3,4,5,6,8}
C_5: {3,4,5,6,8}	$\subset$	{1,3,4,5,6,8}
Star: {3,6,7,8}	$\subset$	{1,3,5,6,7,8}
C_5: {1,5,6,7,8}	$\subset$	{1,3,5,6,7,8}

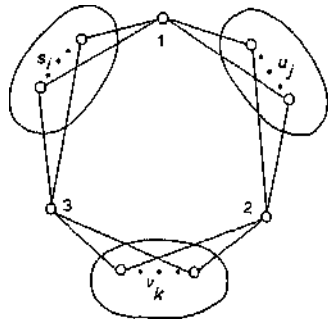

Fig. 4. A graph G where all 2-clubs are 2-clans.

2-cliques. They are:

$\{1, 3, s_i\}$, $\{1, 2, u_j\}$, $\{2, 3, v_k\}$

and

$\{s_i, 1, u_j\}$, $\{u_j, 2, v_k\}$, $\{s_i, 3, v_k\}$

There are two 2-cliques which are not 2-clans: $L \in \mathcal{L}_2$, They are: $\{1, 2, 3\}$ and $\{s_i, u_j, v_k\}$. Note that as subgraphs they are nullgraphs i.e. totally disconnected. All their points are isolated. It should be noted, that these latter 2-cliques do not contain any 2-clubs, or, for that matter, any subgraph of G of diameter 2. This illustrates the more general situation where there can exist w-cliques $L \in \sim\mathcal{L}_n$, which contain no n-clubs $N \in \mathcal{N}_n$, but at most parts of $N \in \mathcal{N}_n$.

Case III. $\sim\mathcal{L}_n = \varnothing$ *All n-cliques are n-clans*

As $\mathcal{L}_n \subset \mathcal{N}_n$ we immediately have $\mathcal{L}_n = \mathcal{N}_n = \mathcal{M}_n$ and therefore $\sim\mathcal{L}_n \overline{=} \varnothing$ implies $\sim \mathcal{N}_n = \varnothing$9£,, = *4>*. If there are no n-cliques which are not n-clans, then there are no n-clubs which are not n-clans. (We have seen in the former

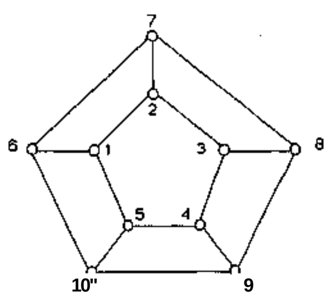

Fig. 5. A graph G where all 2-cliques are 2-clans.

case that the reverse need not be true). It is hard to characterize such graphs beyond the obvious statement that all *n*-cliques have diameter *n.*

For $n = 2$ an example is given in Fig. 5. In this all 2-clubs are 2-cliques and conversely. All three clustersystems coincide as $\mathcal{L}_2 = \mathcal{N}_2 = \mathcal{M}_2$. The common elements of these systems are: (a) all the stars of degree 3: e.g. {1, 2, 3, 7} etc.; (b) all the cycles C_4: e.g. {1, 2, 6, 7} etc.; (c) the two cycles C_5: {1, 2, 3,4, 5} and {6, 7,8,9,10}.

5. Conclusions and suggestions for further research

We may conclude that the two classes of *n*-cliques $\mathcal{L}_n$ and *n*-clubs $\mathcal{N}_n$ of a graph *G* are classes of clusters which are in general but loosely interrelated and have a significance of their own. The latter, the *n*-clubs, are maximal subgraphs *N* of *G* with respect to internal reachability of points within distance *n,* i.e. independent of the connection of the points of *N* with order points in *G: G - N.* In that sense n-clubs are essentially local concepts: their reachability as diameter *n* subgraphs is not effected by changes in the subgraph *G - N* and the connection of *G - N* with *N.* This independence of the environment given by the outer network *G - N* can be seen as a certain local autarchy. In short, *N* as a subgraph would have at most diameter *n* in any other graph *G.*

The *n*-cliques *L* are global concepts in *G* in the sense that their reachability of points within distance *n* can involve points external to *L,* Hence their reachability can be determined outside *L* in *G - L:* elimination of points from *G - L,* or lines in the subgraph *G - L* or connecting *G - L* and *L,* can effect the reachability of points in *L.*

The *n*-clans *M* of *G,* when they exist, combine these local and global aspects as they are cliques as well as clubs. However, the class $\mathcal{M}_n^{\wedge}$ of *n*-clans may be empty for a graph *G.*

Finally, n-clubs *N* are always contained in some *n*-clique *L.* In that sense they are "smaller" than *n*-cliques. Only *n*-clans, as *n*-clubs have the "size" of an *n*-clique; Moreover different *n*-clubs can be contained in the same *n*-clique and different *n*-cliques can contain the same *n*-clubs.

The interrelationship of the three classes $\mathcal{L}_n$, $\mathcal{N}_n$ and $\mathcal{M}_n$ therefore can be manifold. One perspective for further research is therefore to characterize graphs *G* according to the nature of that relationship. In the cases I, II and III, given in the last section our characterization hardly proceeded beyond that provided by the definitions.

Then the development of adequate (computer-) algorithms for the production of the systems of *n*-cliques *L*, *n*-clans *M* and *n*-clubs *N* of any graph *G* invites further research. The problem is satisfactorily solved for the detection of *n*-cliques. A well-known method is given by Auguston and Minker (1970). A reputedly faster algorithm than that referred to by them was recently published by Bron and Kerbosch (1973).

Therefore, the problem of detecting the system of *n*-clans of a graph reduces to sorting out the *n*-cliques of diameter *n* from the n-cliques of that graph (Alba, 1973). The development of an algorithm for the detection of the system of *n*-clubs of a graph may well be more cumbersome. Our first cursory assessment of this problem suggests that it may be of the order of enumerating the subgraphs corresponding to all subsets of points within the *n*-cliques of a graph.

Further research may also concern possible generalizations. One obvious generalization is that of (*m*, *n*)-clans $(m \geq n)$ of a graph: *n*-cliques which are *m*-clubs. Another generalization extends these concepts to directed graphs, with the introduction of directed cliques, clubs and clans.

Notes

The conventional notation of set theory is used. In particular $S \subseteq G$ will denote set inclusion, $S \subset C$ proper inclusion and $S = G$ identity of sets.

[2] Although the cluster concepts to be introduced in this paper suggest more appropriate names, we shall resist the temptation to do so and accept this part of the nomenclature as established.

[3] It should be noted that, although for each $N \in \mathcal{N}_n$ we have $N \subseteq L$ for some $L \in \mathcal{L}_n$, in general $\mathcal{N}_n \subseteq \mathcal{L}_n$, does not hold, as some elements N $(\in \mathcal{N}_n)$ are not *n*-cliques L and therefore do not belong to $\mathcal{L}_n$.,

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