Path, Cycle, Walk

Walk, trail, path, cycle

- **Walk:** A **walk** is a list of vertices and edges
  $[v_0,e_1,v_1,\ldots\ldots\ldots,\ e_k,\ v_k]$
  such that, for $(1 \le i \le k)$, the edge (e_i) has endpoints (v_{i-1}) and (v_i).

- **This walk joins** v_0 and v_k, hence called (v_0-v_k) walk

- **Trail:** A **trail** is a **walk** with **no repeated edge**. – edges are distinct

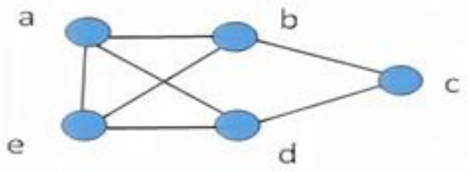
- **Path:** a sequence of **distinct** vertices such that two consecutive vertices are adjacent.

- **Cycle:** a closed path.

here $(v_0=v_n)$, however all other vertices are distinct

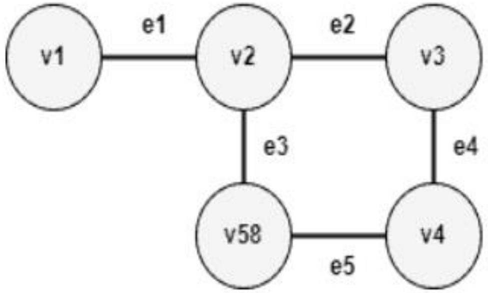
Examples

- (a, d, c, b, e) is a path
- (a, b, e, d, c, b, e, d) is not a path; it is a walk
- (*a, d, c, b, e, a*) is a cycle




```
    b
   / \
  /   \
 a-----c
 |
 |
 u
```

WALK AND TRAIL EXAMPLES

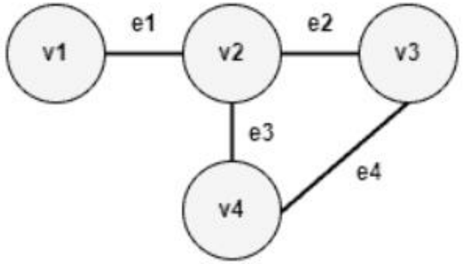


v1 - (e1) - v2 - (e2) - v3 (Open Walk; Trail)

v1 - (e1) - v2 - (e2) - v3 - (e2) - v2 (Open Walk)

v2 - (e2) - v3 - (e4) - v4 - (e5) - v5 - (e3) - v2 (Closed Walk)

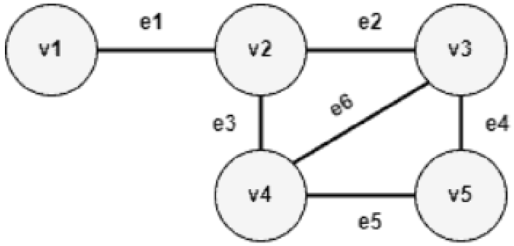
PATH EXAMPLES



v1 - (e1) - v2 - (e2) - v3

v1 - (e1) - v2 - (e3) - v4 - (e4) - v3

CYCLIC GRAPH



CYCLE EXAMPLES

v2 - (e2) - v3 - (e4) - v5 - (e5) - v4 - (e3) - v2

v2 - (e2) - v3 - (e6) - v4 - (e3) - v2

v3 - (e4) - v5 - (e5) - v4 - (e6) - v3

Walk, trail, path, cycle

- A **u,v-walk** or **u,v-trail** has first vertex **u** and last vertex **v**; these are its **endpoints**.
- A **u,v-path**: a **u,v-trail** with **no repeated vertex**.
- The **length** of a walk, trail, path, or cycle is **its number of edges**.
- A walk or trail is **closed** if its endpoints are the same.

Lemma 1

Lemma: Every u-v walk contains a u-v path

Proof:

Prove by Induction

Principle

To prove that a statement $P(n)$ is true for **all** $n \geq n_0$:

Step 1: Base Case

Show that the statement is true for the smallest value.

$$P(n_0) \text{ is true.}$$

Step 2: Induction Hypothesis

Assume the statement is true for some arbitrary value k.

$$P(k) \text{ is true.}$$

This assumption is called the **Induction Hypothesis**.

Step 3: Induction Step

Using the induction hypothesis, prove

$$P(k+1) \text{ is true.}$$

Conclusion

If the base case is true and

$$P(k) \Rightarrow P(k+1),$$

then

$$\boxed{P(n) \text{ is true for all } n \geq n_0.}$$

Prove by Induction

Simple Example

Claim

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \geq 1$.

Base Case

For $n = 1$,

$$1 = \frac{1(2)}{2} = 1$$

Induction Hypothesis

Assume

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Induction Step

$$\begin{aligned} 1 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Hence,

$$P(k+1) \text{ is true.}$$

Conclusion

Therefore,

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$$

for all $n \geq 1$.

Lemma 1

Lemma: Every u-v walk contains a u-v path

Proof: Use induction on the length (L) of a (u,v)-walk (W).

Basis step:(L = 0).

Having **no edge**, (W) consists of a single vertex (u = v).

This vertex is a (u,v)-path of length (0).

Lemma: Every u-v walk contains a u-v path

Induction step: (L >= 1)

- Suppose that the claim holds for walks of length **less than L**.
- If (W) has **no repeated vertex**, then its vertices and edges form a **(u,v)-path.**

```
Vertices:  u, a, b, c, v

Edges:     e₁      e₂      e₃      e₄

u —e₁— a —e₂— b —e₃— c —e₄— v
```

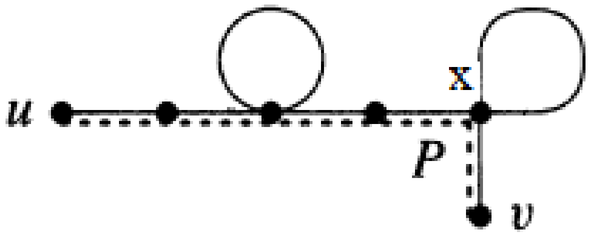
The corresponding walk is

$$W = u, e_1, a, e_2, b, e_3, c, e_4, v.$$

Therefore, W is already a u, v-**path**.

Lemma: Every u-v walk contains a u-v path

- Induction step: (L >= 1)
- If (W) has a **repeated vertex** (x), then deleting the edges and vertices between two appearances of (x) (leaving one copy of (x)) yields a **shorter (u,v)-walk** (W') contained in (W), i.e. (u,v) path.
- By the induction hypothesis, (W') contains a (u,v)-path (P), and this path (P) is contained in (W).



- Delete the closed subwalk between two occurrences of the repeated vertex (w).
- The resulting shorter walk is (W').
- Applying the induction hypothesis to (W') yields the desired (u,v)-path (P).

Connected and Disconnected graphs

Definition:

•A graph (G) is **connected** if it has a (u,v)-path whenever (u,v € V(G)) (otherwise, (G) is **disconnected**).

•If G has a (u,v)-path, then u is **connected to** v in G.

•The **connection relation** on V(G) consists of the ordered pairs (u,v) such that (u) is connected to (v).

• Distinction between *connection* and *adjacency*:

G has a *u*, *v*-path	*u v* ϵ *E*(*G*)
u and *v* are connected	*u* and *v* are adjacent
u is connected to *v*	*u* is joined to *v*
	u is adjacent to *v*

We can prove that a graph is connected by showing that from each vertex, there is a path to one particular vertex.

Relation -- recap

A **relation** is simply a **rule that tells us whether two objects are related**.

A relation on a set S is a subset of $S \times S$.

Friendship

Suppose

$$A = \{\text{Alice}, \text{Bob}, \text{Charlie}\}.$$

If Alice is friends with Bob and Bob is friends with Charlie,

then

$$R = \{(\text{Alice}, \text{Bob}), (\text{Bob}, \text{Alice}), (\text{Bob}, \text{Charlie}), (\text{Charlie}, \text{Bob})\}.$$

"Less than"

On

$$A = \{1, 2, 3\},$$

define

$$aRb \iff a < b.$$

Then

$$R = \{(1,2), (1,3), (2,3)\}.$$

Relation -- recap

A **relation** is simply a **rule that tells us whether two objects are related**.

A relation on a set S is a subset of $S \times S$.

A relation R on a set S is an equivalence relation if

1. $(x, x) \in R$ (R is reflexive)
2. $(x, y) \in R$ implies $(y, x) \in R$ (R is symmetric)
3. $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$ (R is transitive)

An equivalence relation defines a partition of the base set S into equivalence classes. Elements are in relation iff they are within the same class.

An **equivalence class**

Definition

Suppose $\sim$ is an equivalence relation on a set A.

The **equivalence class** of an element $a \in A$ is

$$[a] = \{x \in A : x \sim a\}.$$

In words,

> **The equivalence class of a is the set of all elements that are equivalent to a.**

Theorem 1

Theorem

The connection relation is an equivalence relation.

Proof

1. Reflexive

Every vertex is connected to itself.

- A single vertex (path of length 0) is a u, u-path.

Hence,

$$u \sim u.$$

Therefore, the relation is **reflexive**.

2. Symmetric

Suppose

$$u \sim v.$$

Then there exists a u, v-path

$$u = v_0, v_1, \ldots, v_k = v.$$

Traversing this path in the reverse direction,

$$v = v_k, v_{k-1}, \ldots, v_0 = u,$$

gives a valid v, u-path.

Hence,

$$v \sim u.$$

Therefore, the relation is **symmetric**.

3. Transitive

Suppose

$$u \sim v$$

and

$$v \sim w.$$

Then there exist

- a u, v-path P_1,
- a v, w-path P_2.

Concatenate the two paths:

```
u -------- v -------- w
     P₁          P₂
```

This gives a u, w-walk.

By the lemma:

> **Every walk contains a path.**

Therefore, the u, w-walk contains a u, w-path.

Hence,

$$u \sim w.$$

Therefore, the relation is **transitive**.

Components

A maximal connected subgraph of G is a subgraph which is connected, and is not **contained in any subgraph of G**

The components of a graph (G) are its maximal connected subgraphs.

- A component is trivial if it has no edges;
  - otherwise, it is nontrivial.

- The **equivalence classes** of the **connection relation** on V(G) are exactly the vertex sets of the components of G.

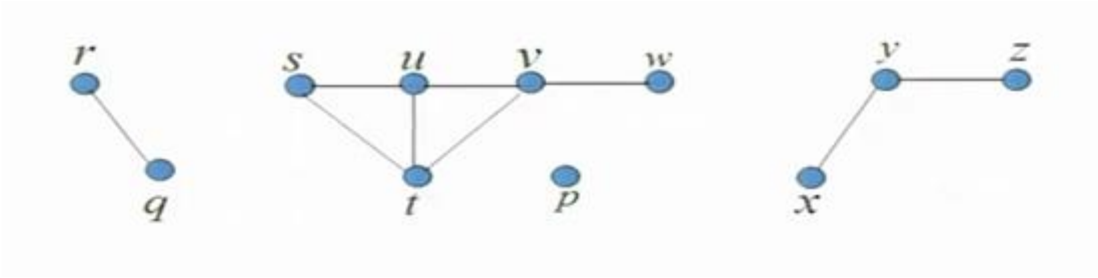
Components

Example

- The graph below has **four components**, one being an **isolated vertex**.
- The vertex sets of the components are

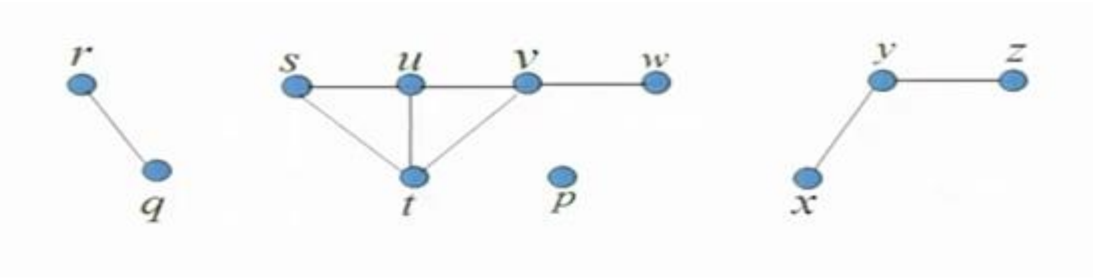
$$\{p\},\ \{q,r\},\ \{s,t,u,v,w\},\ \{x,y,z\};$$

these are the **equivalence classes of the connection relation**.



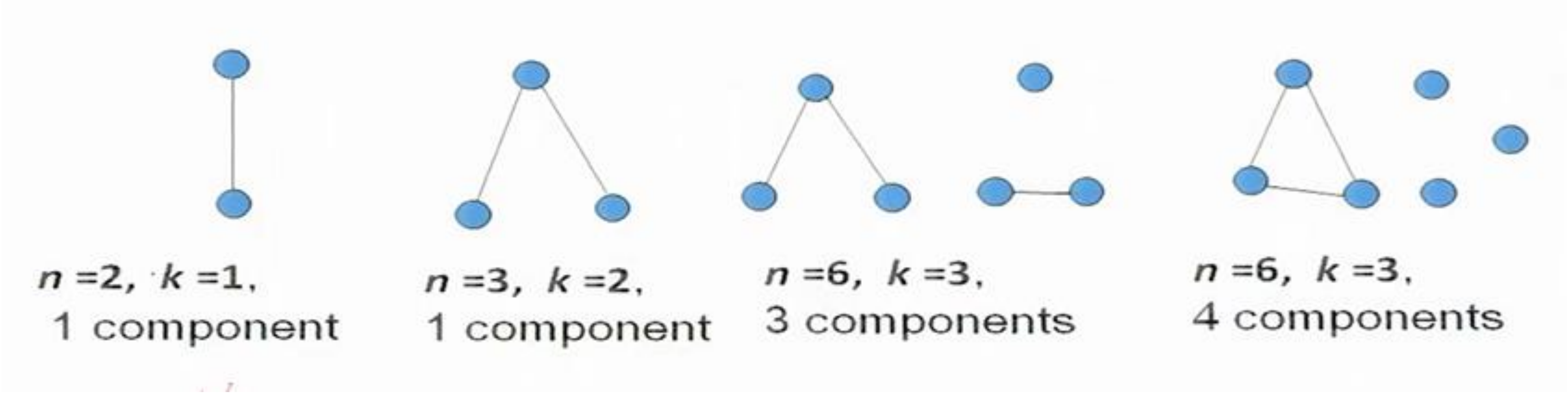
Components

- Components are pairwise disjoint; no two share a vertex. Adding an edge with endpoints in distinct components combines them into one component.
- Thus adding an edge decreases the number of components by 0 or 1, and deleting an edge increases the number of components by 0 or 1.



Theorem

Theorem: Every graph with n vertices and k edges has at least n-k connected components.



Theorem

Every graph with n vertices and k edges has at least n-k connected components.

Proof:

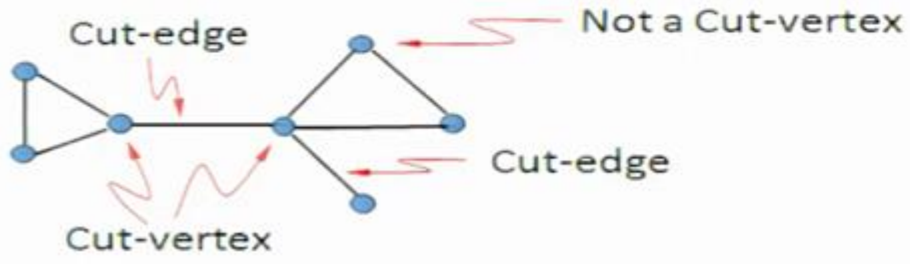
1. Start with the graph consisting of the **n vertices but no edges**.
   Every vertex is isolated.
   Hence, there are **(n) components**.
2. Now **add the k edges one by one**.
3. When an edge is added, **only two cases are possible**:
   Case 1: The endpoints are **already in the same component**.
   The number of components **does not change**.
   Case 2: The endpoints belong to **different components**.
   The **edge merges** these two components.
   Hence, the number of components **decreases by exactly one**.
4. Therefore, **each added edge can decrease the number of components by at most one**.
5. Since there are (k) edges, the total decrease in the number of components is at most (k).

Hence,

$$\text{Number of components} \geq n - k.$$

Cut edge, Cut vertex

- A ***cut-edge*** or ***cut-vertex*** of a graph is an edge or vertex whose deletion increases the number of components.



- *G-e* or *G-M* : The subgraph obtained by deleting an edge *e* or set of edges *M*
- *G-v* or *G-S* : The subgraph obtained by deleting a vertex *v* or set of vertices *S*

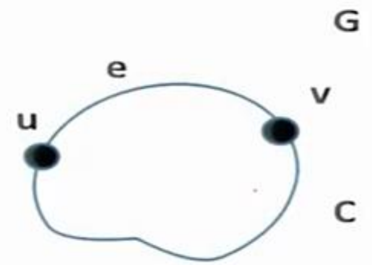
Theorem

- An edge e of a graph G is a **cut edge (bridge) if and only if** e belongs to **no cycle**.

($\Rightarrow$) If e is a cut edge, then e belongs to no cycle.

Proof (by contradiction)

Suppose $e = uv$ is a cut edge. $\Rightarrow$ **e belongs to a cycle**



Assume, to the contrary, that e lies on a cycle.

```
u ----- v
 \     /
  \   /
   \_/
```

Removing e still leaves another path from u to v (the rest of the cycle).

Hence, every path that previously used e can instead use this alternate u-v path.

Therefore, deleting e **does not disconnect** the graph

This contradicts the assumption that e is a cut edge.

Hence,

e belongs to no cycle.

($\Leftarrow$) If e belongs to no cycle, then e is a cut edge.

Let

$$e = uv,$$

and suppose e belongs to no cycle. → **e not a cut edge**

Assume that deleting e does **not** disconnect the graph.

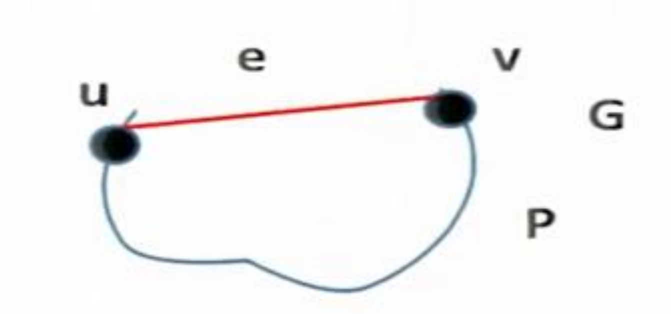
Then, in $G - e$, there is still a path from u to v.

Adding the edge e back gives

- the edge uv, and
- another u-v path.

Together they form a cycle containing e.

This contradicts the assumption that e belongs to no cycle.



Hence,

$$e \text{ is a cut edge.}$$

Theorem

Theorem: Every closed odd walk contains a odd cycle

- Use induction on the length L of the closed odd walk W
- L=1. A closed walk of length 1 traverses a cycle of length 1.
- We need to prove the claim holds **if it holds for closed odd walks shorter than W**.

- **Suppose that the claim holds for closed odd walks shorter than W.**

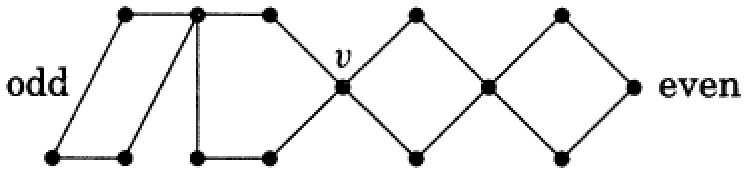
- **If odd walk (W) has no repeated vertex** (other than first = last), then (W) itself forms a cycle of odd length.

- Otherwise, (**W has a repeated vertex**):

- **Need to prove:** If repeated, (W) includes a shorter closed odd walk.

Theorem

- If walk W has a **repeated vertex (v)**, then we view W as starting at (v) and break walk W into two (v,v)-walks.
- Since walk W has odd length, one of these walk is odd and the other is even.
- The odd walk is shorter than (W); **by the induction hypothesis, it contains an odd cycle**, and this cycle appears inside W).
- Odd = Odd + Even



Maximal Path

- A **maximal path** in a graph G is a path (P) in G that **is not contained in any longer path**.

- When a graph is finite, no path can be extended forever. Therefore, maximal (non-extendible) paths always exist.

Claim:

- u is an endpoint of maximal path P
- Show that “All the neighbors of u must already a vertex of P”

Lemma

Lemma: If every vertex of graph G has degree at least 2, then G contains a cycle.

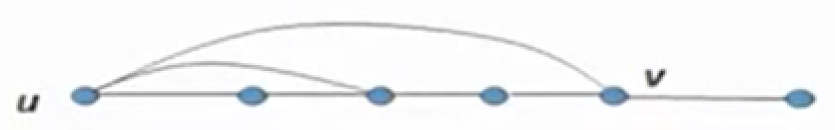
Proof:

Let (P) be a maximal path in (G), and let (u) be an endpoint of (P).

Since (P) cannot be extended, every neighbor of (u) must already be a vertex of (P).

Since (u) has degree at least (2), it has a neighbor v € V(P) via an edge not in P.

The edge (uv) completes a cycle with the portion of P from (v) to (u).



Concept of Extremality

- The proof of this Lemma is an **example of an important technique of proof** in graph theory that we call **extremality.**

- When considering graph structures of a given type, **choosing an example that is extreme in some sense** may yield useful additional information.

- For example, since a **maximal path (P) cannot be extended**, we obtain the **extra information that every neighbor of an endpoint of (P) belongs to (V(P))**

Lemma

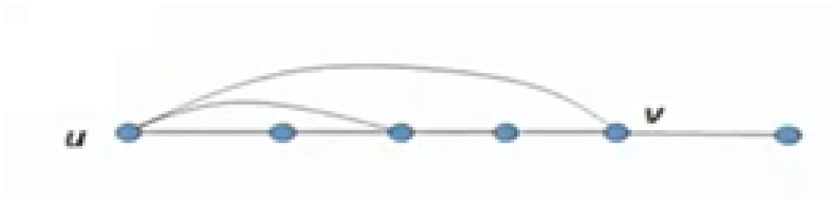
Lemma:

If (G) is a simple graph in which every vertex has degree at least (k), then (G) contains a path of length at least (k).

If (k >=2), then G also contains a cycle of length at least (k+1).

Proof: Let (u) be an endpoint of a maximal path (P) in (G).

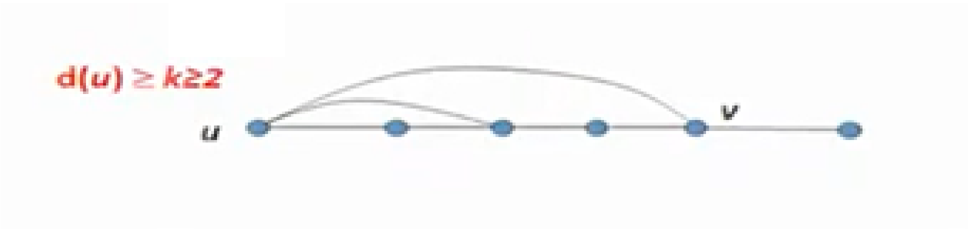
- Since (P) does not extend, every neighbor of u is in V(P).
- Since (u) has at least (k) neighbors and (G) is simple, maximal path P therefore has at least (k) vertices other than (u)
- Hence path length at least (k).



Lemma

If (k >=2), then G also contains a cycle of length at least (k+1).

- If (k >= 2), then the **edge from u to its farthest neighbor v along (P)** completes a sufficiently long cycle (of length k+1) with the portion of P from v to u.



Extremal Problems

- Extremal Problems

- An extremal problem asks for the maximum or minimum value of a function over a class of objects.

- For example, the maximum number of edges in a simple graph with (n) vertices is

$$\boxed{\binom{n}{2} = \frac{n(n-1)}{2}}.$$

Theorem

Theorem: The minimum number of edges in a connected graph with (n) vertices is (n-1)

Proof:

Every graph with n vertices and k edges has at least **(n-k) components**

Hence, every **n-vertex graph** with fewer than (n-1) edges (say, **(n-2) edges**) has at least two **(two)** components and is disconnected.

Add one more edge, make (n-1) edges – graph becomes connected

Every connected (n)-vertex graph has at least (n-1) edges.

Theorem

Theorem: Let G be a graph of order n. If d(u)+d(v) >=(n-1) for every two non-adjacent vertices u and v of G, then G is connected

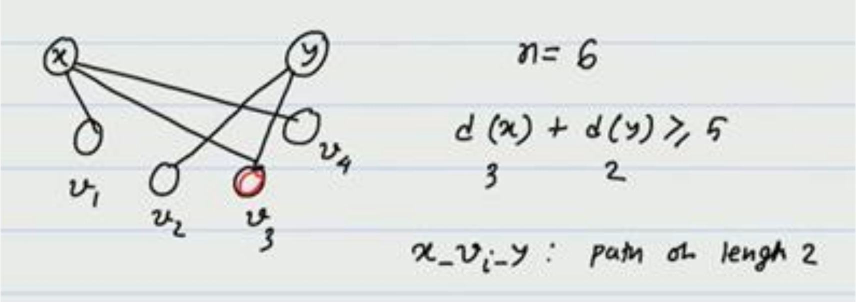
Proof: We need to prove that **every two vertices (x, y) of G is connected by an path**

Let (x, y) € G.

Case: 1 if (x, y) € E, then (x, y) are adjacent nodes

Case 2: if (x, y) does not € E

Here d(x)+d(y)>=n-1 implies that there must be a vertex v_i, which is adjacent to both x and y.



x-v_i-y → Path of length 2

G is connected

Pigeonhole principle: If more than n objects are placed into n boxes, then at least one box must contain two or more objects.

Theorem

If G is a graph of order n with δ(G)>=(n-1)/2, then G is connected

Theorem

If G is a graph of order n with $\delta(G)>=(n-1)/2$, then G is connected

Proof: For every non-adjacent vertices u and v

$d(u)+d(v)=(n-1)$

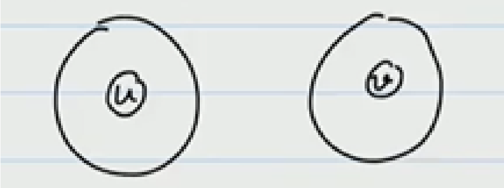
Theorem

Theorem: If G is disconnected graph, then G’ is connected

Proof: if (u,v) € G, we must find a path in G’ connecting (u,v)

Case 1: u and v are in two different components in G

G



u and v are not adjacent in G, so u and v are adjacent in G’

Case 2: u and v are in same component in G.

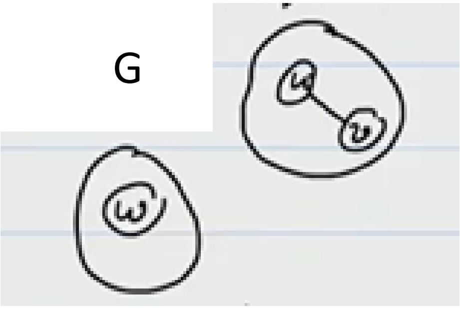
u and v may or may not be adjacent in G

(i) u and v are not adjacent in G =➔ adjacent in G'

(ii) u and v are adjacent in G

Let **w is a vertex in another component** of G

G



In G', w is adjacent to both u, v

In G', **path uwv** connects u and v

Hence G' is connected