Graph Theory: Tutorial

Tutorial

1. Let G be a connected graph with exactly $2k > 0$ odd-degree vertices. Prove that $E(G)$ can be partitioned into exactly k trails, and that fewer than k trails are impossible.

2. Let G be a connected graph in which every vertex has even degree, and suppose $|E(G)|$ is even. Prove that the edges of G can be colored red and blue so that

$$d_R(v) = d_B(v)$$

   for every vertex v.

3. Determine whether the sequence

$$(5, 4, 4, 3, 3, 3, 2)$$

   is graphic. Show all Havel–Hakimi reductions. If it is graphic, construct one realization.

4. Let

$$d_1 \geq d_2 \geq \cdots \geq d_n$$

   be nonnegative integers with even sum, $d_1 \leq n - 1$, and

$$d_1 - d_n \leq 1.$$

   Prove that the sequence is graphic.

5. Let $G = (V, E)$ be connected, with a distinguished vertex r. Every $v \neq r$ chooses a parent $p(v)$ such that $vp(v) \in E(G)$. Suppose there is a function

$$d : V \to \mathbb{N}$$

   with

$$d(r) = 0, \qquad d(p(v)) < d(v) \quad \text{for every } v \neq r.$$

   Let

$$T = \{vp(v) : v \neq r\}.$$

   Prove that T is a spanning tree of G.

6. How many labelled trees on vertex set $[n]$ contain a fixed edge, say $\{1, 2\}$?

7. For the tree with edge set

$$\{12, 13, 34, 35, 56, 57\},$$

   find its Prüfer sequence.

   Then reconstruct the labelled tree corresponding to the Prüfer sequence

$$(3, 3, 5, 5, 2).$$

8. Consider the tree

$$E(T) = \{12, 23, 24, 45, 46, 67, 68, 69\}.$$

Find:

(a) all leaves;

(b) the degree sequence;

(c) a longest path;

(d) the diameter;

(e) the eccentricity of every vertex;

(f) the radius;

(g) the center.