Problem 1: Trail Decomposition

Let G be connected with exactly $2k > 0$ odd-degree vertices.

Prove that $E(G)$ can be partitioned into exactly k trails.

Also prove that fewer than k trails are impossible.

Problem 1: Hint

Goal: turn the graph into an Eulerian graph.

- Pair the $2k$ odd vertices.
- Add one artificial edge for each pair.
- What happens to all degrees?
- Then remove the artificial edges from an Euler circuit.

Problem 1: Solution

Pair the odd vertices and add k artificial edges.

All degrees become even.

The new connected multigraph has an Euler circuit.

Deleting the k artificial edges splits the circuit into k trails.

Problem 1: Minimality

Each trail has at most two odd endpoints.

So t trails can account for at most $2t$ odd-degree vertices.

Since G has $2k$ odd vertices,

$$2t \geq 2k.$$

Therefore $t \geq k$.

Problem 2: Balanced Edge Coloring

Let G be connected.

Every vertex has even degree, and $|E(G)|$ is even.

Prove that the edges can be colored red and blue so that

$$d_R(v) = d_B(v)$$

for every vertex v.

Problem 2: Hint

Use an Euler circuit.

Write its edges in order:

$$e_1, e_2, \ldots, e_m.$$

Color them alternately.

Where is the condition m even used?

Problem 2: Solution

Since all degrees are even, G has an Euler circuit.

Let the circuit be

$$e_1, e_2, \ldots, e_m.$$

Color odd-indexed edges red and even-indexed edges blue.

Problem 2: Local Balance

Each visit to a vertex pairs:

$$\text{one entering edge} \quad + \quad \text{one leaving edge.}$$

Consecutive edges have opposite colors.

At the start vertex, e_m and e_1 also have opposite colors because m is even.

Hence $d_R(v) = d_B(v)$ for every v.

Problem 3: Havel–Hakimi

Determine whether

$$(5, 4, 3, 3, 2, 2, 2, 1)$$

is graphic.

Show all Havel–Hakimi reductions.

If graphic, construct one realization.

Problem 3: Hint

Use Havel–Hakimi.

1. Remove the largest degree.
2. Subtract 1 from the next many entries.
3. Reorder.
4. Repeat.

Stop if a negative number appears.

Problem 3: Reductions

$$(5, 4, 3, 3, 2, 2, 2, 1)$$

$$\to (3, 2, 2, 1, 1, 2, 1)$$

$$\to (3, 2, 2, 2, 1, 1, 1)$$

$$\to (1, 1, 1, 1, 1, 1)$$

$$\to (1, 1, 1, 1, 0) \to (1, 1, 0, 0) \to (0, 0, 0).$$

So the sequence is graphic.

Problem 3: One Realization

Use vertices $1, 2, \ldots, 8$.

One realization has edge set

$$\{12, 13, 14, 15, 16, 23, 24, 27, 34, 57, 68\}.$$

Degrees are

$$5, 4, 3, 3, 2, 2, 2, 1.$$

Problem 4: Near-Regular Sequences

Let

$$d_1 \geq d_2 \geq \cdots \geq d_n$$

be nonnegative integers with even sum.

Assume

$$d_1 \leq n-1, \qquad d_1 - d_n \leq 1.$$

Prove that the sequence is graphic.

Problem 4: Hint

Use induction with Havel–Hakimi.

The sequence has only two possible values:

$$r \quad \text{and} \quad r-1.$$

After one Havel–Hakimi reduction, check the same property again.

Problem 4: Solution Skeleton

Apply Havel–Hakimi to a largest entry r.

Delete r and reduce the next r entries.

After reordering:

- entries are still nonnegative;
- the sum is still even;
- maximum and minimum differ by at most 1.

Induct on the length of the sequence.

Problem 5: Parent Pointers

Let $G = (V, E)$ be connected with root r.

Each $v \neq r$ chooses a parent $p(v)$ with $vp(v) \in E(G)$.

Suppose there is $d : V \to \mathbb{N}$ such that

$$d(r) = 0, \qquad d(p(v)) < d(v).$$

Let

$$T = \{vp(v) : v \neq r\}.$$

Prove that T is a spanning tree.

Problem 5: Hint

Two things to prove:

1. Every vertex reaches r by following parents.
2. There is no cycle.

For acyclicity, choose a vertex of maximum d on a hypothetical cycle.

Problem 5: Connectedness

Start at any vertex v.

Follow parent pointers:

$$v,\ p(v),\ p^2(v),\ldots$$

The d-values strictly decrease.

So the process must stop.

It stops only at r.

Hence every vertex is connected to r in T.

Problem 5: Acyclicity

Suppose T has a cycle C.

Choose $x \in C$ with maximum $d(x)$.

On the cycle, one incident edge must come from a child y with

$$p(y) = x.$$

Then

$$d(x) < d(y),$$

contradicting maximality of $d(x)$.

So T is connected and acyclic.

Problem 6: Fixed Edge in a Labelled Tree

How many labelled trees on vertex set $[n]$ contain a fixed edge, say $\{1, 2\}$?

Problem 6: Hint

Use symmetry.

Cayley gives n^{n-2} labelled trees.

Each tree has $n - 1$ edges.

All $\binom{n}{2}$ possible edges occur equally often.

Problem 6: Solution

Total tree-edge incidences:

$$(n-1)n^{n-2}.$$

Divide equally among all possible edges:

$$\binom{n}{2}.$$

So a fixed edge occurs in

$$\frac{(n-1)n^{n-2}}{\binom{n}{2}} = 2n^{n-3}$$

trees.

Problem 7: Prüfer Code

For the tree with edge set

$$\{12, 13, 34, 35, 56, 57\},$$

find its Prüfer sequence.

Then reconstruct the tree with Prüfer sequence

$$(3, 3, 5, 5, 2).$$

Problem 7: Hint

Forward direction:

- Delete the smallest leaf.
- Record its neighbour.

Reverse direction:

- Take the smallest label absent from the current sequence.
- Join it to the first sequence entry.
- Delete that first entry.

Problem 7: Forward Solution

Tree edges:

$$\{12, 13, 34, 35, 56, 57\}.$$

Deleted leaves:

$$2, 1, 4, 3, 6.$$

Recorded neighbours:

$$1, 3, 3, 5, 5.$$

So the Prüfer sequence is

$$(1, 3, 3, 5, 5).$$

Problem 7: Reverse Solution

For sequence $(3, 3, 5, 5, 2)$, the reconstructed edges are

$$13,\ 43,\ 35,\ 65,\ 52,\ 27.$$

So the tree edge set is

$$\{13, 34, 35, 56, 25, 27\}.$$

Problem 8: Tree Structure

Consider the tree

$$E(T) = \{12, 23, 24, 45, 46, 67, 68, 69\}.$$

Find leaves, degree sequence, longest path, diameter, eccentricities, radius, and center.

Problem 8: Hint

Draw the tree first.

Central chain:

$$1 - 2 - 4 - 6 - 7$$

Branches:

$$3 \text{ at } 2, \quad 5 \text{ at } 4, \quad 8, 9 \text{ at } 6.$$

The center is near the middle of a longest path.

Problem 8: Basic Data

Leaves:

$$\{1, 3, 5, 7, 8, 9\}.$$

Degrees:

$$d(6) = 4, \quad d(2) = d(4) = 3,$$

all leaves have degree 1.

Degree sequence:

$$(4, 3, 3, 1, 1, 1, 1, 1, 1).$$

Problem 8: Distances

One longest path is

$$1 - 2 - 4 - 6 - 7.$$

So the diameter is

$$4.$$

Eccentricities:

$$\epsilon(4) = 2,$$

$$\epsilon(2) = \epsilon(5) = \epsilon(6) = 3,$$

$$\epsilon(1) = \epsilon(3) = \epsilon(7) = \epsilon(8) = \epsilon(9) = 4.$$

Problem 8: Radius and Center

The radius is the minimum eccentricity.

$$\mathrm{rad}(T) = 2.$$

Only vertex 4 has eccentricity 2.

So the center is

$$\{4\}.$$