Problem 1

Suppose a simple graph G on n vertices is isomorphic to its complement $\overline{G}$.

Show that

$$n \equiv 0 \text{ or } 1 \pmod{4}.$$

Problem 1: Hint

Suppose $G \cong \overline{G}$.

Hint.
Compare:

$$e(G), \qquad e(\overline{G}), \qquad \binom{n}{2}.$$

Use that isomorphic graphs have the same number of edges.

Problem 1: Solution

Since $G \cong \overline{G}$,

$$e(G) = e(\overline{G}).$$

Also,

$$e(G) + e(\overline{G}) = \binom{n}{2}.$$

Hence

$$e(G) = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}.$$

Problem 1: Solution

Now $e(G)$ is an integer, so

$$4 \mid n(n-1).$$

Since n and $n-1$ are consecutive,

$$n \equiv 0 \text{ or } 1 \pmod 4.$$

Problem 2

Let A be the adjacency matrix of a simple graph G.

Show that $(A^k)_{ij}$ equals the number of walks of length k from v_i to v_j.

Deduce that the number of triangles in G is

$$\frac{\mathrm{tr}(A^3)}{6}.$$

Problem 2: Hint

Show that $(A^k)_{ij}$ counts length-k walks.

Hint.
Induct on k.

For the last step of a walk:

$$(A^{k+1})_{ij} = \sum_r (A^k)_{ir} A_{rj}.$$

Problem 2: Solution

Base case:

$$A_{ij}$$

counts walks of length 1.

Induction step:

$$(A^{k+1})_{ij} = \sum_r (A^k)_{ir} A_{rj}.$$

This chooses the penultimate vertex v_r.

Problem 2: Solution

So $(A^3)_{ii}$ counts closed walks of length 3 starting at v_i.

Each triangle is counted:

$$3 \text{ starting vertices} \times 2 \text{ directions} = 6.$$

Therefore

$$\#\text{triangles} = \frac{\text{tr}(A^3)}{6}.$$

Problem 3

Let G be connected and let $e \in E(G)$.

Prove that e is a cut-edge if and only if every closed walk in G uses e an even number of times.

Problem 3: Hint

Prove both directions.

Hint.
If e is a cut-edge, then $G - e$ has two components.

If e is not a cut-edge, then e lies on a cycle.

Problem 3: Solution

Assume e is a cut-edge.
Then $G - e$ has two components, say X and Y.

A closed walk crossing from X to Y using e must later cross back using e.

Thus e is used an even number of times.

Problem 3: Solution

Conversely, suppose e is not a cut-edge.
Then e lies on a cycle.

Traversing that cycle once gives a closed walk using e exactly once.

So the stated property fails.

Hence e is a cut-edge.

Problem 4

Show that every graph G has a spanning bipartite subgraph containing at least half of the edges of G.

Problem 4: Hint

Find a large bipartite subgraph.

Hint.
Choose a partition

$$V(G) = X \cup Y$$

maximizing the number of crossing edges.

Ask: what happens if one vertex has more edges inside its part than across?

Problem 4: Solution

Choose X, Y maximizing the number m of crossing edges.

For every vertex v:

$$\#\text{crossing edges at } v \geq \frac{d(v)}{2}.$$

Otherwise, move v to the other part.

Problem 4: Solution

Sum over all vertices:

$$2m \geq \sum_v \frac{d(v)}{2}.$$

Using handshaking,

$$\sum_v d(v) = 2e(G).$$

So

$$2m \geq e(G), \qquad m \geq \frac{e(G)}{2}.$$

Keep only the crossing edges.

Problem 5

Let G have three independent vertex classes A, B, C, each of size n.

Suppose every vertex has degree at least $n+1$.

Prove that G contains a triangle.

Problem 5: Hint

Prove by contradiction.

Hint.
Assume no triangle exists.

Let p be the smallest number of neighbours a vertex has in one of the other two classes.

Use a vertex attaining p.

Problem 5: Solution

Assume no triangle.
Let p be the minimum cross-degree.

Then $p \geq 1$.
Otherwise some vertex has neighbours in only one other class, so degree at most n.

Problem 5: Solution

Suppose $a \in A$ has exactly p neighbours in C.
Since $d(a) \geq n+1$,

$$d_B(a) \geq n+1-p.$$

Choose $c \in C$ adjacent to a.

Problem 5: Solution

No neighbour of a in B can be adjacent to c.
Otherwise we get a triangle:

$$a - b - c - a.$$

So

$$d_B(c) \leq n - d_B(a).$$

Problem 5: Solution

Using $d_B(a) \geq n + 1 - p$:

$$d_B(c) \leq n - (n + 1 - p) = p - 1.$$

But p was the minimum cross-degree.
Contradiction.
Therefore G contains a triangle.