

Network Flow

Flow Network

A flow network $G = (V, E)$ is a directed graph where each edge $(u, v) \in E$ has a capacity $c(u, v) > 0$. A source vertex s and a sink vertex t .

Def. A flow f in G is a real valued function $f : V \times V \rightarrow \mathbb{R}$ such that:

1. Capacity Rule

$$f(u, v) \leq c(u, v)$$

that is, flow on an edge cannot exceed the capacity of the edge.

2. Skew Symmetry

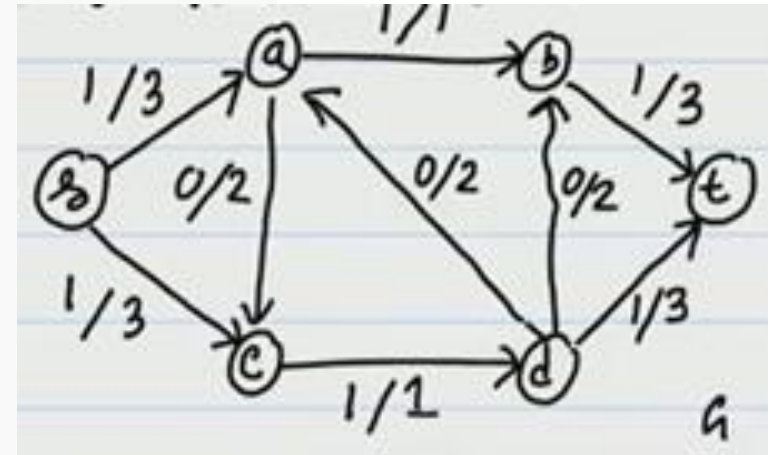
$$f(u, v) = -f(v, u)$$

3. Flow Conservation

for every vertex $v \notin \{s, t\}$

$$\sum_{u \in V} f(u, v) = \sum_{w \in V} f(v, w)$$

Amount of flow into a node equals the amount of flow out of it.



Flow Network

Problem

Given a flow network G , find maximum flow from s to t .

The total amount of flow from s to t is 2.

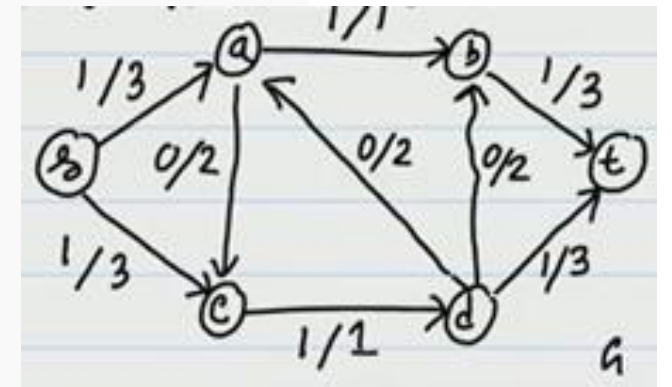
Total outgoing flow from s is 2.

Total incoming flow to t is 2.

The value of a flow f is

$$|f| = \sum_{u \in V} f(s, u) = \sum_{u \in V} f(u, t)$$

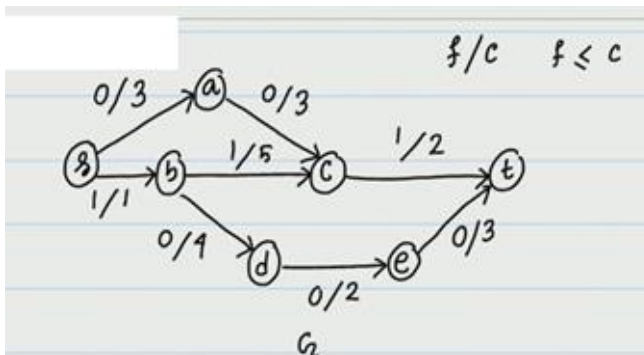
The total flow from the source is the same as the total flow into sink.



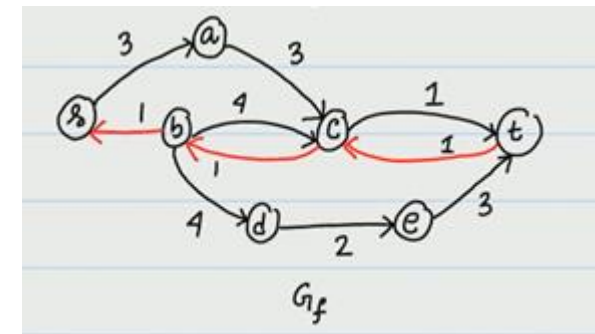
Residual Network & Augmenting path

Residual Network

- Consider an **arbitrary flow f** in a network G .
- The **residual network G_f** has the **same vertices as the original network**,
- **One or two edges** for each edge (u,v) in the original network.
- If $f(u,v) < c(u,v)$ then there is a **forward edge (u,v)** with capacity $c_f(u,v) = c(u,v) - f(u,v)$
- If $f(u,v) > 0$, there is a **backward edge (v,u)** with capacity $c_f(v,u) = f(u,v)$.



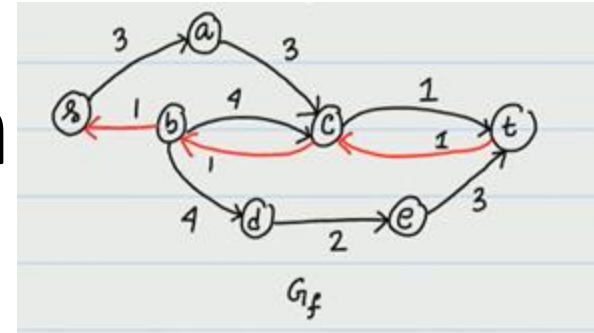
Flow network



Residual network

Residual Network & Augmenting path

Residual network



- An **augmenting path** is simply a path from **source s** to **sink t** in the residual network, **whose purpose is to increase the flow**.

$$P = (s, a, c, t)$$

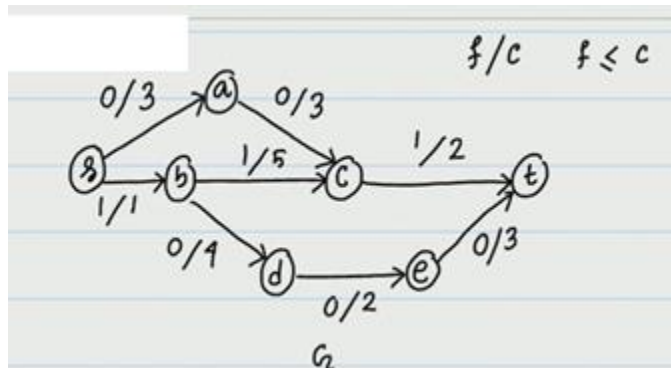
- The **path capacity (tolerance)** of a path is the **min capacity on an edge along that path**.

$$\delta(P) = \min \{3, 3, 1\} = 1$$

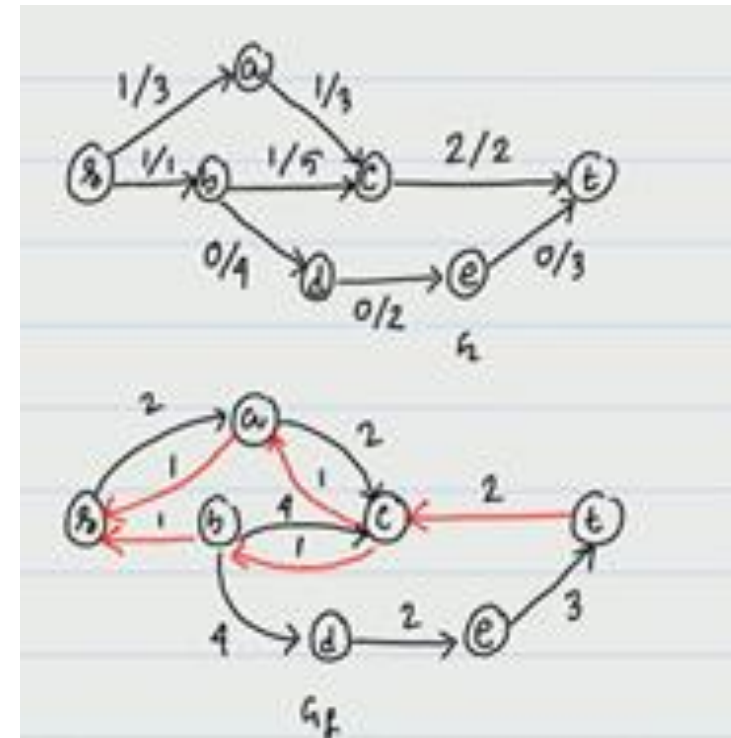
- By considering the augmenting path $P = (s, a, c, t)$, we can **increase the flow by $\delta(P) = 1$ unit** as the path capacity $\delta(P) = 1$.

Residual Network & Augmenting path

- Increasing the flow along the path P with 1 unit yields the following flow:



Flow network

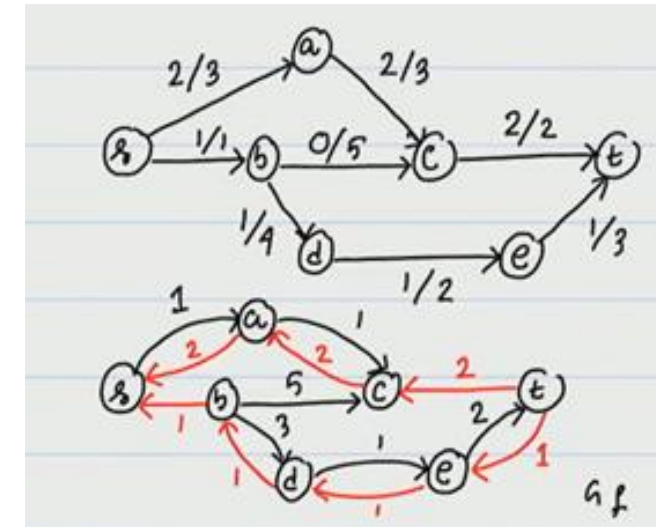
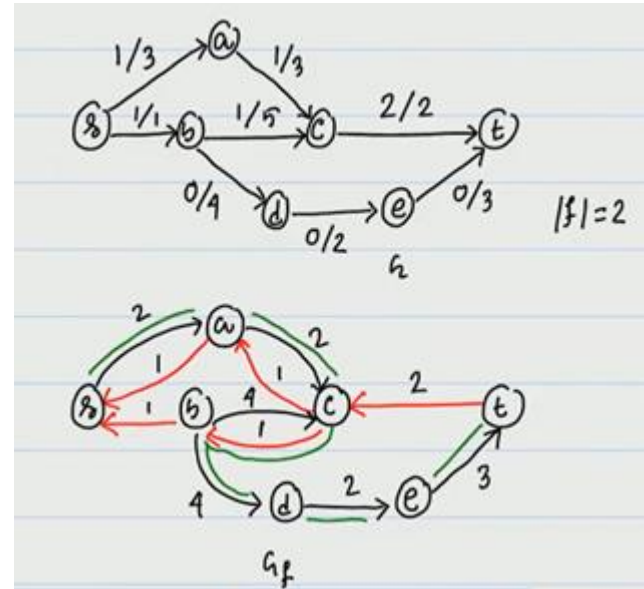


New Flow network

Residual network

Residual Network & Augmenting path

- **Augmenting Path Step**
- By considering the augmenting path
- $P = (s, a, c, b, d, e, t)$
- We can increase the flow by
- $\delta(p) = \min\{2, 2, 1, 4, 2, 3\} = 1$ unit
- Flow after augmenting 1 unit of flow:
- Augmenting path does not exist in this residual network. **STOP**



Discount flow for backward edges

Augmenting path Algorithm

Augmenting Path Algorithm

Ford & Fulkerson, 1956

- **Input:** Network G
- **Output:** Max flow f

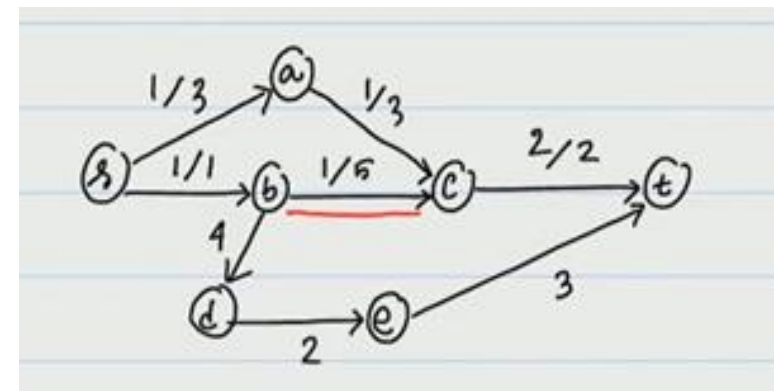
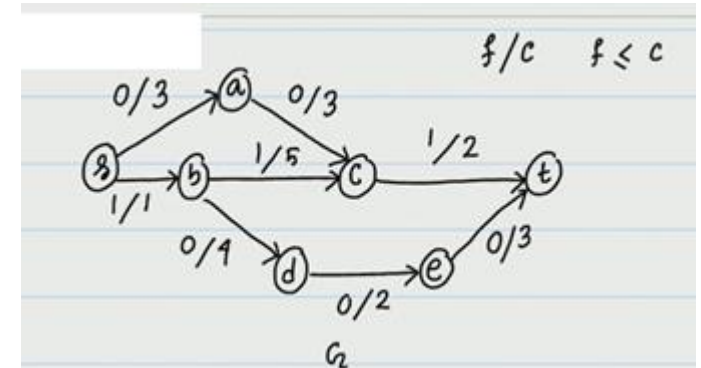
$$f = 0$$

```
while ( $G_f$  contains an augmenting path  $p$  in  $G_f$ ) {  
    Identify an augmenting path  $p$  in  $G_f$ ;  
     $\delta = \min\{c_f(u, v) \mid (u, v) \in p\}$ ;  
    augment  $\delta$  units flow along  $p$  & update  $G_f$ ;  
}
```

Recover max flow from the final residual network.

Importance of Residual network

- Augmenting Paths
- $p1 = \{s, b, c, t\}$; $\delta(p1) = 1$
- $p2 = \{s, a, c, t\}$; $\delta(p2) = 2$
- $|f| = 2$



Define (s,t)-Cut

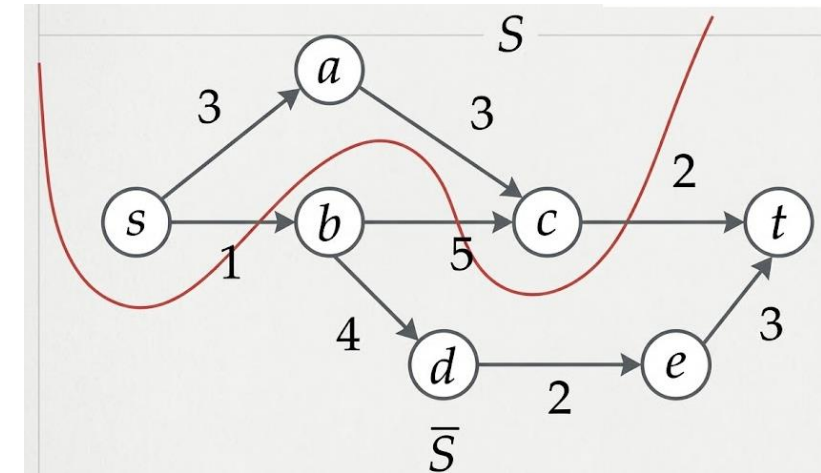
A **cut** $(S, \bar{S})$ partitions the vertex set V into two subsets $S \subseteq V$ & $\bar{S} = V - S$, and it contains all edges with one endpoint in S and the other in $\bar{S}$.

Ex $S = \{s, a, c\}; \bar{S} = \{b, d, e, t\}$

The cut $(S, \bar{S})$ is an **s-t cut** if $s \in S$ and $t \in \bar{S}$.

The **capacity** of the **s-t cut** is

$$\begin{aligned} C(S, \bar{S}) &= \sum_{u \in S, v \in \bar{S}} c(u, v) \\ C(S, \bar{S}) &= C(c, t) + C(s, b) \\ &= 2 + 1 = 3 \end{aligned}$$



(S, S') cut = $\{(s, b), (c, t), (b, c)\}$

Net flow across the cut (S, S')

If f is a flow in G , then the **net flow across the cut** $(S, \bar{S})$ is defined to be

$$f(S, \bar{S}) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$

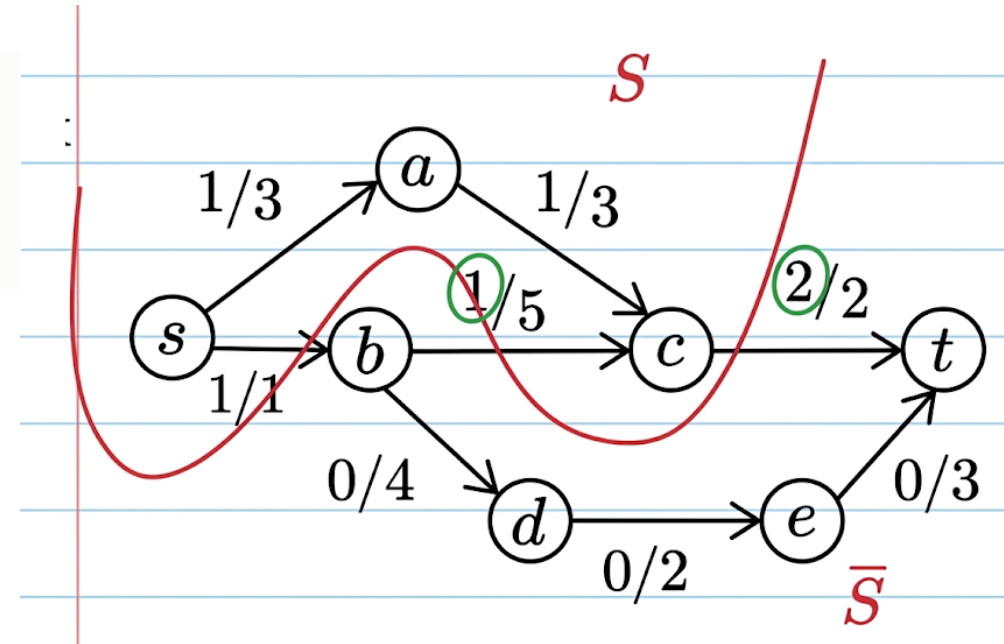
$$S = \{s, a, c\} \quad \bar{S} = \{b, d, e, t\}$$

$$f(S, \bar{S}) = f(s, b) + f(c, t) - f(b, c)$$

$$= 1 + 2 - 1$$

$$= 2$$

flow value $|f| = 2$



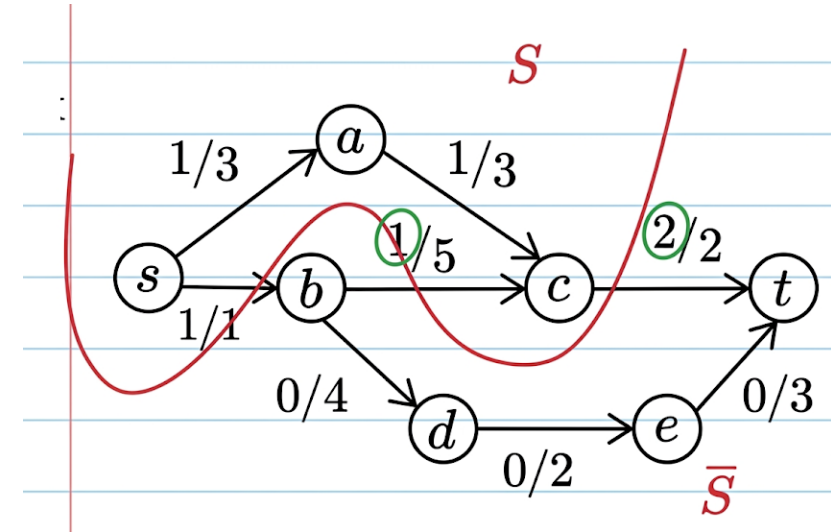
Lemma

Lemma 1 Let f be a flow in a flow network G with source s and sink t , and let $(S, \bar{S})$ be an $s-t$ cut of G . Then the net net flow across $(S, \bar{S})$ is $|f| = f(S, \bar{S})$.

Proof

$$\begin{aligned}
 |f| &= \sum_{e \text{ out of } s} f(e) \\
 &= \sum_{v \in S} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right)
 \end{aligned}$$

$$\begin{aligned}
 S &= \{a, c, s\} \\
 \bar{S} &= \{b, d, e, t\}
 \end{aligned}$$



Lemma

If f is a flow in G , then the **net flow across the cut** $(S, \bar{S})$ is defined to be

$$f(S, \bar{S}) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$

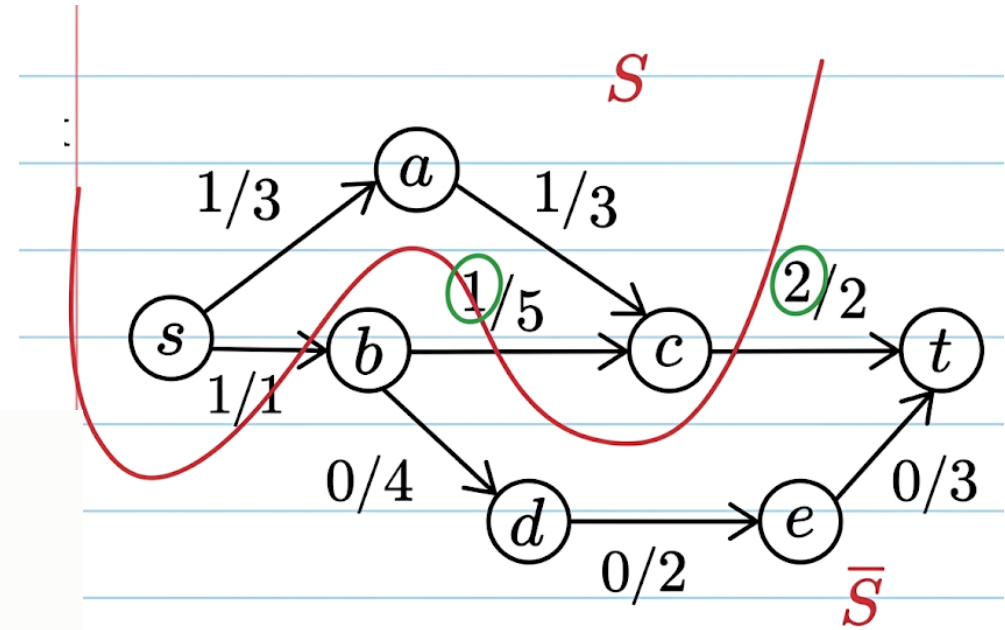


Illustration of Lemma 1

$$\begin{aligned} |f| &= \sum_{e \text{ out of } s} f(e) = f(s, a) + f(s, b) \\ &= [f(s, a) + f(s, b)] + [f(a, c) - f(s, a)] + [f(c, t) - f(a, c) - f(b, c)] \\ &= f(s, b) + f(c, t) - f(b, c) = f(S, \bar{S}) \end{aligned}$$

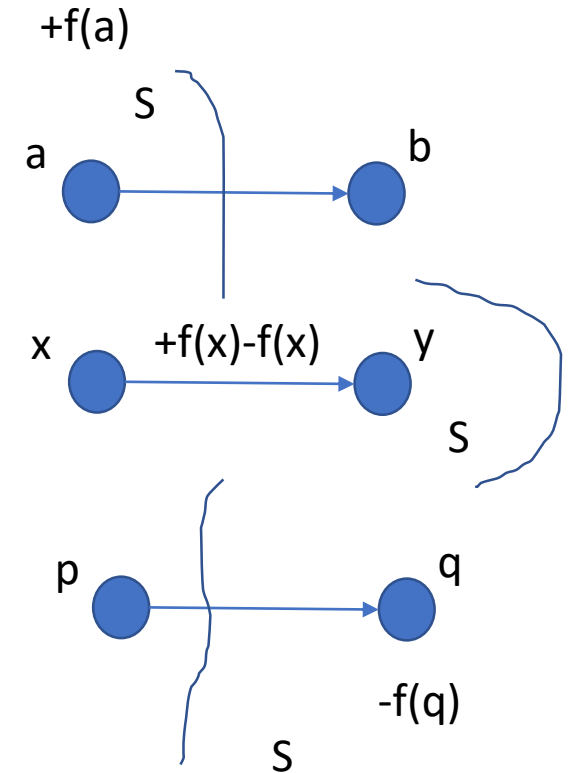
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$$\begin{aligned}
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 &= \sum_{v \in S} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right) \\
 &= \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) = f(S, \bar{S})
 \end{aligned}$$

$$\begin{aligned}
 S &= \{a, c, s\} \\
 v &= a, c, s
 \end{aligned}$$



Few Lemmas

Lemma 1: $|f| = f(S, \bar{S})$.

Lemma 2: Let f be any flow and let $(S, \bar{S})$ be any s - t cut. Then

$$|f| \leq C(S, \bar{S})$$

Proof:

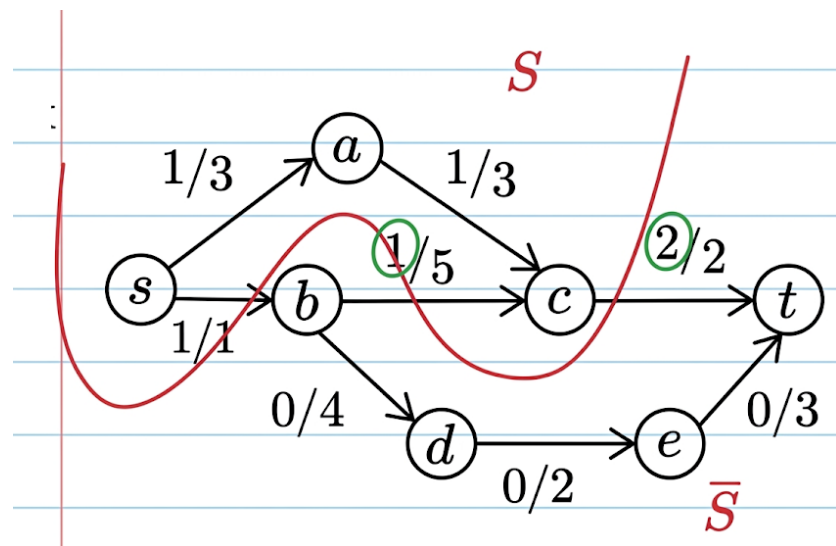
$$\begin{aligned} |f| &= \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) \\ &\leq \sum_{e \text{ out of } S} f(e) \quad \text{① Capacity Rule: } f(e) \leq c(e) \\ &\leq \sum_{e \text{ out of } S} c(e) = C(S, \bar{S}) \end{aligned}$$

Ex: $S = \{s, a, c\}$

$\bar{S} = \{b, d, e, t\}$

$$C(S, \bar{S}) = 3, \quad f(S, \bar{S}) = 2 = |f|$$

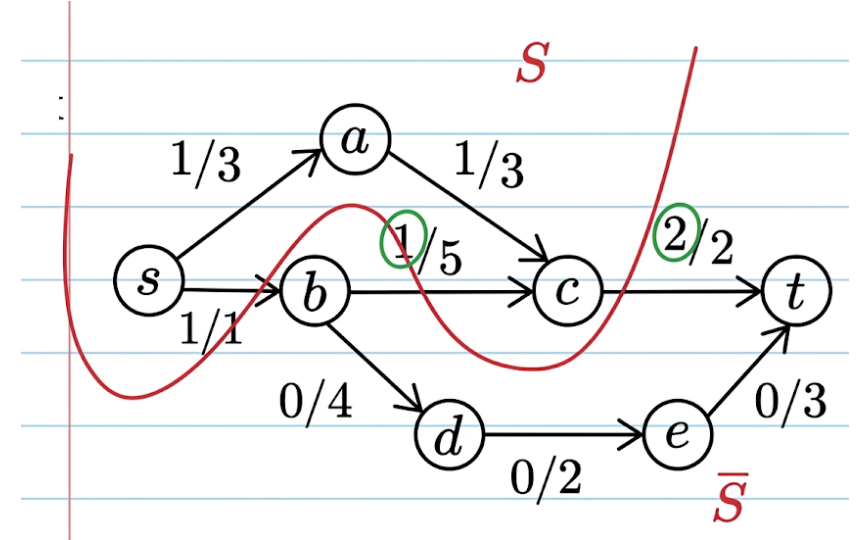
$$|f| < C(S, \bar{S})$$



Few Lemmas

Lemma 3: Let f be any flow in G and $(S, \bar{S})$ be any s - t cut.

If $|f| = C(S, \bar{S})$, then f is **max flow** & $(S, \bar{S})$ is **min-cut**.



Max-Flow Min Cut

Max-Flow Min-Cut Theorem

If f is a flow in a flow network G with source s & sink t , then the following are equivalent:

- (A) f is a max-flow in G
- (B) The residual network G_f contains no augmenting path.
- (C) $|f| = C(S, \bar{S})$ for some cut $(S, \bar{S})$.

Proof Strategy

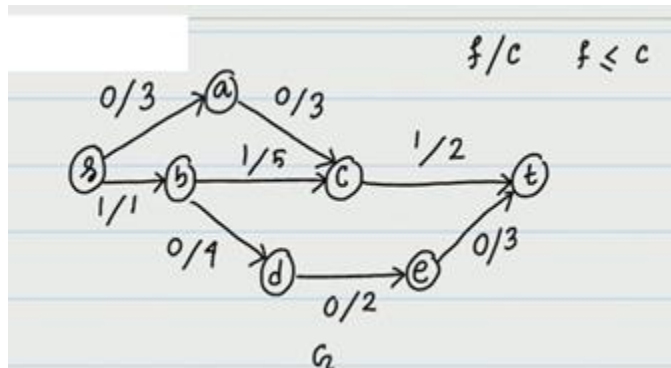
$$(A) \implies (B) \implies (C) \implies (A)$$

- $(A) \implies (B)$:

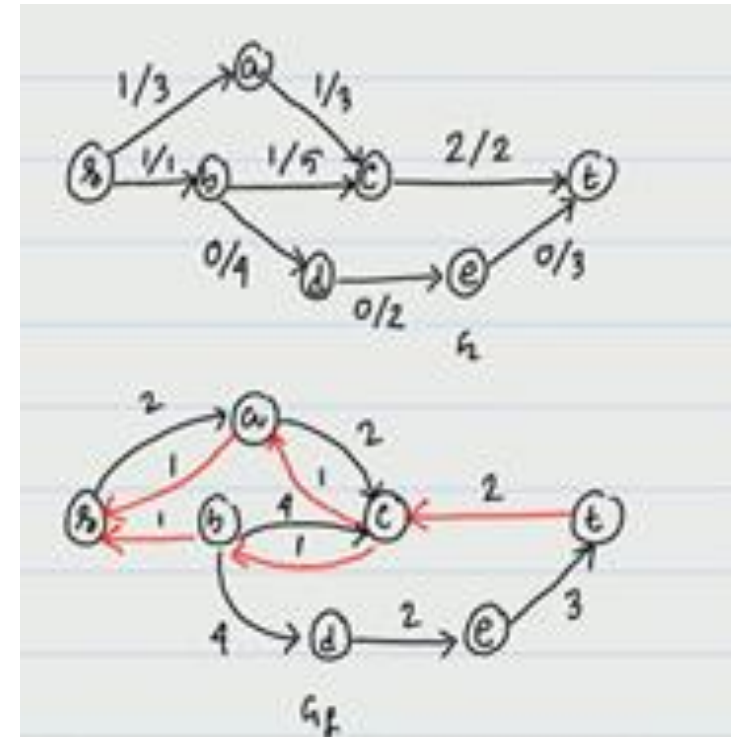
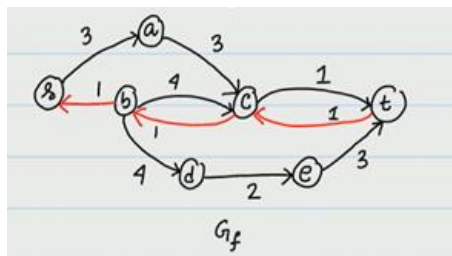
Suppose for the sake of contradiction that f is a max flow in G , but G_f has an augmenting path P . Then we can augment $\delta(P) > 0$ units of flow along P , obtaining a larger flow value. This contradicts the assumption that f is a max-flow.

Residual Network & Augmenting path

- Increasing the flow along the path p with 1 unit yields the following flow:



Flow network



New Flow network

Residual network

Max-Flow Min Cut

$$(B) \implies (C) \quad |f| = c(S, \bar{S})$$

Suppose G_f has no augmenting path.

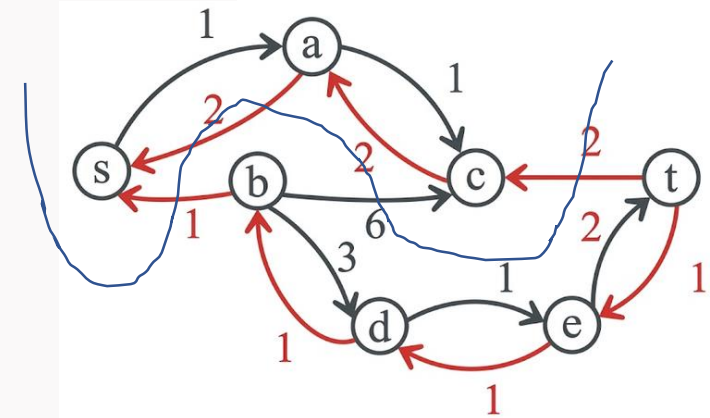
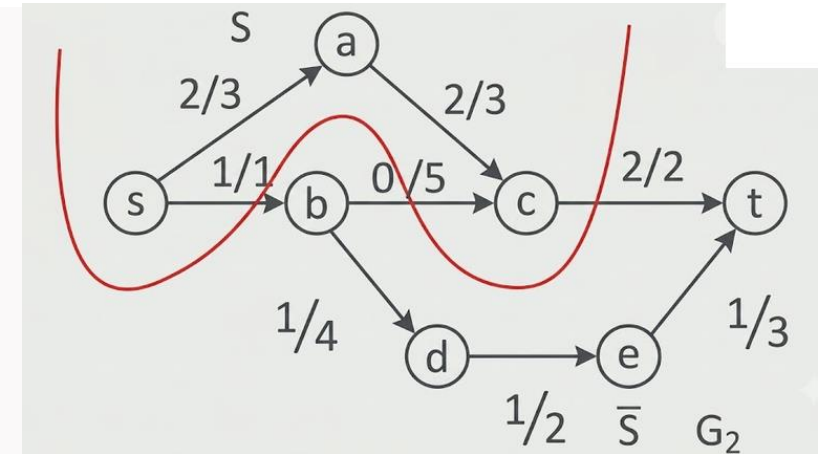
Define

$$S = \{v \in V : \text{there exists a path from } s \text{ to } v \text{ in } G_f\}$$

$$\text{and } \bar{S} = V \setminus S.$$

$(S, \bar{S})$ is an $s - t$ cut as $s \in S$ & $t \in \bar{S}$.

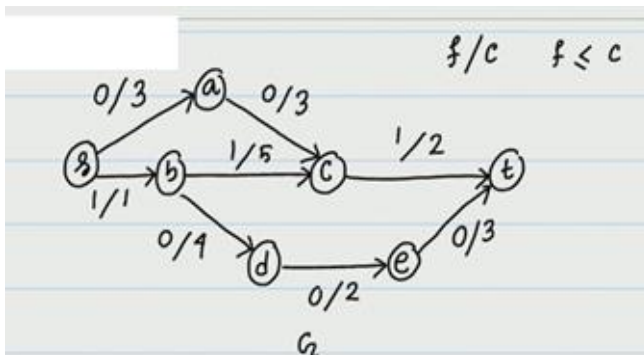
- For each edge $(u, v) \in (S, \bar{S})$, we have $f(u, v) = c(u, v)$.
Otherwise, $(u, v) \in E(G_f)$, which would place v in S .
- For each edge $(v, u) \in (\bar{S}, S)$, we must have $f(v, u) = 0$.
Otherwise, we would have $(u, v) \in E(G_f)$ with capacity $c_f(u, v) = f(v, u) > 0$, which would place v in S .



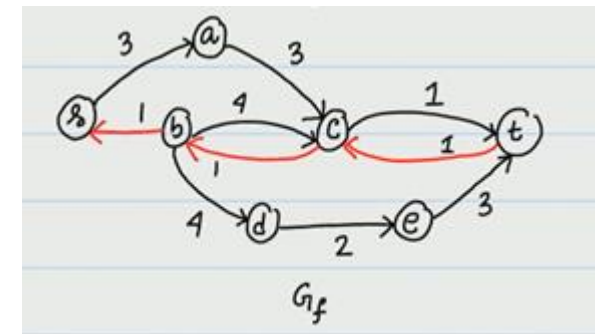
Residual Network & Augmenting path

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- Consider an **arbitrary flow f** in a network G .
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- If $f(u,v) < c(u,v)$ then there is a **forward edge (u,v)** with capacity $c_f(u,v) = c(u,v) - f(u,v)$
- If $f(u,v) > 0$, there is a **backward edge (v,u)** with capacity $c_f(v,u) = f(u,v)$.



Flow network



Residual network

Max-Flow Min Cut

Therefore

$$\begin{aligned} |f| &= \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) && \text{Using Lemma 1: } |f| = f(S, \bar{S}) \\ &= \sum_{e \text{ out of } S} f(e) \\ &= \sum_{e \text{ out of } S} c(e) && (v, u) \in (\bar{S}, S) \implies f(v, u) = 0 \\ &= C[S, \bar{S}] && (u, v) \in (\bar{S}, S), \quad f(u, v) = c(u, v) \end{aligned}$$

(C) $\Rightarrow$ (A)

By Lemma 2, $|f| \leq C(S, \bar{S})$ for all cuts $(S, \bar{S})$.

The condition $|f| = C(S, \bar{S})$ thus implies f is **max**.

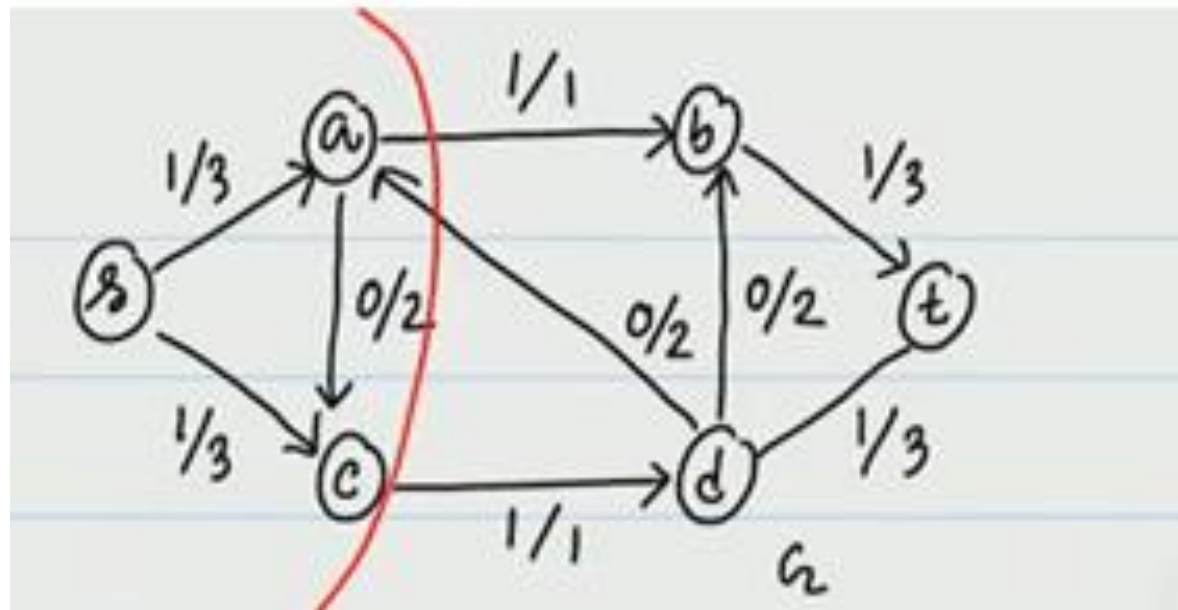
Now suppose (C) holds: $|f| = C(S, \bar{S})$ for some specific cut $(S, \bar{S})$. Combine this with weak duality applied to *every other* cut $(S', \bar{S}')$:

$$C(S, \bar{S}) = |f| \leq C(S', \bar{S}') \quad \text{for all cuts } (S', \bar{S}')$$

Max-Flow Min Cut

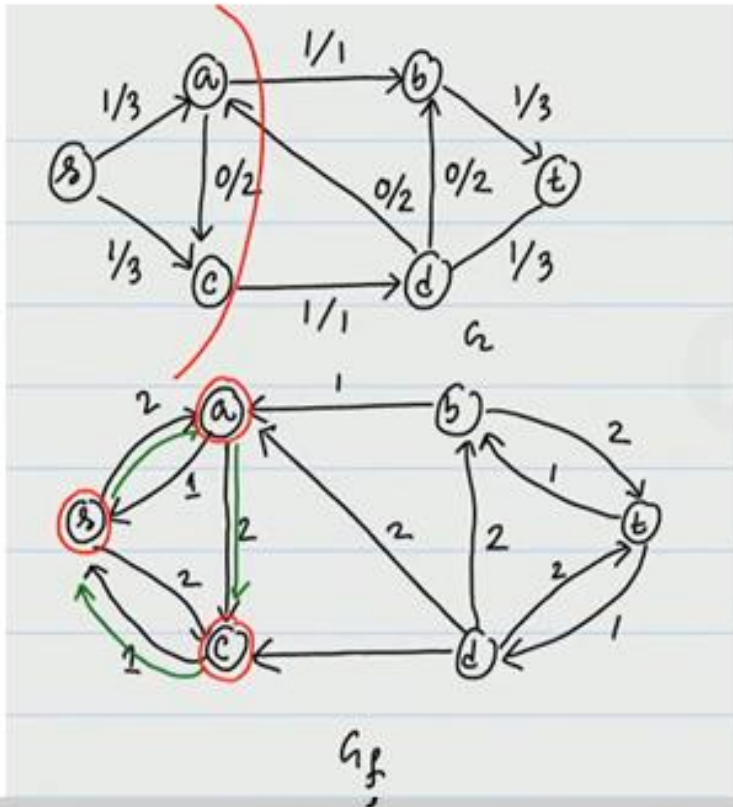
$$|f| = C(S, \bar{S}) \implies \begin{cases} |f'| \leq |f| \text{ for all flows } f' \\ C(S, \bar{S}) \leq C(S', \bar{S}') \text{ for all cuts } (S', \bar{S}') \end{cases} \implies \begin{array}{l} f \text{ is max-flow (A)} \\ (S, \bar{S}) \text{ is min-cut} \end{array}$$

Example



Example

Ex.



NO augmenting path in G_f

Max flow

$$|f| = 2$$

$$S = \{s, a, c\} \quad ; \quad \bar{S} = \{b, d, e\}$$

$$\begin{aligned} c(s, \bar{S}) &= c(a, b) + c(c, d) \\ &= 1 + 1 = 2 \end{aligned}$$

$$c(s, \bar{S}) = |f| = 2$$