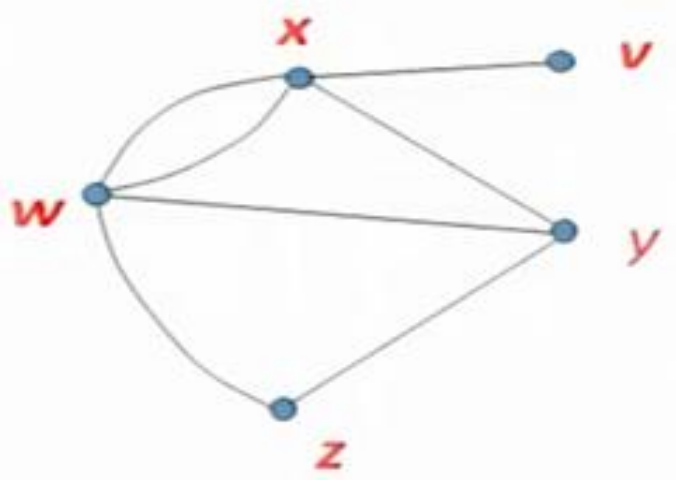
Graphical sequence

Degree sequence

- The ***Degree Sequence*** of a graph is the list of vertex degrees, *usually* written in non-increasing order, as $d_1 \geq .... \geq d_n$.

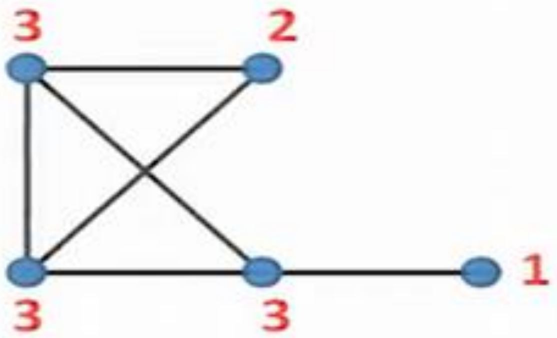
- Example:



Degree sequence:

$d(w), d(x), d(y), d(z), d(v)$

Example



Degree sequence:

1, 3, 3, 3, 2

Or 2, 3, 3, 3, 1

Or 1, 2, 3, 3, 3

etc

Given a graph G, it is easy to find a degree sequence

But given a sequence of non-negative integers, when is it a degree sequence of some graph ?

Theorem

Theorem: The non negative integers d_1, d_2,…, d_n are the vertex degrees of some graph if and only if sum of the degrees is even

Proof: *Necessity* (only if)

- When some graph G has these numbers $d_1, \ldots, d_n$ as its vertex degrees, the degree-sum formula implies that $\sum d_i = 2e(G)$, which is even.

Theorem

Proof: ***Sufficiency*** **(if)**

- Suppose that $\sum d_i$ is even.
- We construct a graph with vertex set $v_1, \ldots, v_n$ and $d(v_i) = d_i$ for all i.
- Since $\sum d_i$ is even, the number of odd values is even.
- First form an arbitrary pairing of the vertices in $\{v_i : d_i \text{ is odd}\}$.
- For each resulting pair, form an edge having these two vertices as its endpoints
- The remaining degree needed at each vertex is even and nonnegative; satisfy this for each i by placing $[d_i/2]$ loops at v_i.

Graphic sequence

- A *graphic sequence* is a list of nonnegative numbers that is the degree sequence of some **simple** graph.

- A simple graph "realizes" *d*.
    - means: A simple graph with degree sequence *d*.

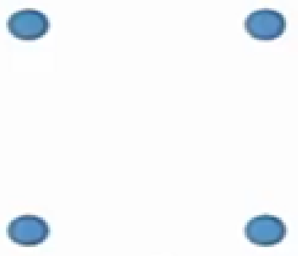
Example:

(i) 1 1 3 – Not graphical sequence since every graph has an even number of odd degree vertices

(ii) 3 3 – Not graphical sequence

Obvious conditions

- $\sum d_i$ must be even
- $d_i \leq n\text{-}1$ for all $i=1,\ldots.,n$
- Although these conditions are necessary, they are not sufficient.
- Ex: 3,3,3,1 $n= 4$

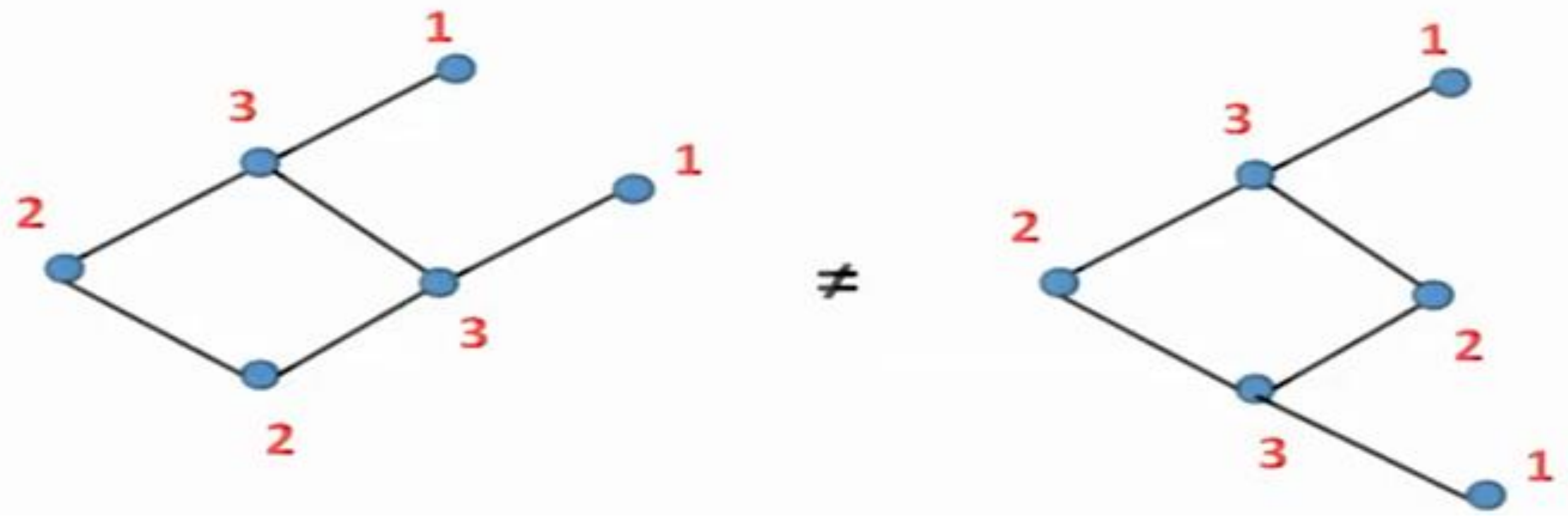


$n = 4$
$\sum d_i = 10$
$d_i \leq 3$

But the sequence is not graphical

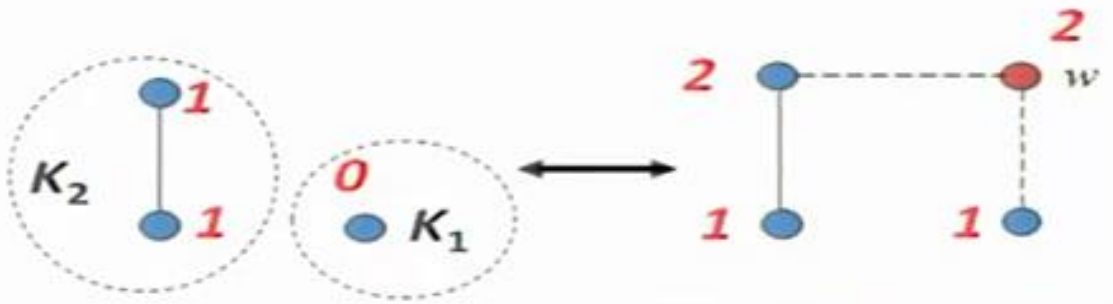
Observations

- A graphic sequence may be a degree sequence for more than one graph.
- Example: 3 3 2 2 1 1



Recursive condition

- The lists (2, 2, 1, 1) and (1, 0, 1) are graphic. The graphic K_2+K_1 realizes 1, 0, 1.
- Adding a new vertex adjacent to vertices of degrees 1 and 0 yields a graph with degree sequence 2, 2, 1, 1, as shown below.
- Conversely, if a graph realizing 2, 2, 1, 1 has a vertex w with neighbors of degrees 2 and 1, then deleting w yields a graph with degrees 1, 0, 1.



Recursive condition

- Similarly, to test 33333221, we seek a realization with a vertex *w* of degree 3 having three neighbors of degree 3.

3 3 3 3 3 2 2 1

Delete this Vertex

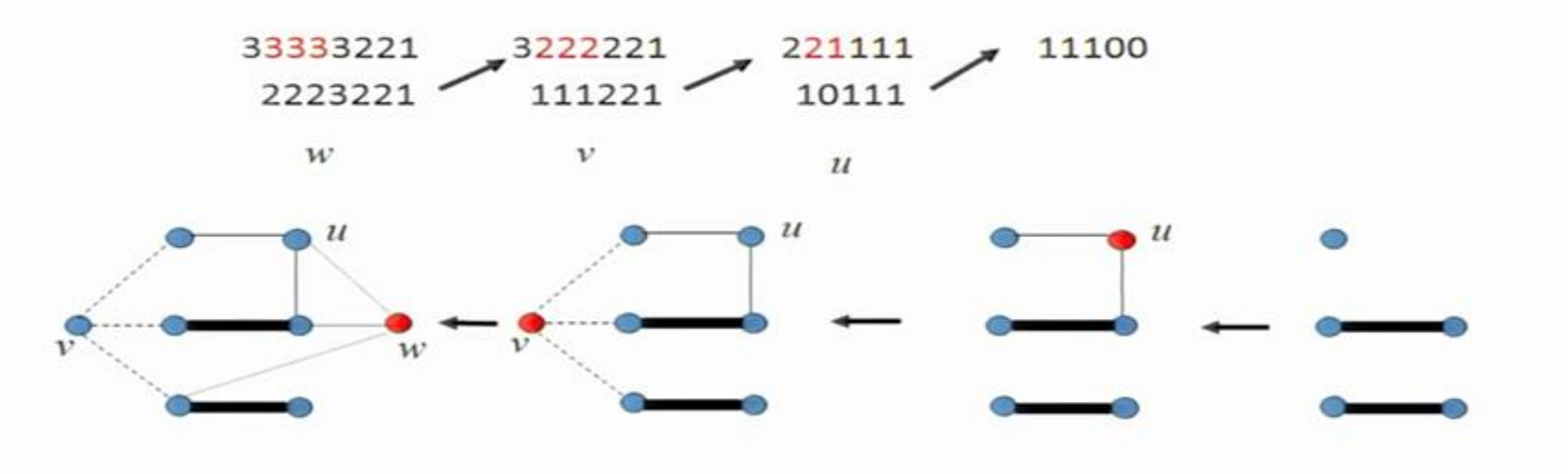
A new degree sequence

2 2 2 3 2 2 1

Recursive condition

- This exists if and only if 2223221 is graphic.
  - We reorder this and test 3222221.
  - We continue deleting and reordering until we can tell whether the remaining list is realizable.
  - If it is, then we insert vertices with the desired neighbors to walk back to a realization of the original list.
  - The realization is not unique.
- The next theorem implies that this recursive test works.

Recursive condition



Havel and Halimi Theorem

Theorem. For $n>1$, an integer list d of size n is graphic if and only if d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1-element graphic sequence is $d_1=0$.

Proof: For $n=1$, the statement is trivial.

- For $n>1$, we first prove that the condition is sufficient.

Given a sequence d-> d1, d2, ….. dn
Obtained d’ from d following the aforementioned procedure
Assume the d’ is graphic

Show that d is graphic

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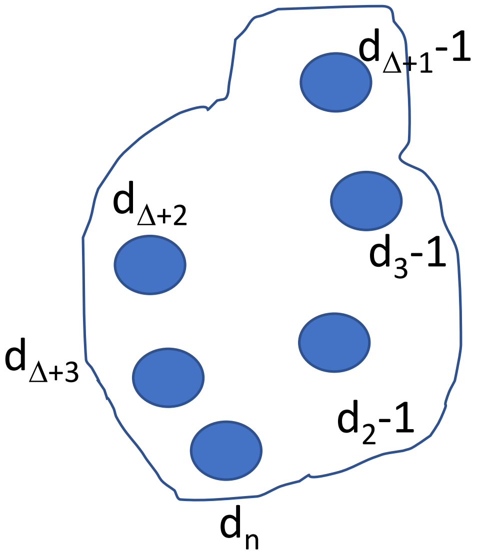

$d : d_1, d_2, \ldots\ d_n$
$d' : \underline{d_2\text{-}1, \ldots\ldots, d_{\Delta+1}\text{-}1}, \ldots\ d_n$

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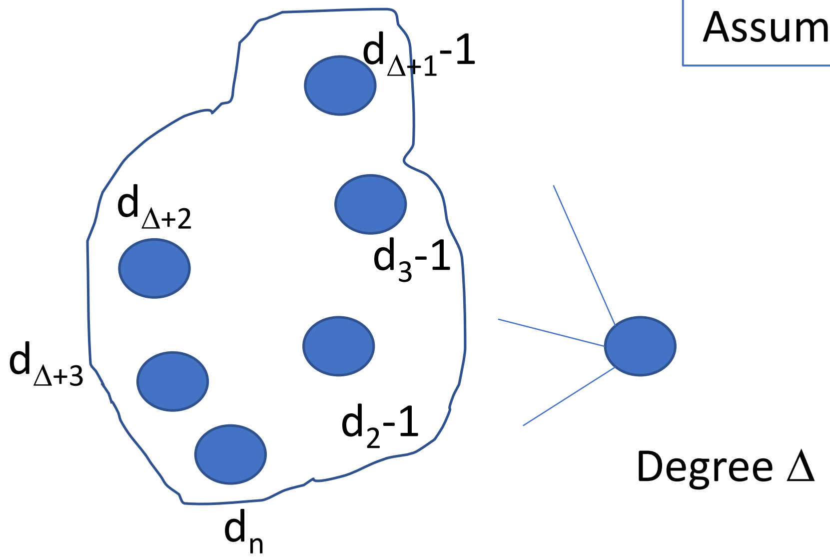

$d : d_1, d_2, \ldots\ d_n$
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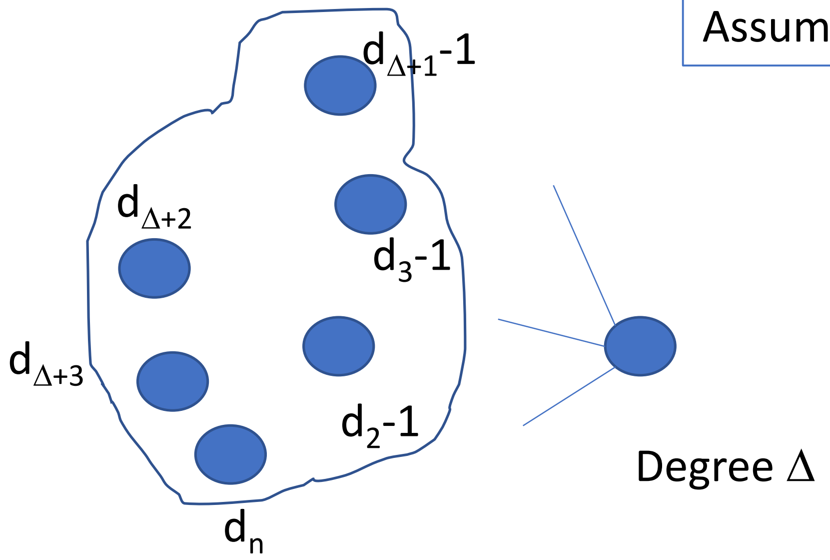

$d: d_1, d_2, \ldots d_n$

$d': \underline{d_2-1, \ldots\ldots, d_{\Delta+1}-1}, \ldots d_n$

d->Δ, d2, d3, ….., dn

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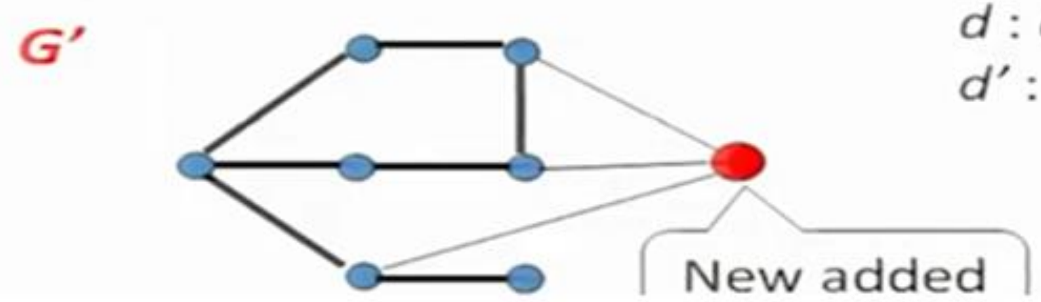
Proof: For $n=1$, the statement is trivial.

- For $n>1$, **we first prove that the condition is sufficient.**
  - Given d with $d_1 \geq \ldots\ldots \geq d_n$ and a simple graph G' with degree sequence d', we add a new vertex adjacent to vertices in G' with degrees $d_2-1,\ldots\ldots,d_{\Delta+1}-1$.
  - These d_i are the Δ largest elements of d after (one copy of) Δ itself, but $d_2-1,\ldots\ldots,d_{\Delta+1}-1$ need not be the Δ largest numbers in d'



$d : d_1, d_2, \ldots\ d_n$

$d' : \underbrace{d_2-1,\ldots\ldots,d_{\Delta+1}-1}_{\text{May not be the } \Delta \text{ largest numbers}},\ldots\ d_n$

Havel and Halimi Theorem

Theorem. For $n>1$, an integer list d of size n is graphic if and only if d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1-element graphic sequence is $d_1=0$.

Sufficient condition

Given a degree sequence d-> d1, d2, ….. dn is **graphic**
Obtained d' from d following the aforementioned procedure

→ Show that d' is graphic

Havel and Halimi Theorem

- **To prove necessity,**
  - We begin with a simple graph ***G*** realizing ***d***, we produce a simple graph ***G'*** realizing ***d'***.
  - Let ***w*** be a vertex of degree Δ in ***G***, and let ***S*** be a set of Δ vertices in ***G*** having the "desired degrees" $d_2, \ldots\ldots, d_{\Delta+1}$

    If $N(w)=S$, then we delete w to obtain G'.

w

$\downarrow$

$d : d_1, \underline{d_2, \ldots d_\Delta, d_{\Delta+1}, \ldots}\ \ d_n$

$d_1=\Delta$

S: Δ vertices

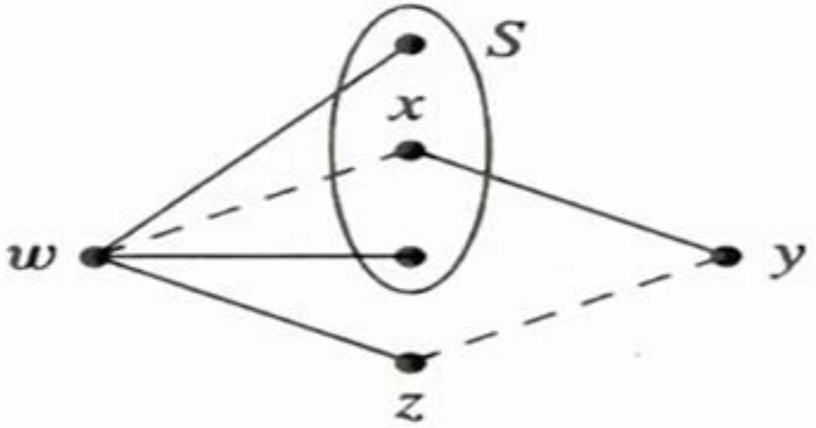
Delete w than we have G'

d': $d_2-1, \ldots\ldots, d_{\Delta+1}-1, \ldots\ d_n$

Havel and Halimi Theorem

Proof: *continue*

- To find the modification when $N(w) \neq S$, we choose $x \in S$ and $z \notin S$ so that $w \leftrightarrow z$ and $w \not\leftrightarrow x$.
- We want to add wx and delete wz, but we must preserve vertex degrees. Since $d(x) \geq d(z)$ and already w is a neighbor of z but not x, there must be a vertex y adjacent to x but not to z. Now we delete $\{wz, xy\}$ and add $\{wx, yz\}$ to increase $|N(w) \cap S|$.



2 switch

- A **2-switch** is the replacement of a pair of edges ***xy*** and ***zw*** in a simple graph by the edges ***yz*** and ***wx***, given that ***yz*** and ***wx*** did not appear in the graph originally.

- If y $\leftrightarrow$ z or w $\leftrightarrow$ x then the 2-switch cannot be performed, because the resulting graph would not be simple. A 2-switch preserves all vertex degrees. If some 2-switch turns H into H*, then a 2-switch on the same four vertices turns H* into H.

The dashed lines above indicate nonadjacent pairs.

- Below we illustrate two successive 2-switches.