Connectivity

Vertex Connectivity

- How many vertices must be deleted to **disconnect** a graph?

- **Definition:** A **separating set** or **vertex cut** of a graph (G) is a set (S$\subseteq$ V(G)) such that (G-S) has more than one component.
- The **connectivity** of (G), written **κ(G)**, is the **minimum size** of a vertex set (S) such that (G-S) is disconnected or **has only one vertex**.

- A graph (G) is **k-connected** if its connectivity is **at least (k)**.

- A graph other than a complete graph is **k-connected** if and only if every **separating set has size at least (k)**.
- We can view "**(k)-connected" as a structural condition**, while "**connectivity (κ)**" is the solution of an **optimization problem**.

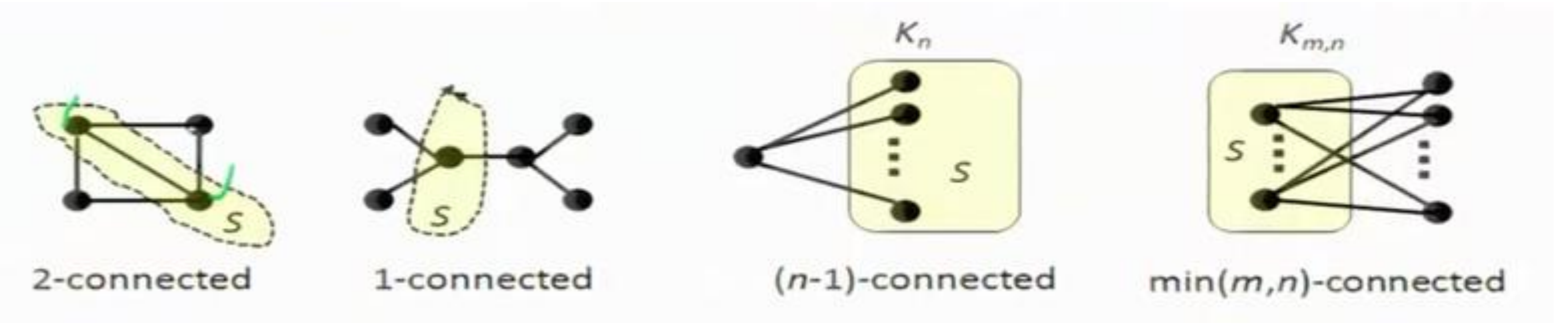
Connectivity of (K_n) and (K_{m,n})

- Because a **clique has no separating set**, we need to adopt a convention for its connectivity. This explains the phrase “or has only one vertex” in the definition. We obtain
- κ(K_n)=n-1, while κ(G)<= n-2 when (G) is not a complete graph. With this convention, most general results about connectivity remain valid on complete graphs.

- Consider a bipartition (X,Y) of (K_{m,n}). Every induced subgraph that has at least one vertex from (X) and from (Y) is connected. Hence every separating set of (K_{m,n}) contains (X) or (Y). Since (X) and (Y) themselves are separating sets (or leave only one vertex), we have
- $\kappa(K_{\{m,n\}})$=min{m,n}.

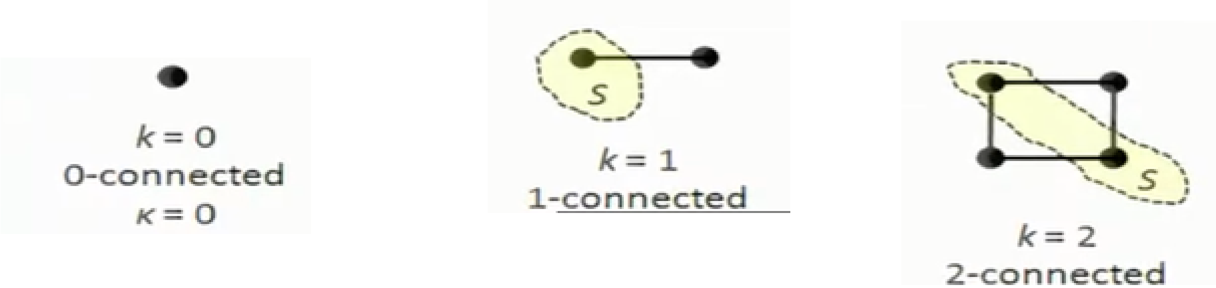
- The connectivity of (K_{3,3}) is (3); the graph is (1)-connected, (2)-connected, and (3)-connected. These conditions are useful.

Examples



K_n
$K_{m,n}$
S
S
S
S
2-connected
1-connected
(n-1)-connected
min(m,n)-connected

0-connected
$2K_2$:
disconnected, so
0-connected



$k = 0$
0-connected
$\kappa = 0$
S
$k = 1$
1-connected
S
$k = 2$
2-connected

Examples

	K_1	K_2	K_3	K_4	K_n (n>3)	C_4	C_n (n>2)
Connectivity κ	0	1	2	3	n-1	2	2
1-connected?	N	Y	Y	Y	Y	Y	Y
2-connected?	N	N	Y	Y	Y	Y	Y
3-connected?	N	N	N	Y	Y	N	N

Edge-Connectivity

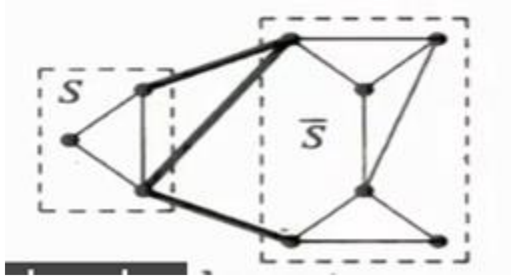
Definition: Edge-Connectivity

A **disconnecting set of edges** is a set ($F \subseteq E(G)$) such that ($G-F$) has **more than one component**.

The **edge-connectivity of (G), written $\kappa'(G)$**, is the **minimum size** of a **disconnecting set** (equivalently, removal of at most (k-1) links keeps the graph connected → k-edge-connected).

A graph is **k-edge-connected** if every **disconnecting set has at least (k) edges**.

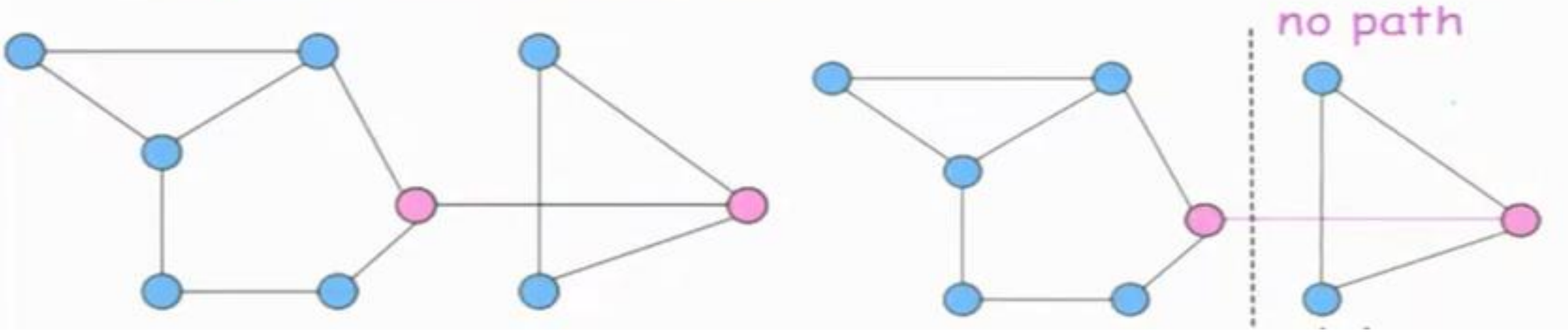
Given ($S,T \subseteq V(G)$), we write ($[S,T]$) for the set of edges having one endpoint in (S) and the other in (T). An **edge cut** is an edge set of the form ($[S,S']$), where (S) is a nonempty proper subset of ($V(G)$) and S' denotes ($V(G)-S$).



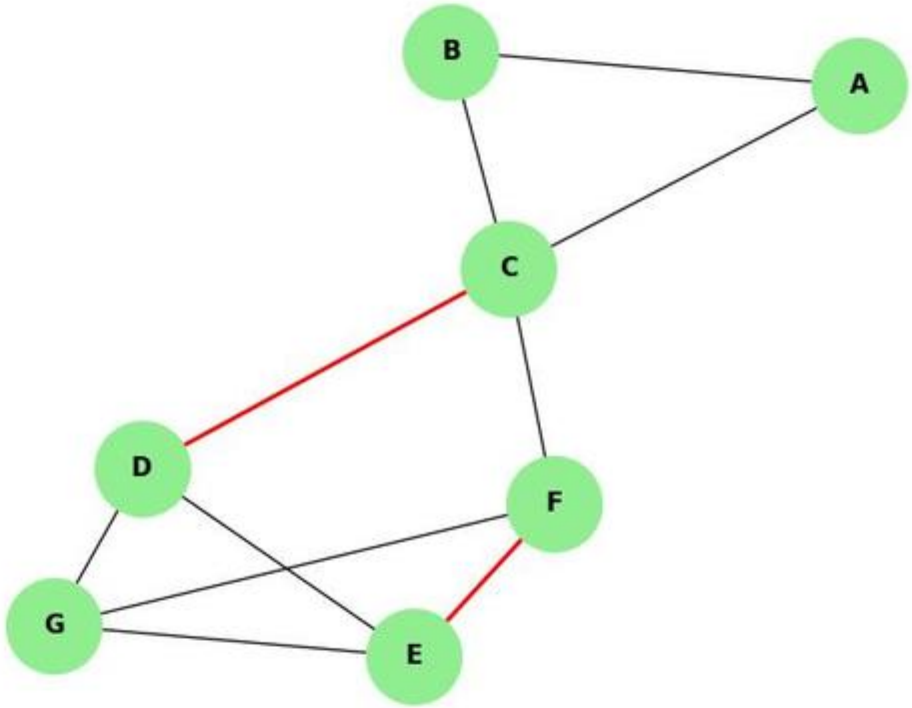
Examples

Definition: vertices *v*, *w* are ***k*-edge connected** if they remain connected whenever **fewer than** *k* edges are deleted.

Example: 1-edge connected

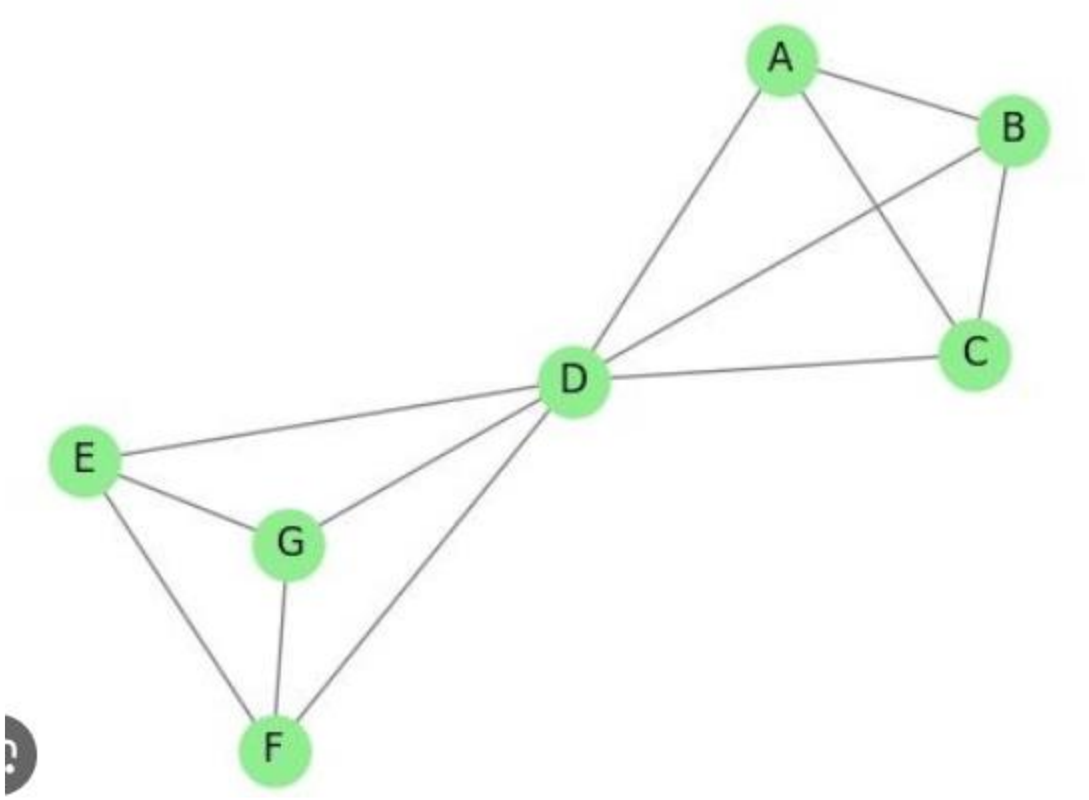


Examples

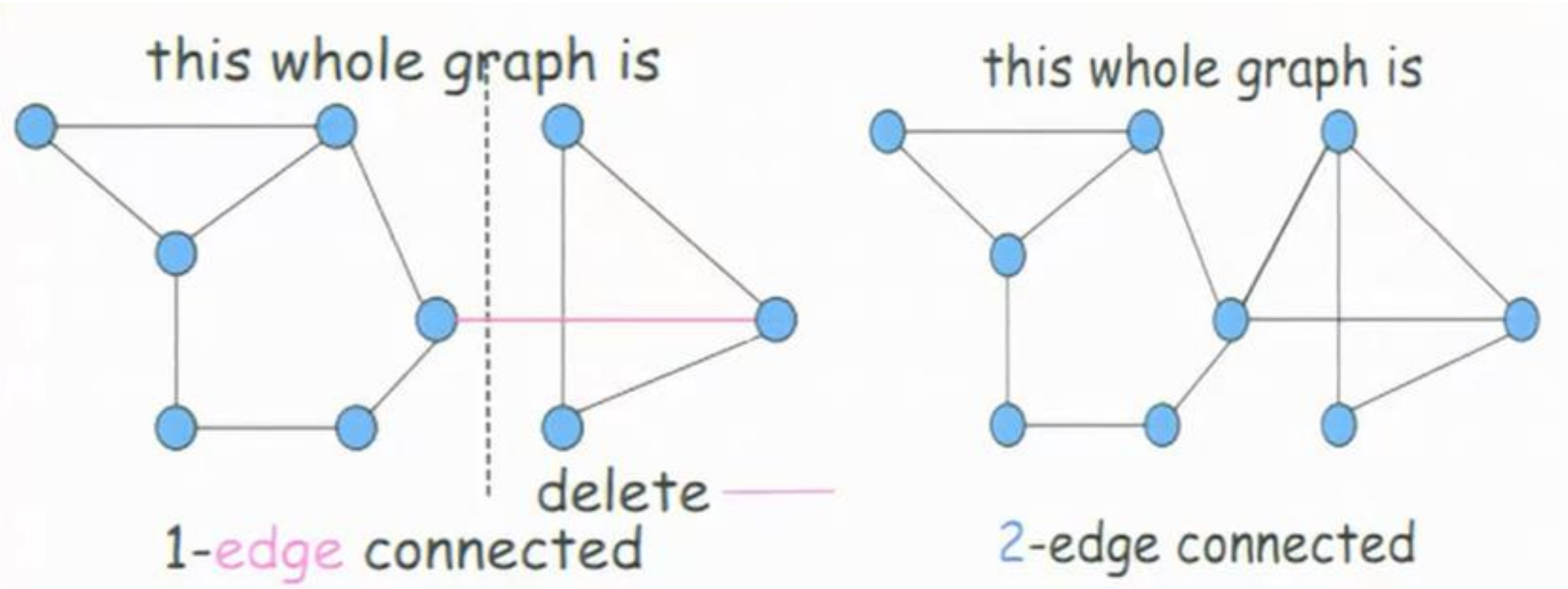


2-edge connected

Examples



3-edge connected

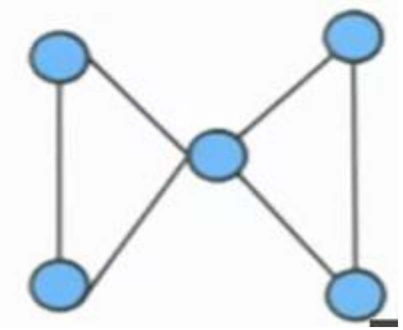

this whole graph is
delete
1-edge connected
this whole graph is
2-edge connected

Observation

- **k-vertex connectedness** defined similarly
- k-vertex connected

  IMPLIES

  k-edge connected not conversely:



2-edge connected
1-vertex connected

Disconnecting set vs. edge cut

Every edge cut is a disconnecting set. Since ($G-[S,S']$) has no path from (S) to (S'). The converse is false, since a **disconnecting set can have extra edges**.

Every **minimal disconnecting set** of edges is an **edge cut** (when ($n(G)>1$)).

Consider (G-F) has **more than one component** for some (**$F\subseteq E(G)$**),

Consider a component (H) of (G-F).

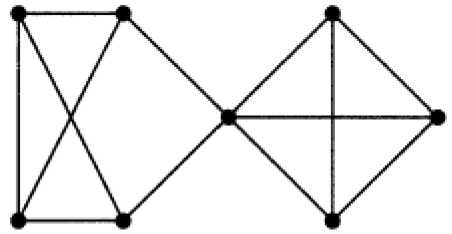
We have deleted **all edges with exactly one endpoint in (H)**. Hence (F) contains the edge cut $[V(H),V(G)-V(H)]$]

Examples

For graph G below,

$\kappa(G) = 1$, $\kappa'(G) = 2$, and $\delta(G) = 3$. Note that no minimum edge cut isolates a vertex.

Each inequality can be arbitrarily weak, When $G = K_m + K_m$, we have $\kappa(G)=\kappa'(G) = 0$ but $\delta(G) = m$-1. When G consists of two m-cliques sharing a single vertex, we have $\kappa'(G)= \delta(G) = m$-1 but $\kappa(G)= 1$.



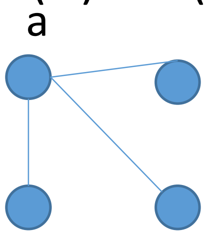
Connectivity and Minimum Degree for Simple Graphs

Theorem (Whitney [1932]): If (G) is a simple graph, then κ(G) <= κ ‘(G) <= δ(G).}

Proof

Proof of (κ’(G)<=δ(G)):

- The edges incident to a vertex of minimum degree form a disconnecting set. Therefore, κ’(G)<= δ(G).

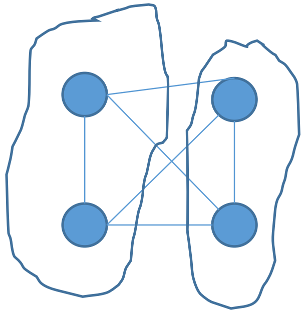


Proof of (κ(G)<= κ '(G)):

- Let **F be a minimum disconnecting (edge) set** of (G) of size (κ'(G)). F is an edge cut **F=[S,V(G)-S]**.

We consider **two cases.**

- Case 1: Every vertex of (S) is **adjacent to every vertex** of (V(G)-S).



Then

- **κ '(G)=|[S,S’]|=|S||S’|>= n-1.**
- Since we already know that **κ(G) <= (n-1)**,
- we obtain κ(G) <= κ '(G).

Connectivity and Minimum Degree for Simple Graphs

Case 2: There exist vertices ($x \in S$) and ($y \in S'$) such that $x\text{-}y \notin E(G)$.

Define

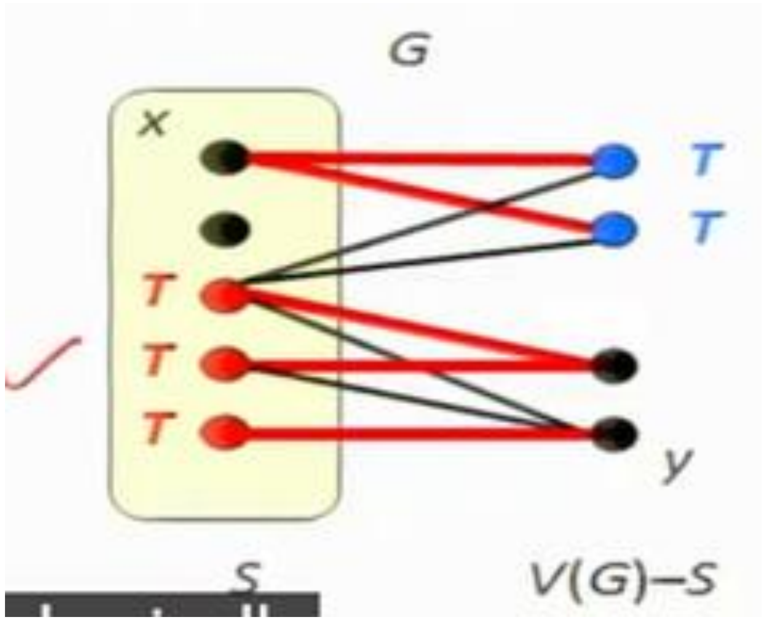
T= **Consists all neighbors of x in S'**

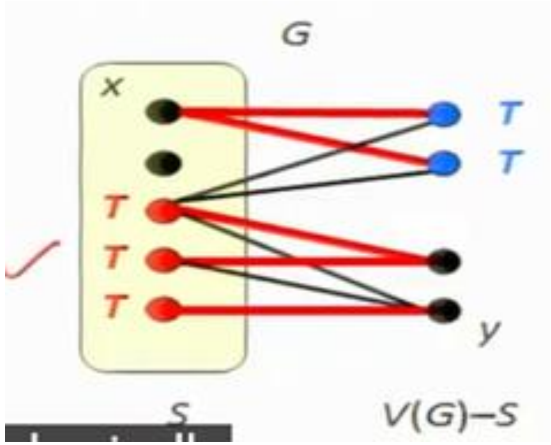
($N(x) \cap S'$)

and

All vertices of (S-{x}) who has neighbors in S'

$\{z \in S-\{x\} : N(z) \cap S' \neq \Phi\}$





Case 2

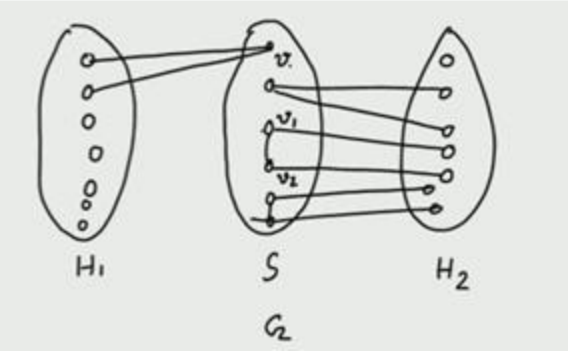
- $T=(N(x)\cap(S'))\cup\{z\in S-\{x\}:N(z)\cap(S')\neq\Phi\}$.

T is a vertex cut because **all (x,y)-paths** would have to cross through (T).

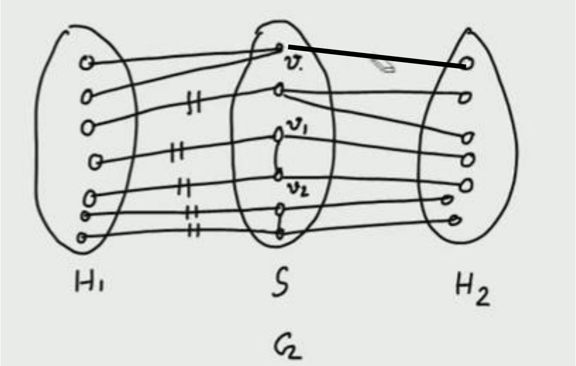
- **The edges F_T** both **incident to (T)** and **in the edge cut [S,S']** form a **disconnecting set.**
- Every vertex of (T) has at least one neighbor, so
- Edge Cut $=|[S,S']| = |F_T| >= |T|$.
- We have therefore found a **vertex cut (T)** with size <= **minimum edge cut ([S,S'])**.
- Hence $\kappa(G) <= |T| <= |[S,S']| = \kappa'(G)$.
- Therefore, $\{\kappa(G) <= \kappa'(G)\}$.
- Combining this with ($\kappa'(G) <= \delta(G)$), we obtain

$\{\kappa(G) <= \kappa'(G) <= \delta(G)\}$.

Theorem



If (G) is a 3-regular graph, then κ(G)=κ'(G)



Proof

Let S be a **minimum vertex cut**

We need to show an **edge-cut** of size (|S|).

Let H_1 & H_2 be two components of G-S.

Since (S) is a minimum vertex cut, each ($v\in$ S) has a neighbor in (H_1) and a neighbor in (H_2). Since (G) is 3-regular, (v) cannot have two neighbors in (H_1), and two in (H_2).

For each (v\in S), delete the edge from (v) to the member of ({H_1,H_2}) where (v) has only one edge.

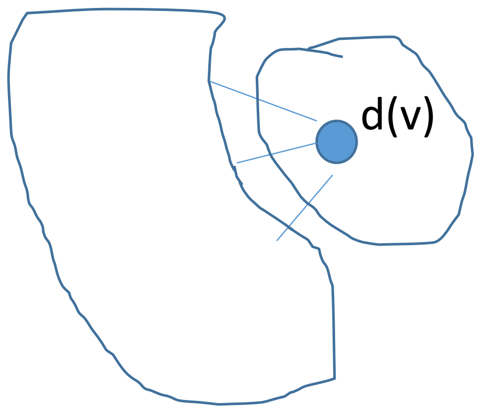
In the other case, delete the edges to (H_1) for both (v_1) & (v_2).

Then (\kappa(G)) edges break all paths from (H_1) to (H_2).

[\therefore\quad \kappa(G)=\kappa'(G)]

Size of an Edge Cut

Lemma: $\kappa'(G)$**< δ(G), a minimum edge cut cannot isolate a vertex (S>1)**



Assume S=1

Single vertex {v}

Minimum Edge cut $\kappa'(G)$ =[S, S'] = d(v) >= δ(G)

Not possible

Size of an Edge Cut

If S is a set of vertices in a graph G, then

$$\boxed{|[S, \bar{S}]| = \sum_{v \in S} d(v) - 2e(G[S])}$$

where $G[S]$ is the subgraph induced by S.

Size of an Edge Cut

Proof

Each edge in $G[S]$ contributes **twice** to

$$\sum_{v \in S} d(v),$$

while each edge in $[S, \bar{S}]$ contributes **once**.

Therefore,

$$\sum_{v \in S} d(v) = |[S, \bar{S}]| + 2e(G[S]).$$

Hence,

$$\boxed{|[S, \bar{S}]| = \sum_{v \in S} d(v) - 2e(G[S])}$$

Size of an Edge Cut

If G is a simple graph and

$$|[S,\bar{S}]| < \delta(G)$$

for some nonempty proper subset S of $V(G)$, then

$$\boxed{|S| > \delta(G).}$$

We have

$$|[S,\bar{S}]| = \sum_{v\in S} d(v) - 2e(G[S]).$$

Since

$$|[S,\bar{S}]| < \delta(G),$$

we get

$$\delta(G) > \sum_{v\in S} d(v) - 2e(G[S]).$$

Size of an Edge Cut

We have

$$|[S,\bar{S}]| = \sum_{v\in S} d(v) - 2e(G[S]).$$

Since

$$|[S,\bar{S}]| < \delta(G),$$

we get

$$\delta(G) > \sum_{v\in S} d(v) - 2e(G[S]).$$

Using

$$d(v) \geq \delta(G)$$

and

$$2e(G[S]) \leq |S|(|S|-1)$$

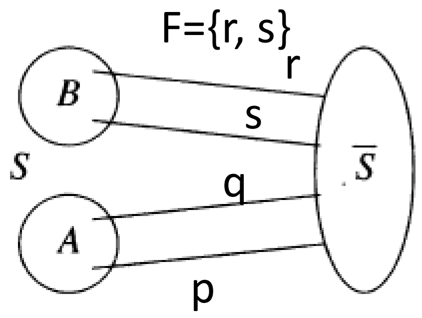
yields

$$\delta(G) > |S|\delta(G) - |S|(|S|-1).$$

This inequality requires $|S| > 1$, so we can combine the terms involving $\delta(G)$ and cancel $|S|-1$ to obtain

$$\boxed{|S| > \delta(G)}.$$

Bond



What is a bond?

A **bond** is a **minimal nonempty edge cut**.

Suppose

$$F = [S, \bar{S}].$$

It is an edge cut because deleting all edges in F disconnects G.

It is **minimal** if we cannot remove any edge from F and still disconnect the graph.

So:

$$F \text{ is a bond} \quad \Longleftrightarrow \quad F \text{ disconnects } G \text{ and no proper subset of } F \text{ disconnects } G.$$

Bond

Theorem: If G is connected, an edge cut F is a bond iff (G-F) has exactly two components

Sufficiency : If (G-F) has exactly two components, → then (F) is a bond

Suppose G-F has exactly two components.

Call them (C_1) and (C_2).

Because G was originally connected, t**he edges in F must connect C_1 and C_2**

Now suppose we remove **some proper subset** F'$\subset$ F from edge set F.

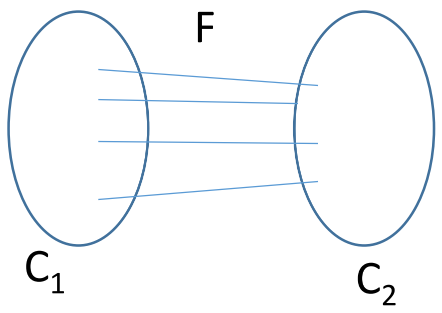
Since F' is a proper subset, at least one edge **e $\in$ F-F'** is still present.

That edge **e** connects C_1 to C_2

Therefore, G-F' is connected.

So no proper subset of (F) can disconnect (G).

Hence F is a minimal edge cut, i.e., a bond.



Bond

Direction 2: If F is a bond, → then (G-F) has exactly two components

Now suppose F is a bond → So (F) is a **minimal** edge cut.

We know (G-F) is disconnected.

Suppose, for contradiction, that (G-F) has **more than two components**

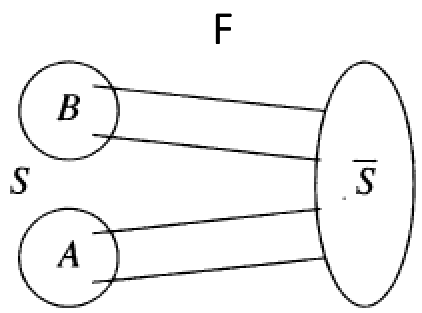
For example: S and S' are two components. So we can write V(G)=S ∪ S'.

More specifically, **suppose (G[S]) itself has at least two components**. S=A ∪ B, where **A and B are the vertex sets of two different components of (G[S]).**

Because (A) and (B) are **different components**, there are **no edges between them [A,B]=Φ.**

Now look at the original **edge cut F=[S, S']**. **It contains all edges going from (S) to (S').**

We can split (F) into two smaller cuts: [A,A'] and [B,B'].



Both of these are contained in (F):

[A, A'] ⊆ F

[B, B'] ⊆ F

And each is itself an edge cut.

Therefore, (F) contains a **proper nonempty subset that is also an edge cut.**

But that **contradicts the fact that F is a bond** (a minimal edge cut).

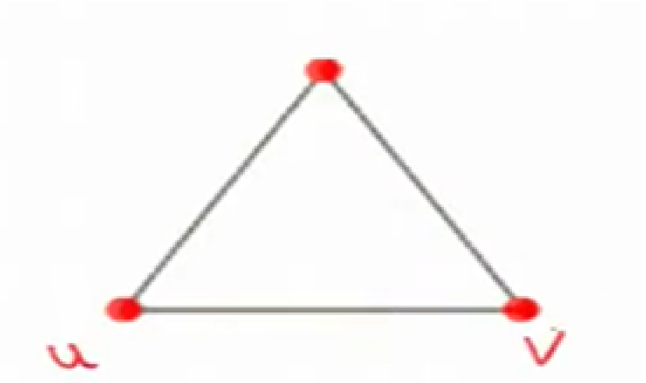
Therefore, (G-F) cannot have more than two components.

Mark the differences

- Disconnecting edges (F)
- Edge cut ([S,S’])
- Edge connectivity ($\kappa'(G)$)
- Bond

Internally disjoint paths

Definition: Two paths from (u) to (v) are internally disjoint, if they have no common internal vertex.

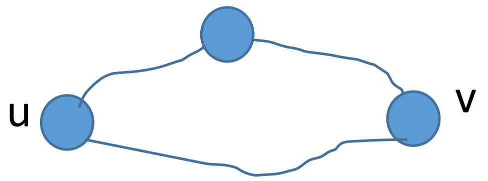


Whitney Theorem

Theorem : A graph (G) having at least three vertices is 2-connected if and only if for each pair (u,v ∈ V(G)), there exist internally disjoint (u,v)-paths in (G).

Proof — Sufficiency: **there exist internally disjoint (u,v)-paths → 2 connected**

- Suppose that for every pair (u,v ∈ V(G)), there **exist internally disjoint (u,v)-paths.**
- Since the two (u,v)-paths have **no common internal vertex**, deleting any one vertex cannot destroy both paths simultaneously.
- Therefore, deleting any one vertex cannot separate (u) from (v).
- Since this is true for **every pair** (u,v), deletion of any one vertex cannot make any vertex unreachable from another.
- Hence, G is 2-connected



Whitney Theorem

Necessity: **Suppose that G is 2-connected. We prove by induction on d(u,v) that G has internally disjoint u,v-paths.**

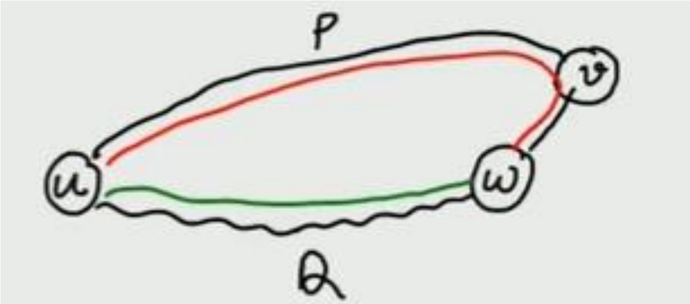
Basis step (d(u,v) = 1): When d(u,v)=1, the **graph G-{uv} is connected**, since **κ'(G) ≥ κ(G) ≥ 2** (G is 2-connected).

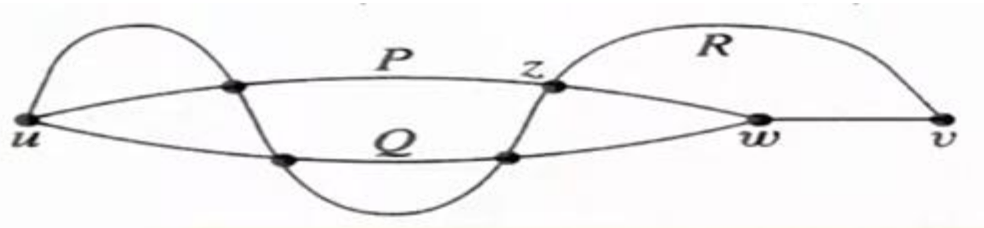
A u,v-path in G-{uv} is internally disjoint in G from the u,v-path formed by **the edge u-v itself.**

Induction step (d(u,v) > 1): Let k=d(u,v). **Let w be the vertex before v on a shortest u,v-path; we have d(u,w)=k-1.**

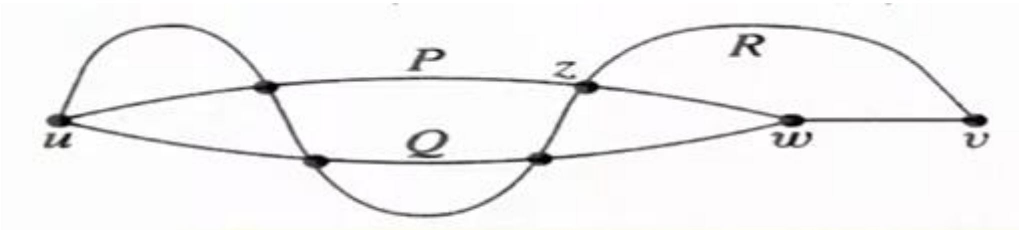
By the **induction hypothesis**, **G has internally disjoint u,w-paths P and Q**.

Case 1: If v ∈ V(P) ∪ V(Q), then we find the desired paths in the cycle P ∪ Q.





Whitney Theorem

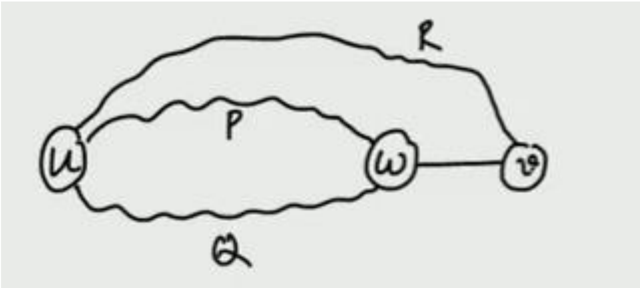


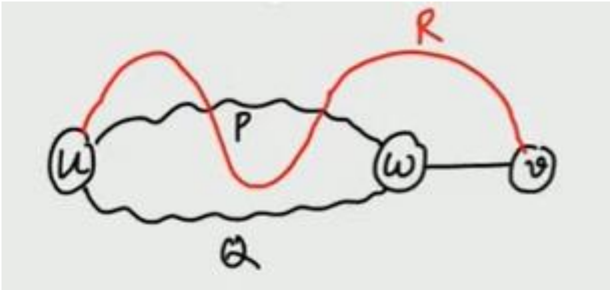
Case 2: Suppose not. If v ∉ V(P) ∪ V(Q),

Since G is 2-connected, **G-w is connected** and contains a **u,v-path R**.

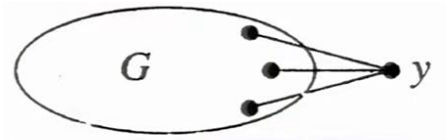
If R avoids P or Q, we are done,

Case 3: R may share internal vertices with both P and Q. Let z be the last vertex of R (before v) belonging to P ∪ Q. By symmetry, we may assume **that z ∈ P**. We combine the **u,z-subpath of P** with the **z,v-subpath of R** to obtain a u,v-path internally disjoint from Q ∪ v.

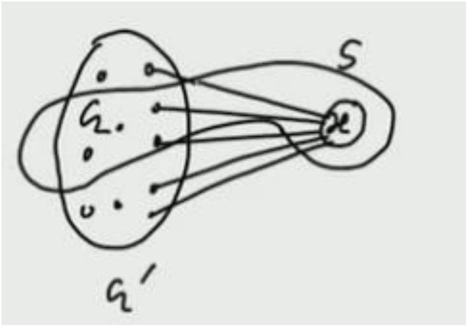




Expansion Lemma



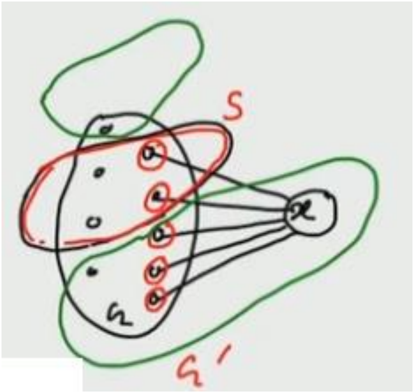
Lemma (Expansion Lemma): If G is a k-connected graph, and G' is obtained from G by adding a new vertex x with at least k neighbors in G, then G' is k-connected.



Proof: We prove that a **separating set S of G'** must have size **at least k**.

Case 1: If x ∈ S, then S-{x} separates G, so |S| ≥ k+1.

Case 2: If x ∉ S and N(x) ⊆ S, then |S| ≥ k.



Case 3: Otherwise, if N(x) ⊄ S, then **x and N(x)-S** lie in a **single component of G'-S**.

Thus again S must separate G and |S| ≥ k.

Theorem

- Theorem: For an n-vertex graph G (with (n >= 1)), the following statements are equivalent (and characterize the trees with (n) vertices):

A)G is connected and has no cycles.

B)G is connected and has (n - 1) edges.

C)G has (n - 1) edges and no cycles.

D)G has no loops and has, for each (u, v \in V(G)), exactly one (u,v)-path.

Theorem

For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs)

(A) G is connected and has no cut-vertex.

(B) For all $x, y \in V(G)$, there are internally disjoint x,y-paths.

(C) For all $x, y \in V(G)$, there is a cycle through x and y.

(D) $\delta(G) >= 1$, and every pair of edges in G lies on a common cycle.

Theorem

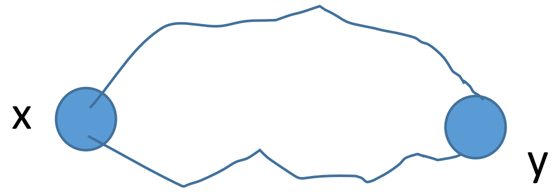
Proof:

(A) G is connected and has no cut-vertex.

(B) For all x, y $\in$ V(G), there are internally disjoint x,y-paths.

(C) For all x, y $\in$ V(G), there is a cycle through x and y.

(A) and (B) are equivalent.



- For (B) $\Leftrightarrow$ (C),
- Note that cycles containing x and y correspond to pairs of internally disjoint (x, y)-paths.

Theorem

For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs)

(A) G is connected and has no cut-vertex.

(B) For all x, y $\in$ V(G), there are internally disjoint x,y-paths.

(C) For all x, y $\in$ V(G), there is a cycle through x and y.

(D) $\delta(G) >= 1$, and every pair of edges in G lies on a common cycle.

Theorem

For D $\Rightarrow$ C,

Given any two vertices (x,y), we prove that there is a cycle containing both x and y.

Since **δ(G) >=1**, every vertex has at least **one incident edge**.

Therefore, **x is not isolated** and **y is not isolated**.

So we can choose:

$e_x = x - x_1$ incident with x, and

$e_y = y - y_1$ incident with y.

Now we want to apply the second part of (D):

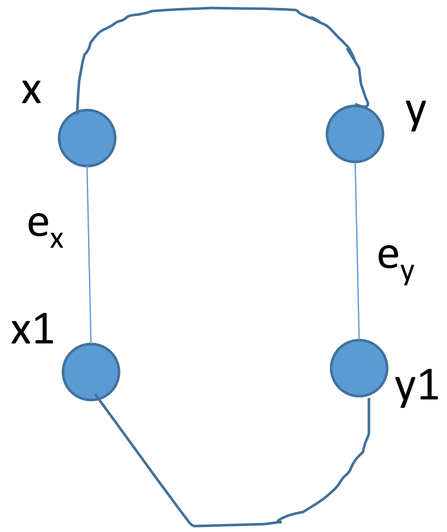
Every pair of edges lies on a common cycle

If e_x and e_y are **different edges**, then they lie on a common cycle C.

Since vertex x is an endpoint of e_x, $x \in C$

Similarly, $y \in C$.

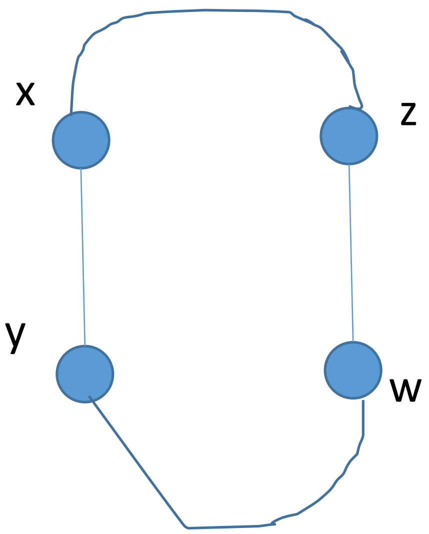
Therefore, cycle C contains both x and y.



Theorem

But what if there is only ONE edge incident to x and y?

- Suppose (x) and (y) are adjacent
- The only edge we can choose for both is e=x-y.
- So we cannot take two different edges incident with (x) and (y).
- But (G) has **at least three vertices**.
- Let the **third vertex be z**.
- Because **δ(G)>=1**, z has some incident edge, say **f=z-w**.
- Now e=x-y and f=z-w are two edges.
- **By D, they lie on a common cycle.**
- Since that cycle contains e=x-y, it contains both (x) and (y).
- Hence there is a cycle through (x) and (y).



Theorem

For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs)

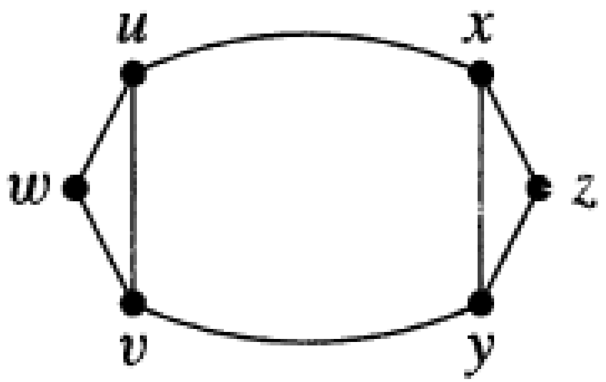
(A) G is connected and has no cut-vertex.

(B) For all x, y $\in$ V(G), there are internally disjoint x,y-paths.

(C) For all x, y $\in$ V(G), there is a cycle through x and y.

(D) $\delta(G) >= 1$, and every pair of edges in G lies on a common cycle.

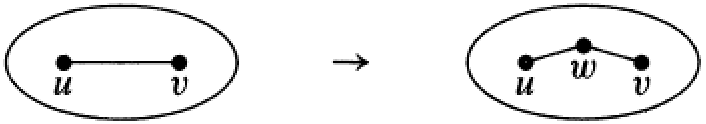
Theorem



Proof: A, C $\Rightarrow$ D

- Since G is connected, **δ(G) >= 1**.
- Now consider **two edges u-v and x-y**.
- Add to G the vertices **w with neighborhood u, v** and **z with neighborhood x, y**.
- Since G is 2-connected, the **Expansion Lemma** implies that the resulting graph G' is **2-connected**.
- Hence **condition C** holds in G', so **w and z** lie on a **cycle C in G'**.
- Since **w, z each have degree 2**, **cycle C** must contain the **paths u-w-v** and **x-z-y** but **not the edges u-v or x-y**.
- **Replacing the paths** u-w-v and x-z-y in cycle C with the edges u-v and x-y yields the **desired cycle through u-v and x-y** in G.

Subdivision of an edge



In a graph G, **subdivision of an edge (u, v)** is the operation of **replacing (u, v)** with a **path u-w-v** through the **new vertex w**

Theorem: If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected.

Proof:

Since **G is 2-connected**, **any two edges of G lie on a common cycle**.

Let **G' be obtained** from G by **subdividing edge (u-v)**.

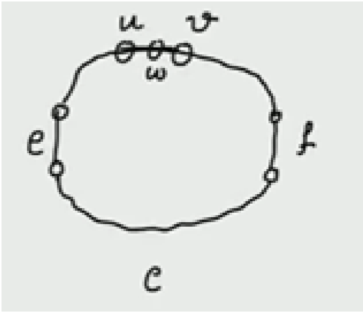
Let edges **e, f ∈E(G')**. It suffices to find a **cycle through edge e and f**.

Case 1: e, f ∈ E(G)

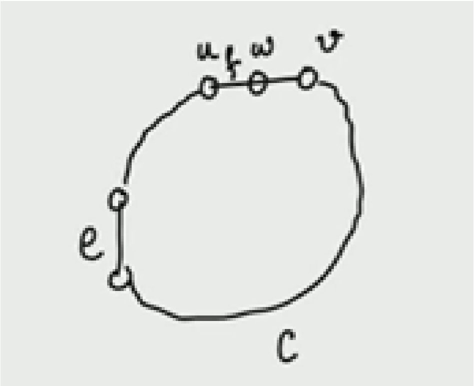
There is a cycle C through edges e and f in G.

C is also a cycle through e and f in G'.

Little modification in C might be necessary

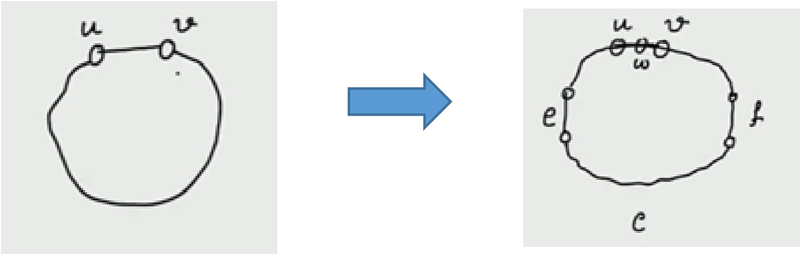


Theorem



- Case 2: $e \in E(G)$, $f \in \{u\text{-}w, w\text{-}v\}$

We modify a cycle passing through edge e and edge (u, v) in G.



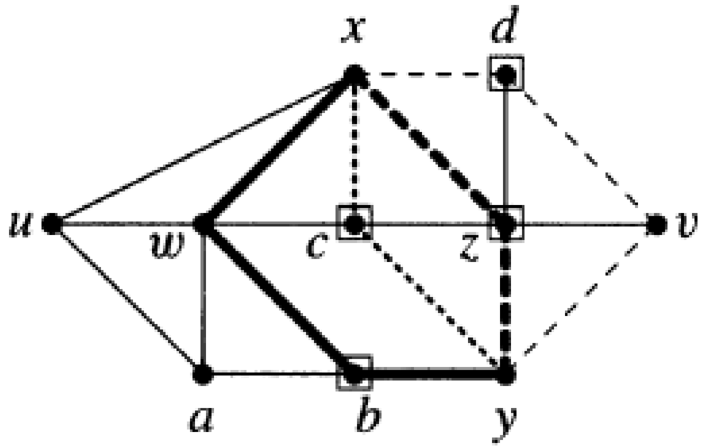
- Case 3: $e, f \in \{u\text{-}w, w\text{-}v\}$
- We modify a cycle passing through (u, v) in G

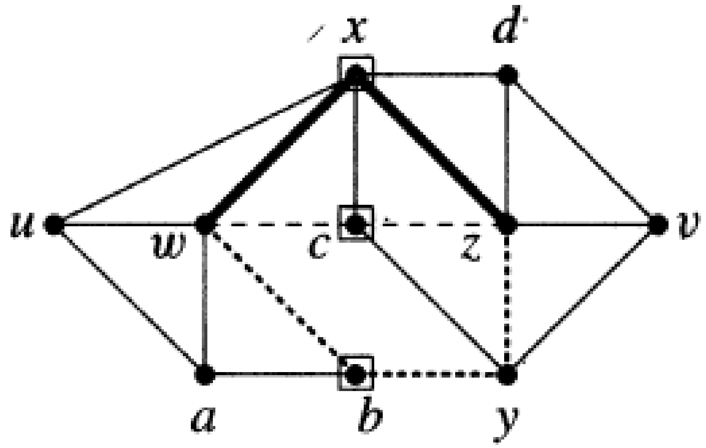
k-Connected and k-Edge-Connected Graphs

- Given x, y $\in$ V(G), a set **S $\subseteq$ V(G) - {x, y}** is an **{x, y}-separator** or **{x, y}-cut** if **G - S** has **no {x, y}-path**.
- Let **κ(x, y)** be the **minimum** size of an **{x, y}-cut**.
- Let λ**(x, y)** be the **maximum size** of a set of **pairwise internally disjoint {x, y}-paths.**
- For **X, Y $\subseteq$ V(G)**, an **X, Y-path** is a path having first vertex in X, last vertex in Y, and no other vertex in X $\cup$ Y

k-Connected and k-Edge-Connected Graphs

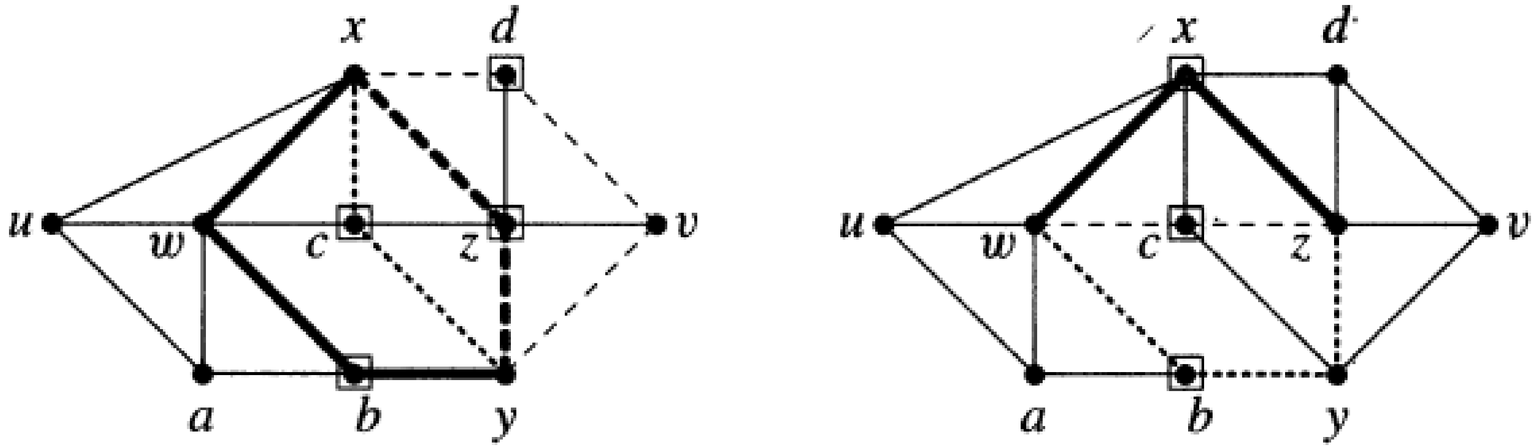
- In the graph G below, the set S = {b, c, z, d} is an x, y-cut of size 4;
- Thus κ(x, y) ≤ 4.
- As shown on the left, G has four pairwise internally disjoint x, y-paths; thus λ(x, y) ≥ 4.





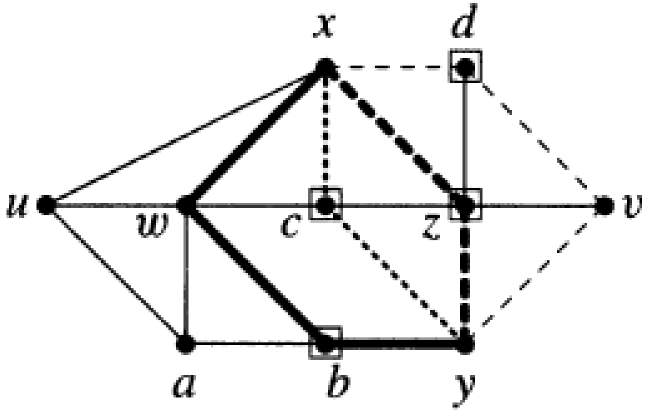
k-Connected and k-Edge-Connected Graphs

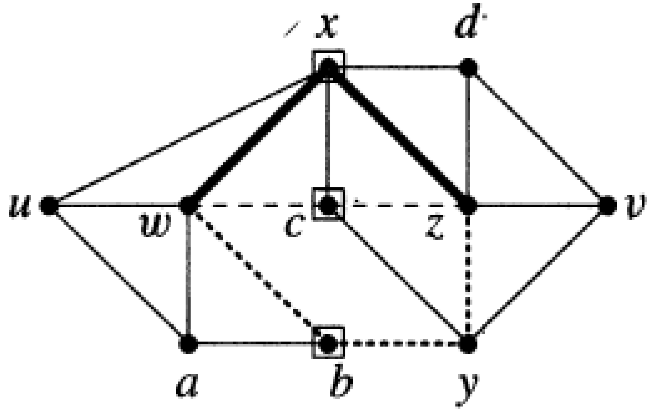
- An {**x, y}-cut** must contain an **internal vertex** of **every x, y-path**,
- and **no single vertex** can cut **two internally disjoint {x, y}-paths**.
- Therefore, always **κ(x, y)>= λ(x, y)**
- Since $\kappa(x, y) \geq \lambda(x, y)$ always, we have $\kappa(x, y) = \lambda(x, y) = 4$.



k-Connected and k-Edge-Connected Graphs

- Consider also **the pair (w, z)**. As shown on the right, **$\kappa(w,z) = \lambda(w,z) = 3$**, with **{b, c, x} being a minimum {w, z}-cut**.
- The graph G is 3-connected; for every pair {u, v $\in$ V(G), we can find three pairwise internally disjoint {u, v}-paths.
- From the equality for internally disjoint paths, we will obtain an analogous equality for edge-disjoint paths. Although $\kappa(w,z) = 3$ above, it takes four edges to break all w, z-paths, and there are four pairwise edge-disjoint w, z-paths.

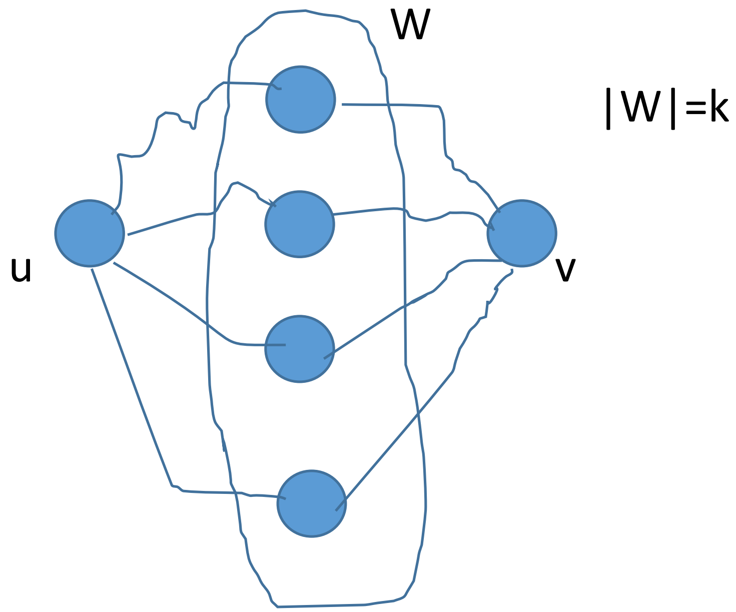




Menger's Theorem

Let u and v be nonadjacent vertices in a graph G.

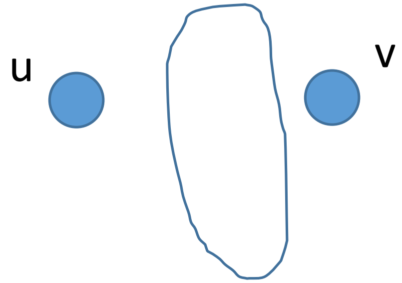
The minimum number of vertices in a u-v separating set equals the maximum number of internally disjoint u-v paths in G

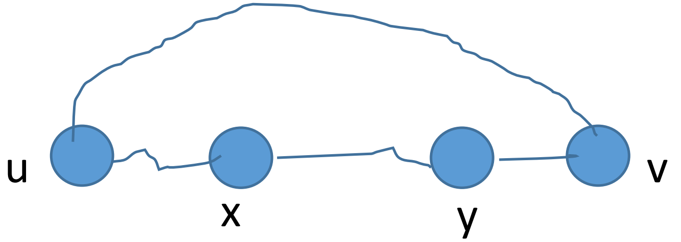


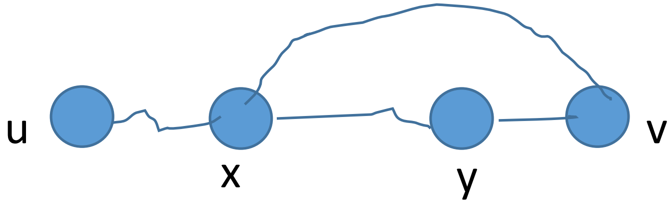
Menger's Theorem

Induction on the size of the graph → number of edges in the graph (m)
Base case: Menger theorem is true for empty graph

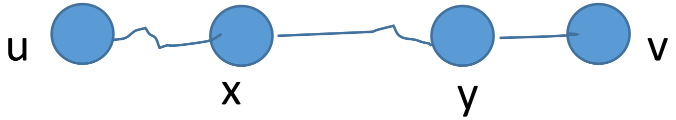
If κ=0
Then λ=0







If κ=1
then λ=1

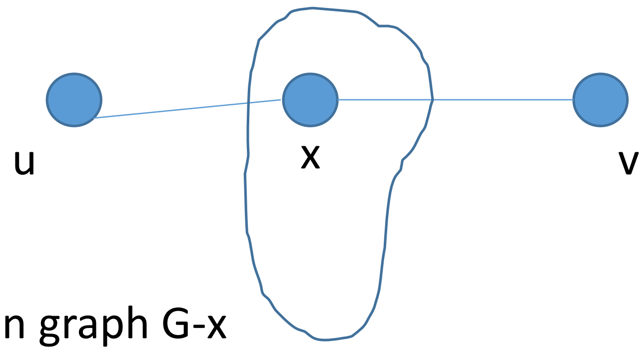


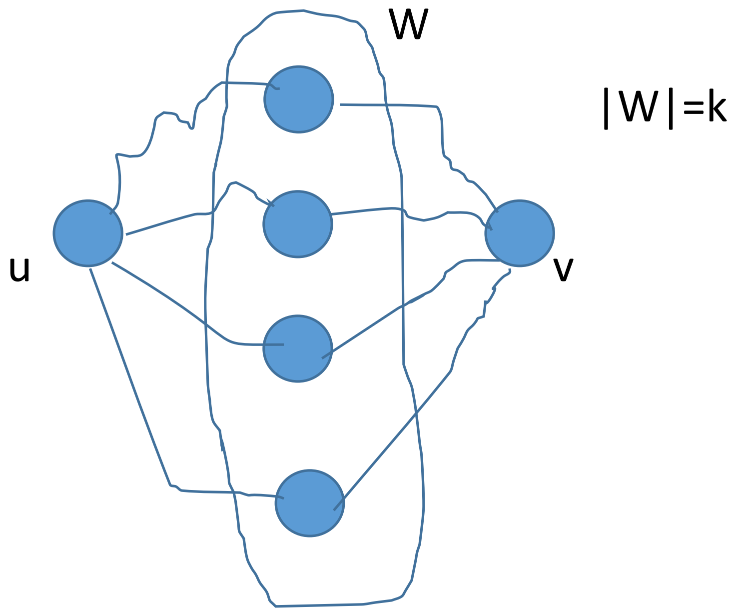
κ=1
λ=1

We need to prove for m>0
Induction hypothesis: In a graph of size <m links, if the minimum number of vertices in a u-v separating set k, there exists maximum k number of internally disjoint u-v paths in G

Menger's Theorem

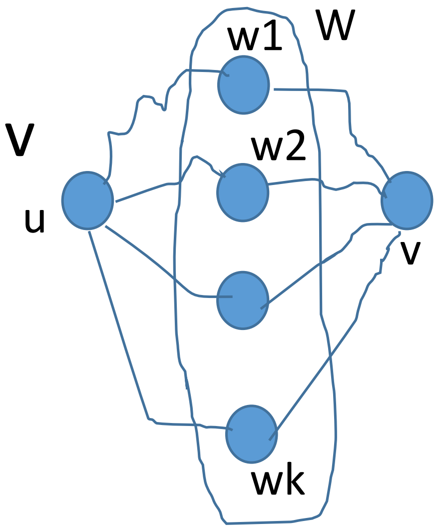
Case 1: There exists a minimum u-v separating set W in G, containing a vertex x, which is adjacent of both u and v





Remove x from G, obtain graph G-x
Graph G-x has size <m
Claim: W-{x} is the minimum u-v separating set in G-{x}
Assume not??
|W-{x}|=k-1
Apply induction hypothesis:
There exists a maximum of (k-1) internally disjoint u-v paths in graph G-x
Now consider the path P=(u-x-v) in G
Hence, these (k-1) internally disjoint u-v paths of graph G-x, together with P makes the maximum k internally disjoint u-v paths in graph G

Menger's Theorem



Case 2: There exists a minimum u-v separating set W in G, containing a vertex, which is not adjacent to u, and a vertex which is not adjacent to v

That can be same vertex → not adjacent to both u, v

Both adjacent → case 1

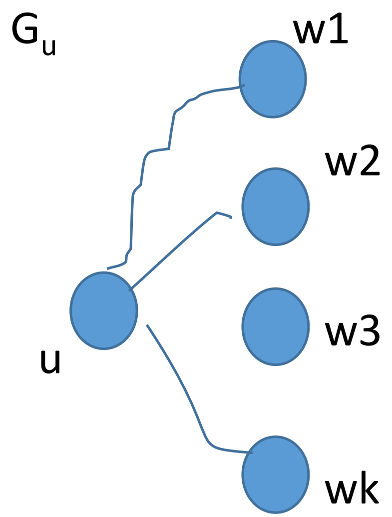
$|W|=k$

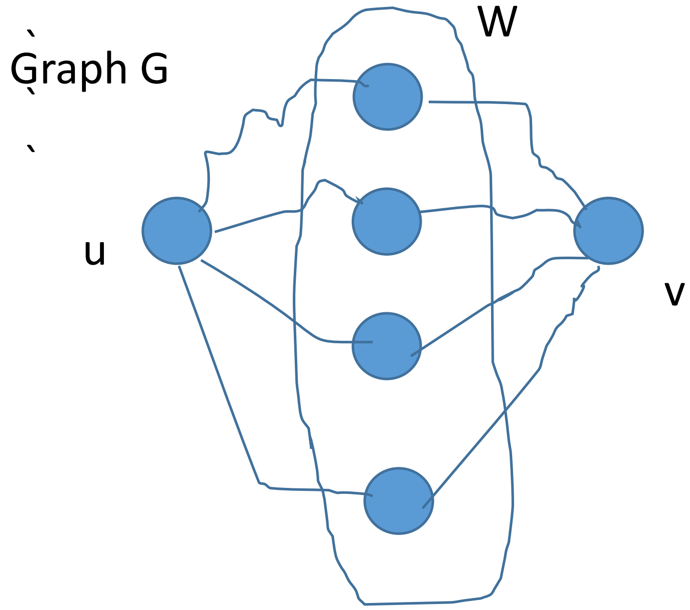
Consider the set W={w1, w2, w3, ……, wk}

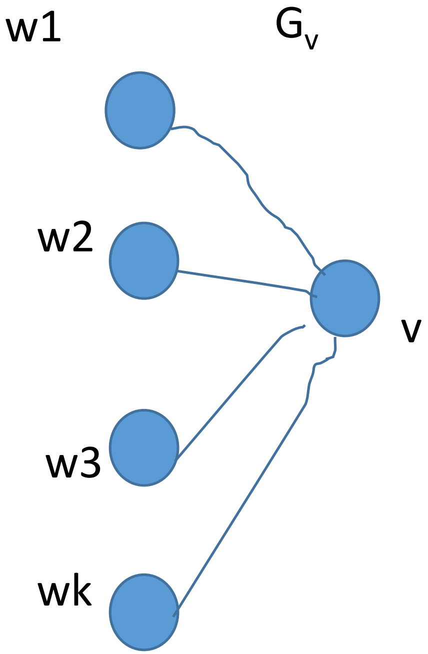
Let $\mathbf{G_u}$ **be the subgraph** of G consisting of all u–w_i paths where only $w_i \in W$ in each path.

Let $\mathbf{G_v}$ **be the subgraph** of G consisting of all v–w_i paths where only $w_i \in W$ in each path.

Menger's Theorem



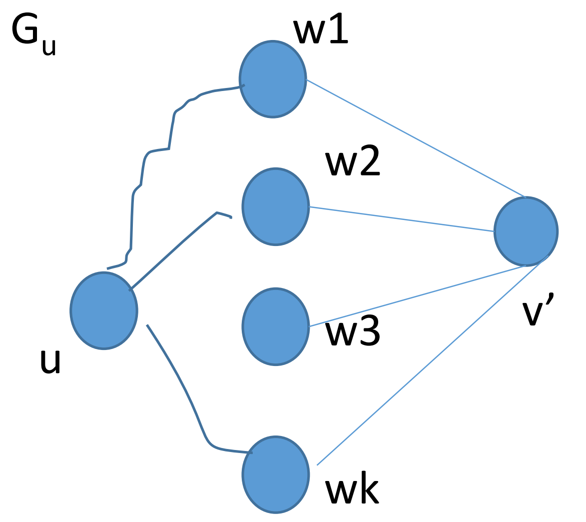


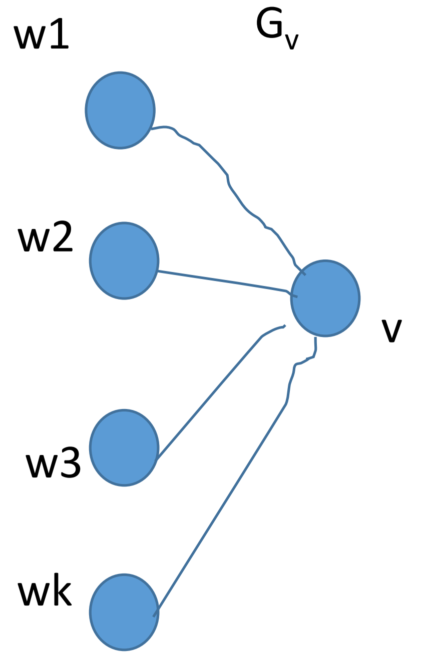


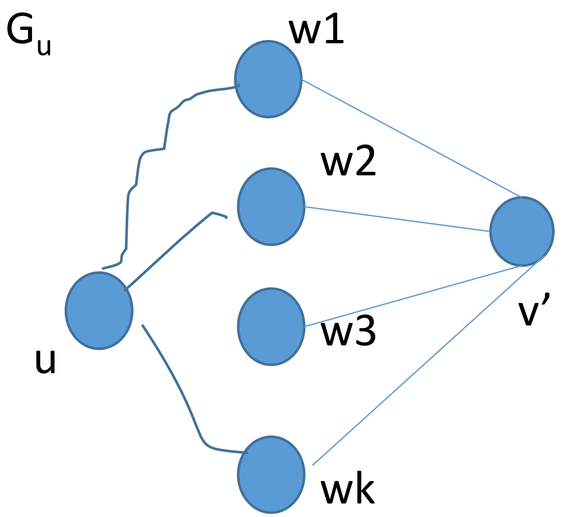
Let **G_u be the subgraph** of G consisting of all u–w_i paths where only $w_i \in W$ in each path.

Let **G_v be the subgraph** of G consisting of all v–w_i paths where only $w_i \in W$ in each path.

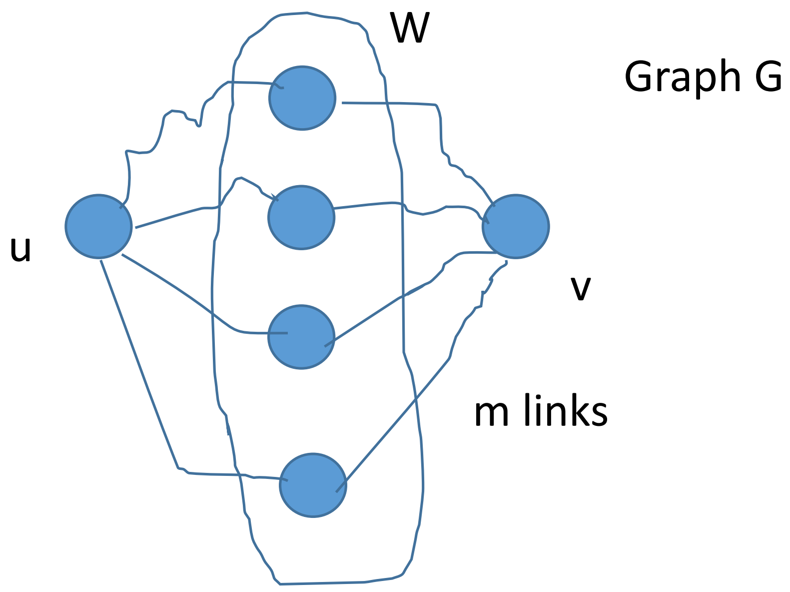
Menger's Theorem



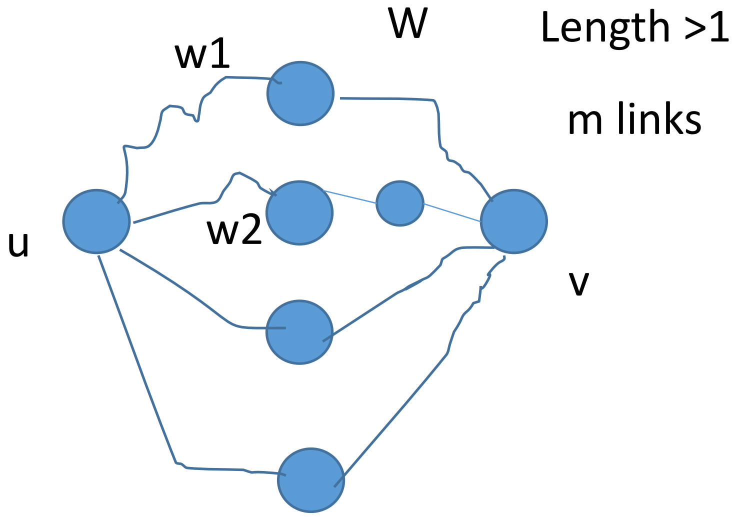




The size of G_u is <m



Possibility 1

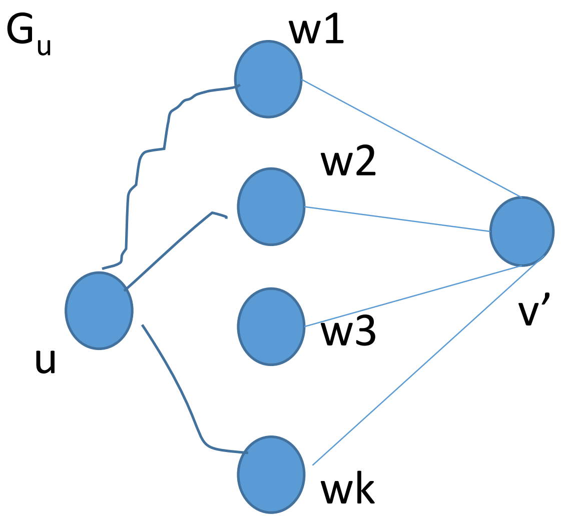


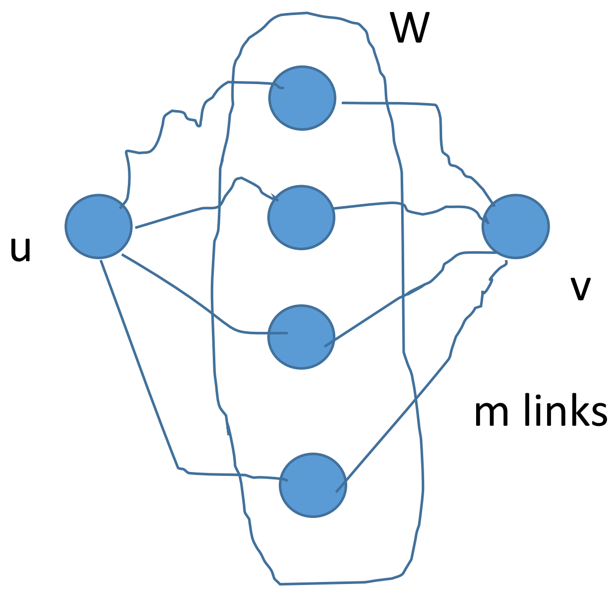
Consider a vertex w2, which is not adjacent to v

adjacent to v

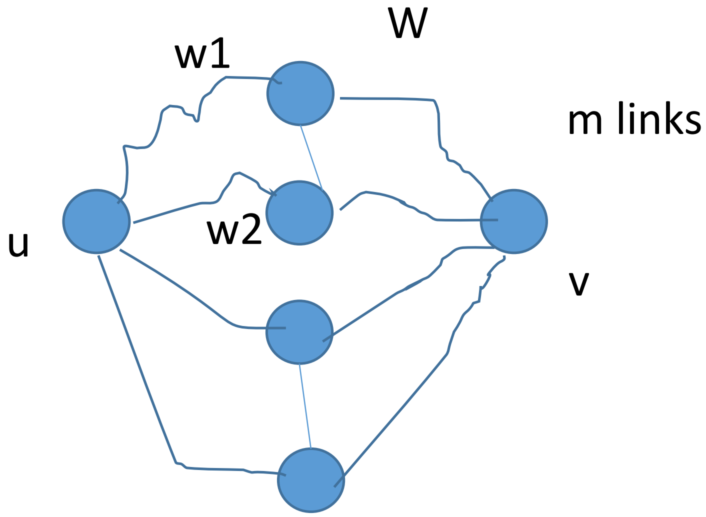
Menger's Theorem

The size of G_u is <m



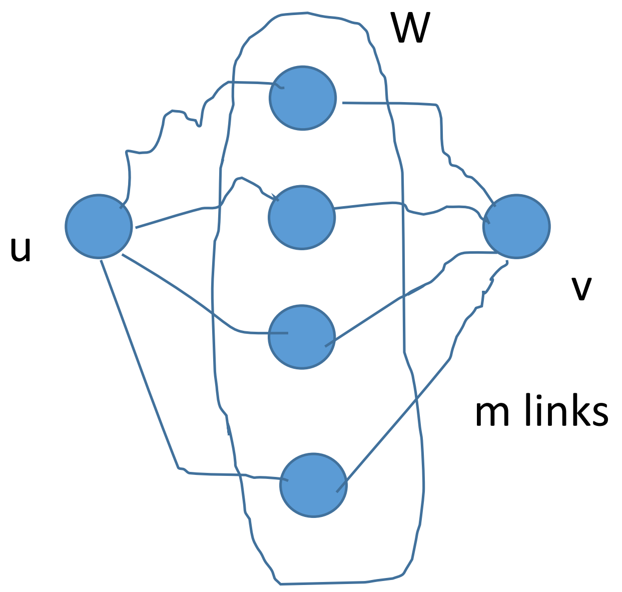


Possibility 2

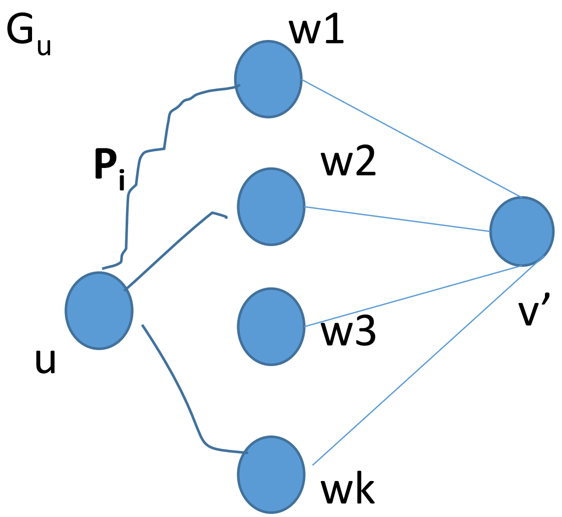


You may have links between W

Menger's Theorem

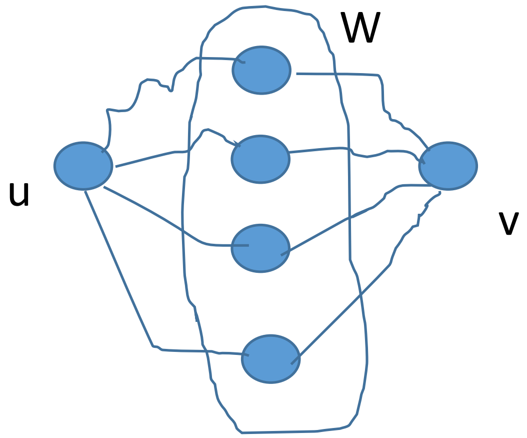


The size of G_u is <m

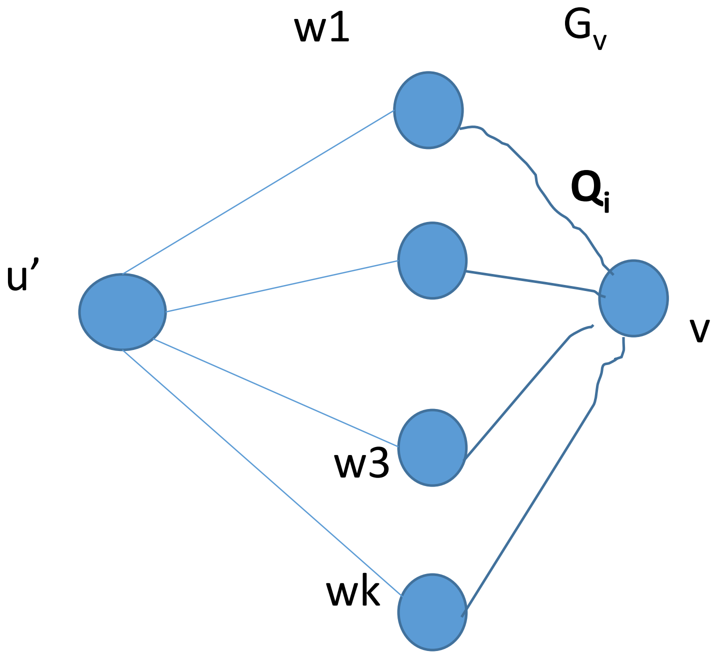


∃ a maximum of **k internally disjoint u–v' paths in G_u** , each one consisting of a **u–w_i path P_i**, followed by w_i-v'

Menger's Theorem

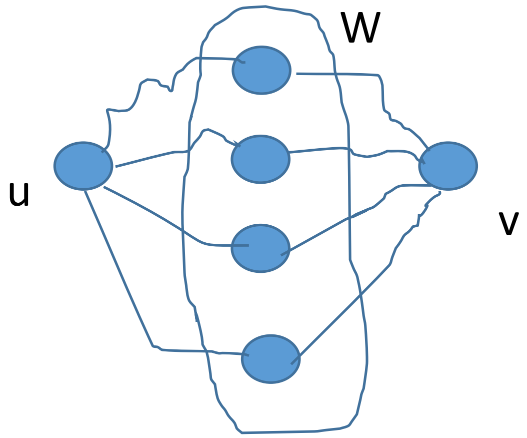


The size of G_v is <m

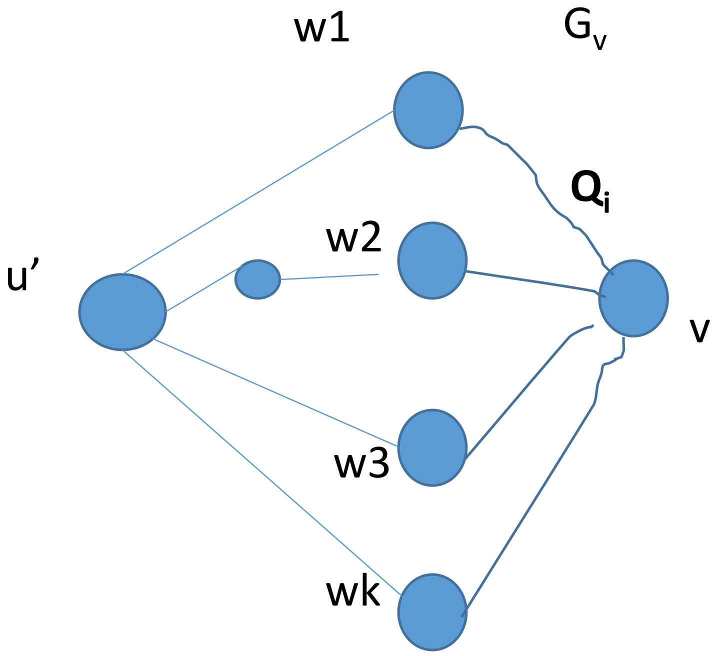


∃ a maximum of **k internally disjoint u'–v paths in G_v** , each one consisting of a **w_i-v path Q_i**, preceded by u'-w_i

Menger's Theorem



The size of G_v is <m

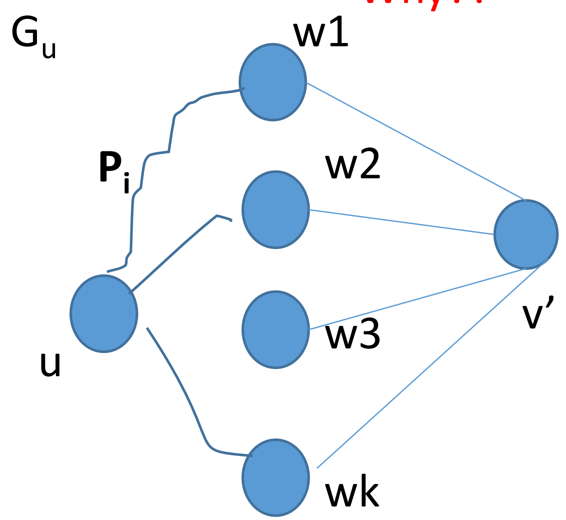


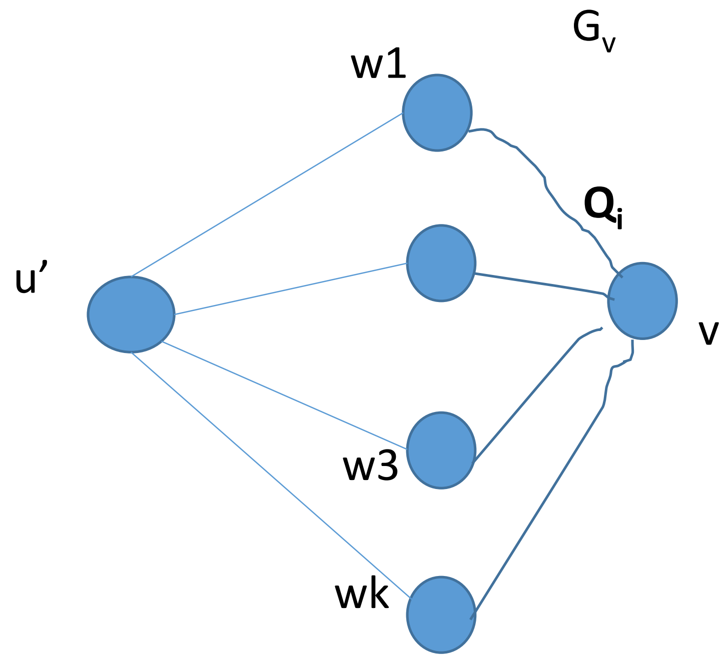
∃ a maximum of **k internally disjoint u'–v paths in G_v**, each one consisting of a **w_i-v path Q_i**, preceded by u'-w_i

Stich each P_i path of G_u and each path Q_i of G_v at the overlapping node w_i

Note: all P_i and Q_i are internally disjoint paths

Why??





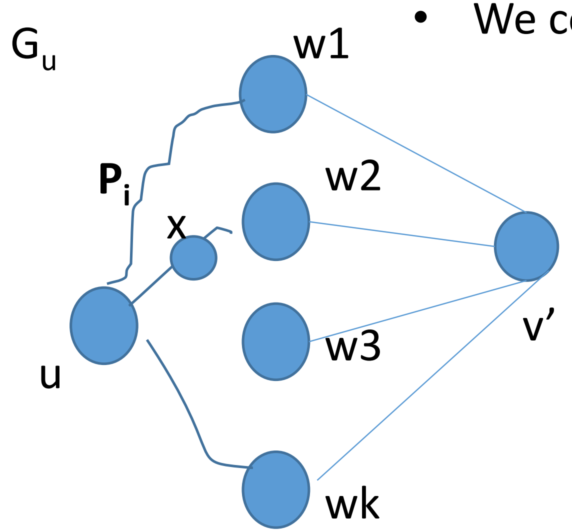
The k paths consisting of P_i followed by Q_i for $1 <= i <= k$ are k internally disjoint paths in G

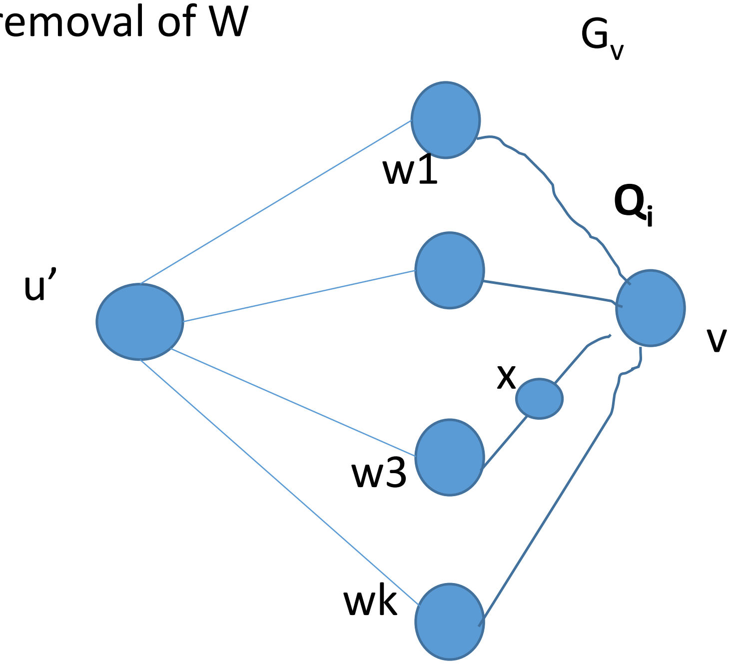
Stich each P_i path of G_u and each path Q_i of G_v at the overlapping node w_i

Note: all P_i and Q_i are internally disjoint paths

Why??

- If there exists an overlapping node x
- We could be able to connect u-v, even after the removal of W

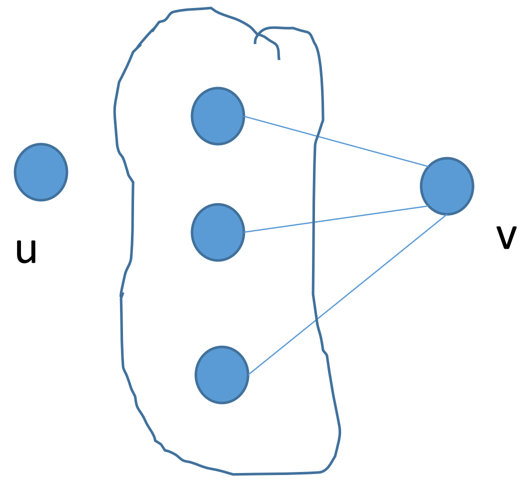




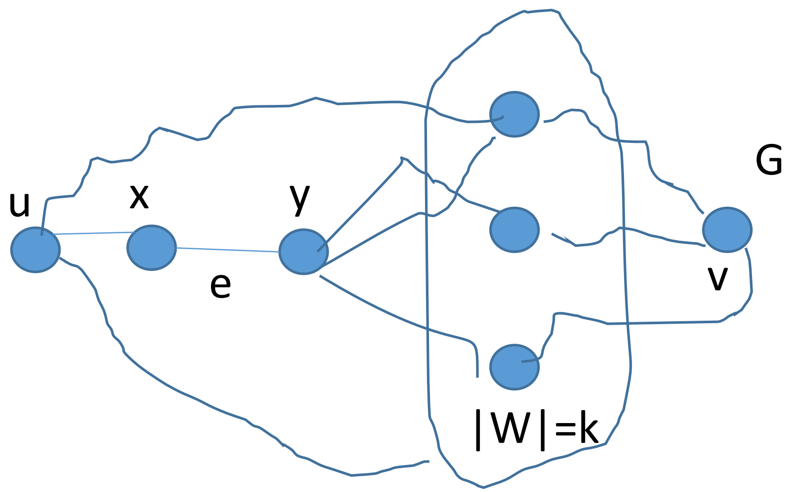
The k paths consisting of P_i followed by Q_i for 1 <= i <= k are k internally disjoint paths in G

Menger's Theorem

Case 3: For every a minimum u-v separating set W in G, either every vertex of W is adjacent to u, but not v, or every vertex of W is adjacent to v, but not u



Menger's Theorem



Let P = (u, x, y, ….., v) be a u–v shortest path.

Presence of (x, y) avoids case 1

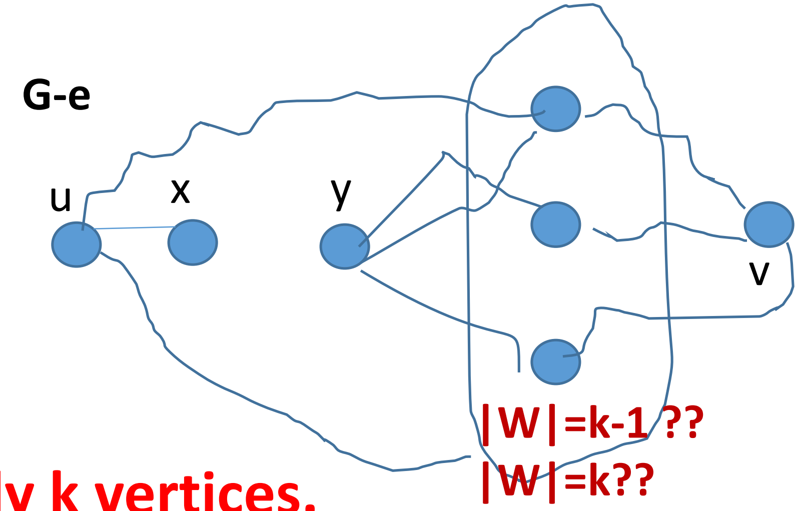
e = x-y

Remove e from G

Claim: Every minimum u–v separating set in G-e contains at least k-1 vertices.

It contains (k-1) separating vertices in G-e. For G, we remove x or y → k vertices in separating set W of G

If it contains (k-2) vertices?? → W would contain (k-1) vertices in G



Let P = (u, x, y, ….., v)$ be a u–v shortest path.

Claim: Every minimum u–v separating set in G-e contains exactly k vertices.

Contradiction

Assume: Every minimum u–v separating set in G-e contains exactly k-1 vertices.

Let S = {s_1, s_2, ….., s_{k-1}} be a minimum u–v separating set in G-e

If S is the minimum u-v separating set for (G-e)

Then S U {x} is a minimum u–v separating set in G.

Note: Either every vertex of W is adjacent to u, but not v, or every vertex of W is adjacent to v, but not u

Thus all vertices of S are adjacent to u.

Also, S U {y}$ is a minimum u–v separating set in G.

→ y is adjacent to u too

Menger's Theorem

Let P = (u, x, y, ….., v) be a u–v shortest path.

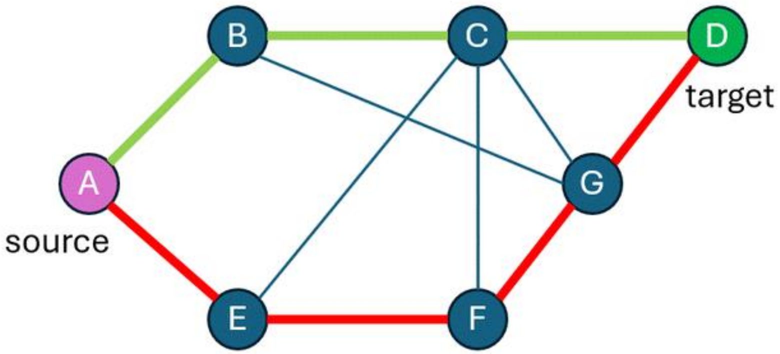
- Both x, and y are adjacent to u
- This is not possible
  - P is a shortest path
- Hence, every minimum separating set of G-e contains **exactly k vertices** (cannot contain>k vertices)

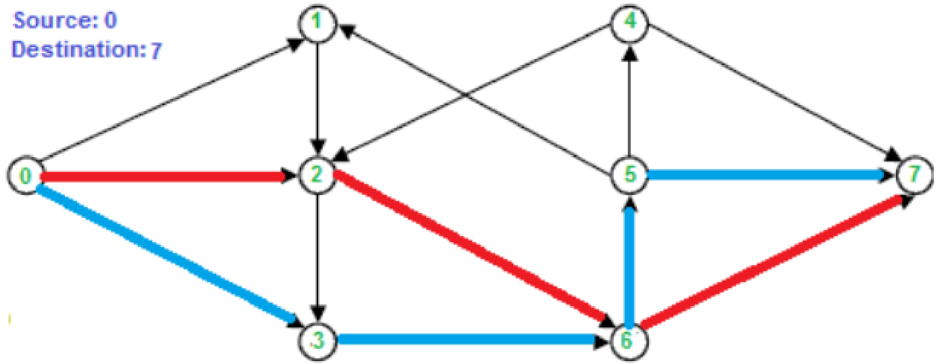
Apply induction hypothesis:

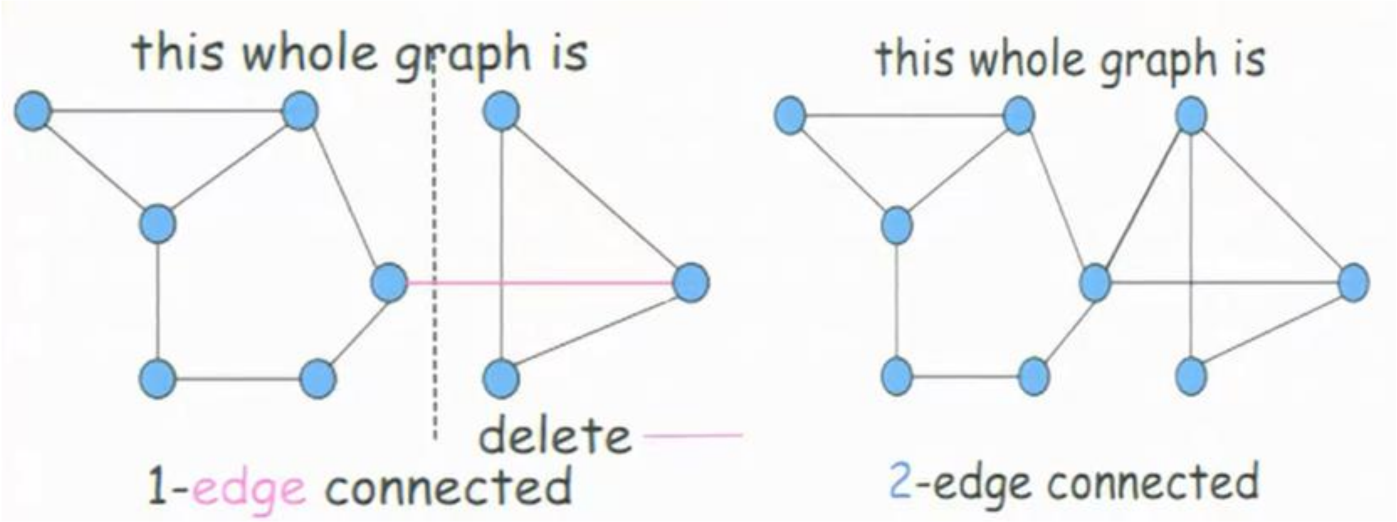
There exists a maximum k internally disjoint path (u, v) path in G-e, **and same for G**

(as $\boldsymbol{\kappa}$**(x, y)>=** $\boldsymbol{\lambda}$**(x, y))**

Edge-disjoint paths and disconnecting set of edges







k-Connected and k-Edge-Connected Graphs

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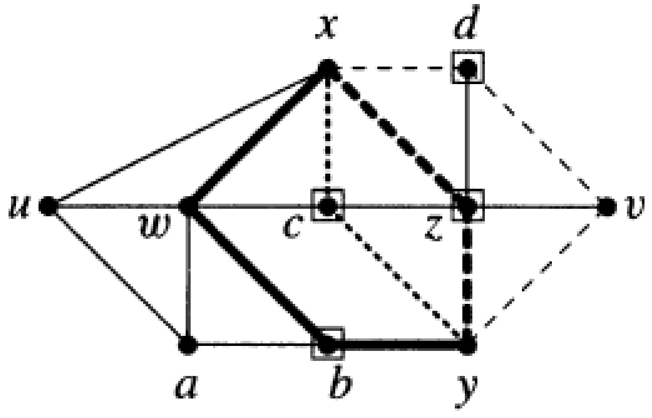
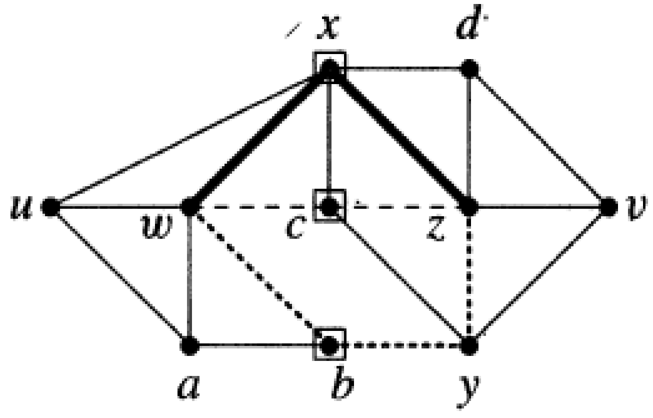
- From the equality for internally disjoint paths, we will obtain an analogous equality for edge-disjoint paths.
- Although κ(w,z) = 3 above, it takes **four edges to break all w, z-paths**, and **there are four pairwise edge-disjoint w, z-paths**.





Path	Edges
P_1	wc, cz
P_2	wx, xz
P_3	wu, ux, xd, dz
P_4	wb, by, yz

The following four edges disconnect w from z:

$$\boxed{\{wc,\ wx,\ ux,\ by\}}$$

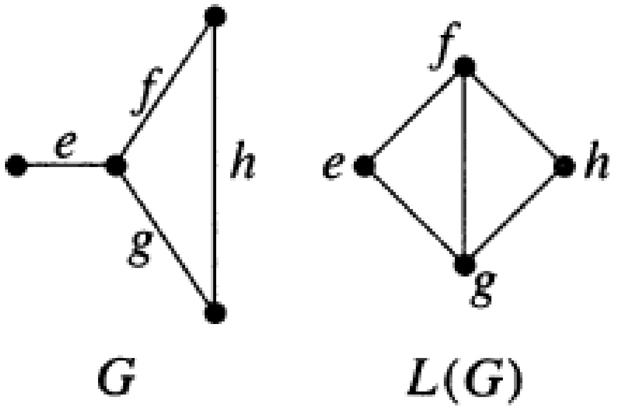
Line Graph

Definition: Line Graph

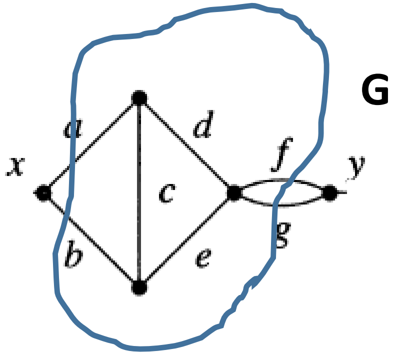
The line graph of a graph G, written L(G), is the graph whose vertices are the edges of G,

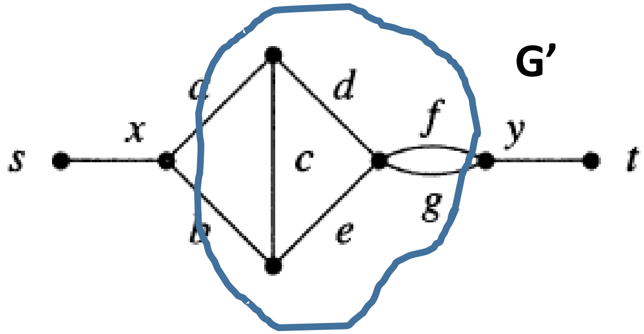
with edge e-f $\in$ E(L(G)) when e=u-v and f=v-w in G (edge **e** and **f** shares a common vertex **v**)

Example



Theorem



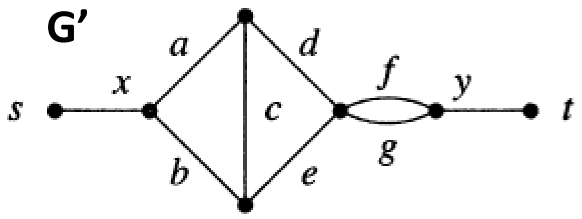


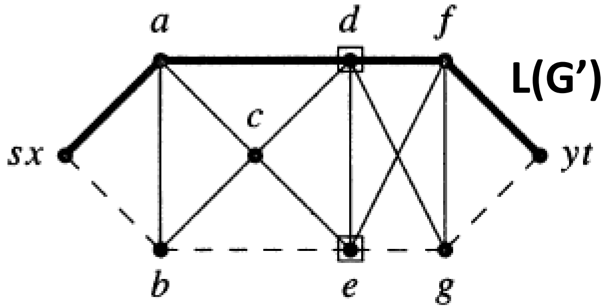
Theorem: If x and y are distinct vertices of a graph G, then the **minimum size of an {x, y}-disconnecting set of edges** equals the **maximum number of pairwise edge-disjoint x, y-paths**.

The construction

- Build graph G' from G by **adding two new vertices (s, t)** and two new edges **s-x and y-t.**
  - This doesn't change $\kappa'(x,y)$ or $\lambda'(x,y)$ in G — the extra pendant edges just give every {x,y}-path (in G) a natural extension to an s,t-path in G' (start with edge s-x, end with edge y-t).

Theorem



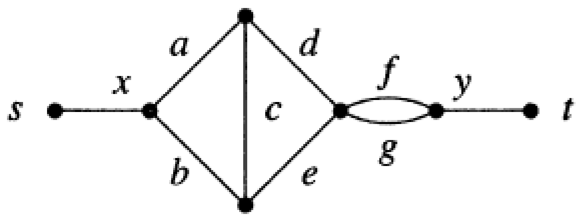


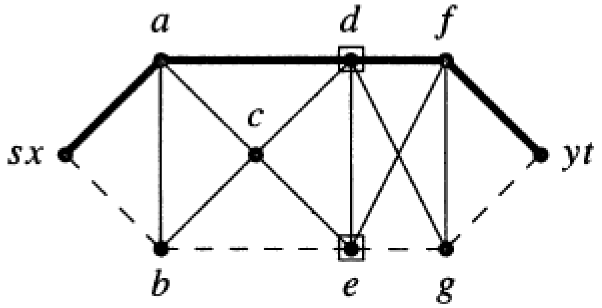
Construct the line graph L(G'). Now:

- Each **edge of G'** becomes a vertex of **L(G')**.

- A **set of edges disconnecting y from x in G** ($\kappa'_G(x,y)$) corresponds **exactly**
  - to **a set of vertices in L(G')** that forms an **sx,yt-cut** (i.e., disconnects the vertex "sx" from the vertex "yt" in L(G’)) ---- ($\kappa_{L(G')}(sx,yt)$)
- **Reason:** In G, a **path from x to y using edges {e1, e2, ….}** becomes, in L(G'), a **path through the vertices {e1, e2, …..}** (each consecutive pair adjacent since they share an endpoint) — starting at vertex **sx** and ending at vertex **yt**.
- Similarly, **edge-disjoint {x, y}-paths in G** ($\lambda'_G(x,y)$) become **internally vertex-disjoint {sx,yt}-paths in L(G’)** …… ($\lambda_{L(G')}(sx,yt)$)
- **Reason:** Since **each edge of G** is used by **at most one path**, the corresponding **vertex of L(G')** appears in **at most one path**.

Theorem





- **Since x ≠ y$**, there's **no edge directly from sx to yt** in L(G’),
- so **sx and yt are non-adjacent vertices** — a requirement for applying the vertex form of Menger's theorem.
- Now apply Menger's theorem to the pair (sx, yt) in L(G')

- $\boldsymbol{\kappa_{L(G')}(sx, yt) = \lambda_{L(G')}(sx, yt)}$

- Translating back through the correspondences in steps 2

- $\kappa'_G(x,y) = \kappa_{L(G')}(sx,yt) = \lambda_{L(G')}(sx,yt) = \lambda'_G(x,y)$

Lemma

Lemma: If x-y is an edge of graph G, then

Deleting a single edge can reduce vertex connectivity by at most 1

$\boldsymbol{\kappa(G)-1 \leq \kappa(G-xy) \leq \kappa(G)}$

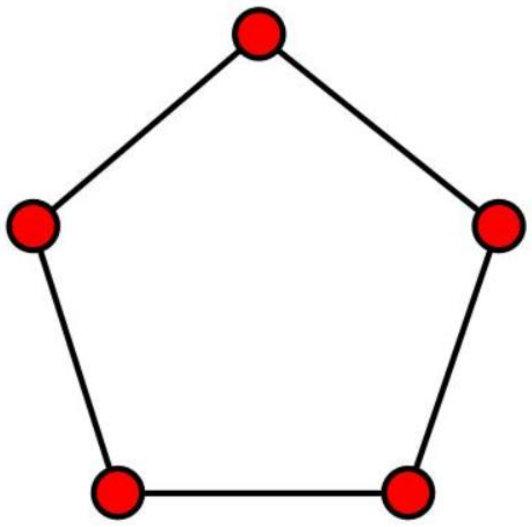
It means:

- connectivity may **stay the same**, or
- connectivity may **decrease by exactly 1**,
- but it cannot decrease by 2 or more.

For a cycle,

$$\boxed{\kappa(C_5) = 2}.$$

Why? You need to remove two vertices to disconnect the cycle.



Now delete one edge.

The cycle becomes a path:

$$P_5.$$

For a path,

$$\boxed{\kappa(P_5) = 1}.$$

So here

$$\kappa(C_5 - xy) = 1$$

and

$$\kappa(C_5) = 2.$$

Therefore,

$$2 - 1 = 1 = \kappa(C_5 - xy).$$

The connectivity **decreased by exactly 1**.

Examples

We know

$$\boxed{\kappa(K_4) = 3}.$$

Now remove one edge, say 12.

The resulting graph is $K_4 - 12$.

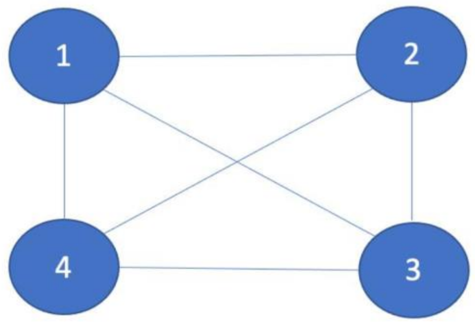
What is its connectivity?

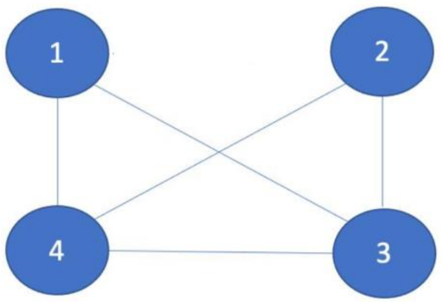
$$\boxed{\kappa(K_4 - 12) = 2}.$$

So again,

$$3 \longrightarrow 2.$$

The decrease is exactly 1.





Examples

The shared vertex is a cut vertex, so

$$\kappa(G) = 1.$$

Now delete an edge from one of the triangles.

The graph is still connected, and there is still a cut vertex.

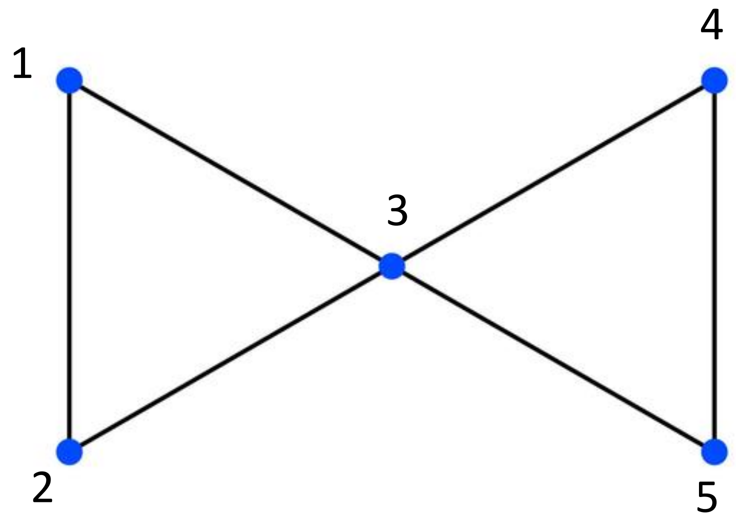
Thus

$$\kappa(G - xy) = 1.$$

So:

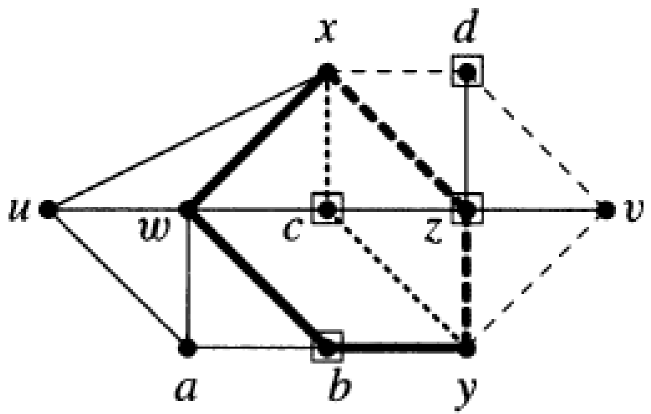
$$\boxed{\kappa(G - xy) = \kappa(G)}.$$

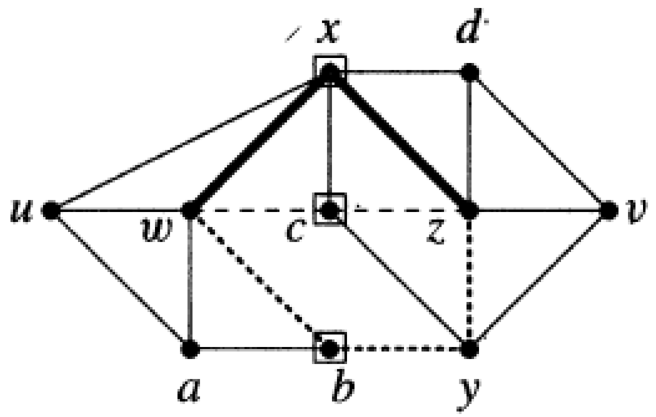
The connectivity decreased by **0**.



k-Connected and k-Edge-Connected Graphs

- Consider also **the pair (w, z)**. As shown on the right, **$\kappa(w,z) = \lambda(w,z) = 3$**, with **{b, c, x} being a minimum {w, z}-cut**.
- The graph G is **3-connected**; for every pair {u, v $\in$ V(G), we can find **three pairwise internally disjoint {u, v}-paths**.





Theorem: The (vertex) connectivity of G equals the maximum k such that λ(x, y) >= k, for all (x, y) ∈ V(G).
The edge-connectivity of G equals the maximum k such that λ'(x, y) >= k for all (x, y) ∈ V(G).

The theorem says

$$\boxed{\kappa(G) = \max\{k : \lambda(x, y) \geq k \text{ for all } x, y \in V(G)\}}$$

where:

- $\kappa(G)$ = vertex connectivity of G;
- $\kappa(x, y)$ = local vertex connectivity between x, y;
- $\lambda(x, y)$ = maximum number of **internally vertex-disjoint** $x - y$ paths.

Let's define

$$K = \max\{k : \lambda(x, y) \geq k \text{ for all } x, y \in V(G)\}.$$

Our goal is to prove

$$\boxed{K = \kappa(G)}.$$

Theorem

$$K = \max\{k : \lambda(x,y) \geq k \text{ for all } x, y \in V(G)\}.$$

Rewrite K

Suppose the values of $\lambda(x,y)$ for all pairs are

$$4,\ 6,\ 5,\ 4,\ 7.$$

The largest k such that **every** value is at least k is clearly the smallest value, namely 4.

Therefore,

$$\boxed{K = \min_{x,y} \lambda(x,y)}.$$

So our goal can be rewritten as

$$\boxed{\min_{x,y} \lambda(x,y) = \kappa(G)}.$$

Now we prove the two inequalities.

Theorem

Part A: Prove $K \leq \kappa(G)$

Remember:

$$K = \min_{x,y} \lambda(x, y).$$

We want to show

$$K \leq \kappa(G).$$

Find a particular pair

By the definition of vertex connectivity, there is a pair u, v such that

$$\kappa(u, v) = \kappa(G).$$

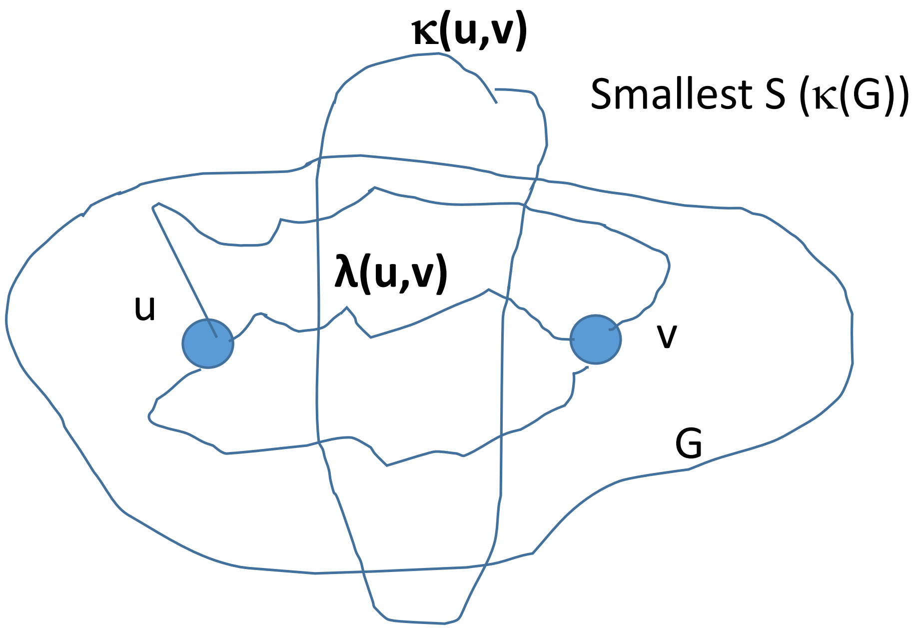
We can choose u, v to be nonadjacent.

Since $uv \notin E(G)$, Menger's theorem gives

$$\lambda(u, v) = \kappa(u, v).$$

Hence

$$\lambda(u, v) = \kappa(G).$$



But K is the **minimum** of all the $\lambda(x, y)$'s.

Therefore,

$$K \leq \lambda(u, v).$$

Since $\lambda(u, v) = \kappa(G)$,

$$\boxed{K \leq \kappa(G)}.$$

Theorem

Part B: Prove $K \geq \kappa(G)$

This part means:

> We must show that **every pair** x, y has at least $\kappa(G)$ internally vertex-disjoint paths.

So take an **arbitrary** pair x, y.

We have two cases.

Case 1: x and y are nonadjacent

Suppose

$$xy \notin E(G).$$

Menger gives

$$\lambda(x,y) = \kappa(x,y).$$

But $\kappa(G)$ is the minimum of all the local connectivities.

Therefore,

$$\kappa(x,y) \geq \kappa(G).$$

Hence

$$\boxed{\lambda(x,y) \geq \kappa(G)}.$$

So every nonadjacent pair is fine.

Case 2: x and y are adjacent

Now suppose

$$xy \in E(G).$$

This is the only interesting case.

We remove the edge xy:

$$G' = G - xy.$$

Now x and y are nonadjacent in G'.

Therefore, by Menger,

$$\lambda_{G'}(x,y) = \kappa_{G'}(x,y).$$

And because $\kappa(G')$ is the minimum local connectivity in G',

$$\kappa_{G'}(x,y) \geq \kappa(G').$$

Therefore,

$$\boxed{\lambda_{G'}(x,y) \geq \kappa(G')}.$$

Step 2: Put xy back

When we return from G' to G, the edge xy comes back.

That edge itself is an $x - y$ path:

$$x \longrightarrow y.$$

It has **no internal vertices**.

Therefore, it is internally vertex-disjoint from all the $x - y$ paths in G'.

So we get one additional path:

$$\boxed{\lambda_G(x, y) = 1 + \lambda_{G'}(x, y)}.$$

Since

$$\lambda_{G'}(x, y) \geq \kappa(G'),$$

we get

$$\lambda_G(x, y) \geq 1 + \kappa(G').$$

Step 3: Use the lemma

$$\boxed{\kappa(G - xy) \geq \kappa(G) - 1}.$$

Since $G' = G - xy$,

$$\kappa(G') \geq \kappa(G) - 1.$$

Therefore,

$$1 + \kappa(G') \geq 1 + \kappa(G) - 1 = \kappa(G).$$

Hence

$$\boxed{\lambda_G(x, y) \geq \kappa(G)}.$$

So the adjacent pair is also fine.

Theorem

For an **arbitrary** pair x, y, we proved

$$\boxed{\lambda(x,y) \geq \kappa(G)}.$$

This is true for every pair.

Therefore the minimum of all the $\lambda(x,y)$'s must also be at least $\kappa(G)$:

$$\boxed{K \geq \kappa(G)}.$$

Remember, $K = \min \lambda(x,y)$.

Theorem

From Part A:

$$\boxed{K \le \kappa(G)}$$

From Part B:

$$\boxed{K \ge \kappa(G)}$$

Therefore,

$$\boxed{K = \kappa(G)}.$$

Since

$$K = \max\{k : \lambda(x, y) \ge k \text{ for every } x, y\},$$

we obtain

$$\boxed{\kappa(G) = \max\{k : \lambda(x, y) \ge k \text{ for all } x, y \in V(G)\}.}$$