

Graph Connectivity – Tutorial Problems

Problem 1. Let $G = K_{m,n}$, where $1 \leq m \leq n$. Determine the vertex connectivity $\kappa(G)$, edge connectivity $\kappa'(G)$, and minimum degree $\delta(G)$. Verify Whitney's inequality

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Problem 2. Let G be a connected graph and let $S \subset V(G)$ be nonempty and proper. Suppose that both $G[S]$ and $G[V(G) \setminus S]$ are connected. Prove that the edge set

$$[S, V(G) \setminus S]$$

is a bond of G .

Problem 3. Let G be a graph with $|V(G)| \geq 3$. Suppose that every pair of vertices of G lies on a common cycle. Prove that G is 2-connected.

Problem 4. Let G be a 4-connected graph. Construct G' by adding two new vertices x and y such that x and y are nonadjacent, and each of x and y is adjacent to at least four vertices of G . Prove that G' is 4-connected.

Problem 5. Let u and v be distinct nonadjacent vertices of a graph G . Suppose that no set of fewer than four vertices separates u from v . What can be concluded about the maximum number of pairwise internally vertex-disjoint u - v paths? In particular, if three pairwise internally vertex-disjoint u - v paths are already known to exist, does this rule out the existence of a fourth such path? Justify your answer using Menger's theorem.

Problem 6. Let G be a connected graph and let $uv \in E(G)$. If $\kappa(G) = k$, prove that

$$k - 1 \leq \kappa(G - uv) \leq k.$$

Give examples showing that either bound can be attained.