GENERATING FUNCTIONS

1. Introduction

The concept of **generating functions** is a powerful tool for solving counting problems. Intuitively put, its general idea is as follows. In counting problems, we are often interested in counting the number of objects of 'size *n*', which we denote by a_n. By varying *n*, we get different values of a_n. In this way we get a **sequence** of real numbers

$$a_0,\ a_1,\ a_2,\ \dots$$

from which we can define a **power series** (which in some sense can be regarded as an 'infinite-degree polynomial')

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots.$$

The above $G(x)$ is the generating function for the sequence a_0, a_1, a_2, ….

In this set of notes we will look at some elementary applications of generating functions. Before formally introducing the tool, let us look at the following example.

Example 1.1.

(IMO 2001 HK Preliminary Selection Contest) Find the coefficient of x^{17} in the expansion of $(1+x^5+x^7)^{20}$.

Solution.

The only way to form an x^{17} term is to gather two x^5 and one x^7. Since there are $C_2^{20} = 190$ ways to choose two x^5 from the 20 multiplicands and 18 ways to choose one x^7 from the remaining 18 multiplicands, the answer is $190 \times 18 = 3420$.

To gain a preliminary insight into how generating functions is related to counting, let us describe the above problem in another way. Suppose there are 20 bags, each containing a \$5 coin and a \$7 coin. If we can use at most one coin from each bag, in how many different ways can we pay \$17, assuming that all coins are distinguishable (i.e. the \$5 coin from the first bag is considered to be different from that in the second bag, and so on)? It should be quite clear that the answer is again 3420 — to pay \$17, one must use two \$5 coins and one \$7 coin. There are $C_2^{20} = 190$ ways to choose two \$5 coins from the 20 bags, and 18 ways to choose a \$7 coin from the remaining 18 bags. Using the notations we introduced at the very beginning, we could say that $a_{17} = 3420$.

2. Techniques of Computation

Let us once again give the definition of a generating function before we proceed.

> **Definition.** Given a sequence a_0, a_1, a_2, …, we define the **generating function** of the sequence $\{a_n\}$ to be the power series
>
> $$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Let us look at a few examples.

Example 2.1.

Find the generating functions for the following sequences. In each case, try to simplify the answer.

(a) 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, …

(b) 1, 1, 1, 1, 1, …

(c) 1, 3, 3, 1, 0, 0, 0, 0, …

(d) C_0^{2005}, C_1^{2005}, C_2^{2005}, …, C_{2005}^{2005}, 0, 0, 0, 0, …

Solution.

(a) The generating function is

$$\begin{aligned} G(x) &= 1 + 1x + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 0x^6 + 0x^7 + \cdots \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 \end{aligned}$$

We can apply the formula for the sum of a geometric series to rewrite $G(x)$ as $G(x) = \dfrac{1-x^6}{1-x}$.

(b) The generating function is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + \cdots.$$

When $|x| < 1$, we can apply the formula for the sum to infinity of a geometric series to rewrite $G(x)$ as $G(x) = \dfrac{1}{1-x}$. In working with generating functions, we shall ignore the question of convergence and simply say $G(x) = \dfrac{1}{1-x}$.

(c) The generating function is $G(x) = 1 + 3x + 3x^2 + 1$, and of course, the binomial theorem enables us to simplify the answer as $G(x) = (1+x)^3$.

(d) The generating function is

$$G(x) = C_0^{2005} + C_1^{2005}x + C_2^{2005}x^2 + \cdots + C_{2004}^{2005}x^{2004} + C_{2005}^{2005}x^{2005},$$

and the binomial theorem once again enables us to simplify the answer as $G(x) = (1+x)^{2005}$.

When dealing with computations of generating functions, we are particularly interested with two things, namely, whether the generating function can be written in *closed form* and whether we can find the coefficient of a certain power of x easily.

To write a generating function in 'closed form' means, in general, writing it in a 'direct' form without summation sign nor '$\cdots$'. For instance, in Example 2.1 (b), $G(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$ is not in closed form while $G(x) = \frac{1}{1-x}$ is. The reason for trying to put a generating function in closed form is as follows. In the more advanced theory of generating functions (beyond the level of this set of notes), we will find that certain combinations of problems correspond to certain operations (e.g. addition, multiplication or more complicated operations) on generating functions. If we can find a generating function in closed form, the computations can be greatly simplified and easily carried out.

On the other hand, we are interested in knowing the coefficient of a certain power of x because, as we have remarked at the very beginning, it often refers to the number of objects of size n, which is usually the thing we wish to find in counting problems.

Clearly, if a generating function is given in 'explicit form', such as

$$G(x) = x + 2x^2 + 3x^3 + 4x^4 + \cdots \quad \text{or} \quad G(x) = \sum_{n=0}^{\infty} \frac{n-1}{2n+1} x^n \ ,$$

then finding a specific coefficient will be easy. However, if a generating function is given in closed form, ingenious tricks are sometimes required in determining certain coefficients. The following example illustrates some common tricks.

Example 2.2.

In each of the following, find the coefficient of x^{2005} in the generating function $G(x)$.

(a) $G(x) = (1-2x)^{5000}$

(b) $G(x) = \dfrac{1}{1+3x}$

(c) $G(x) = \dfrac{1}{(1+5x)^2}$

Solution.

(a) By the binomial theorem, we have

$$G(x) = 1 - C_1^{5000}(2x) + C_2^{5000}(2x)^2 - \cdots - C_{4999}^{5000}(2x)^{4999} + (2x)^{5000}.$$

From this, we see that the coefficient of x^{2005} is $-2^{2005}C_{2005}^{5000}$.

(b) Recalling the formula for the sum to infinity of a geometric series, we have (noting once again that everything is dealt with formally, ignoring questions of convergence)

$$G(x) = \frac{1}{1-(-3x)} = 1 + (-3x) + (-3x)^2 + (-3x)^3 + \cdots = 1 - 3x + 3^2x^2 - 3^3x^3 + \cdots.$$

From this, we see that the coefficient of x^{2005} is -3^{2005}.

(c) As in (b), we know that

$$\frac{1}{1+5x} = 1 - 5x + 5^2x^2 - 5^3x^3 + \cdots.$$

Hence

$$G(x) = (1 - 5x + 5^2x^2 - 5^3x^3 + \cdots)(1 - 5x + 5^2x^2 - 5^3x^3 + \cdots).$$

To form an x^{2005} term, we can multiply 1 with $-5^{2005}x^{2005}$, $-5x$ with $5^{2004}x^{2004}$, 5^2x^2 with $-5^{2003}x^{2003}$ and so on, and finally $-5^{2005}x^{2005}$ with 1. There are altogether 2006 products, each equal to $-5^{2005}x^{2005}$. It follows that the coefficient of x^{2005} is -2006×5^{2005}.

The technique used in Example 2.2 (c) is rather 'ad-hoc' in nature. It will not work if the power 2 is increased to higher powers. (At least it will involve much more analysis.) To deal with higher powers, we shall need an extended version of the binomial theorem. In fact, the generating function in Example 2.2 (b) and (c) can be written as $G(x) = (1+3x)^{-1}$ and $G(x) = (1+5x)^{-2}$ respectively. If we can expand them using more or less the same method as in Example 2.2 (a), then the computation will be much simpler. For this purpose, we attempt to generalize the binomial theorem for positive integral indices. We begin by extending the usual notion of binomial coefficients to non-integer values.

Definition. For any <u>real number</u> u and <u>positive integer</u> k, we define the extended binomial coefficient C_k^u by

$$C_k^u = \frac{u(u-1)(u-2)\cdots(u-k+1)}{k!}.$$

We also define $C_0^u = 1$ for any real number u.

Clearly, if u is a positive integer with $u \geq k$, then the above extended binomial coefficient agrees with the usual binomial coefficient. Also, if u and k are positive integers with $u < k$, then we have $C_k^u = 0$. This is natural from a combinatorial point of view: if $u < k$, there is no way to choose k different objects from a collection of u objects. With the notion of the extended binomial coefficients, we can state the extended binomial theorem as follows.

Theorem (Extended Binomial Theorem). For any real number u, we have

$$(1+x)^u = C_0^u + C_1^u x + C_2^u x^2 + \cdots.$$

Again, if u is a positive integer, we see that the extended binomial theorem agrees with the usual binomial theorem by noting that $C_k^u = 0$ when $u < k$.

Now if we go back to Example 2.2 (c), we see at once, with the extended binomial theorem, that the answer is $5^{2005} \times C_{2005}^{-2}$. Note that we have

$$C_{2005}^{-2} = \frac{(-2)\times(-3)\times(-4)\times\cdots\times(-2005)\times(-2006)}{2005!} = \frac{-2006!}{2005!} = -2006,$$

so that the answer agrees with what we obtained in Example 2.2 (c).

Since we often have to compute extended binomial coefficients of the form C_k^u where u is a negative integer, it is useful to relate the extended binomial coefficients with the usual binomial coefficients. We have the following.

Theorem. For positive integers n and r, we have

$$C_r^{-n} = (-1)^r C_r^{n+r-1}.$$

Hence we have $C_{2005}^{-2} = (-1)^{2005} C_{2005}^{2+2005-1} = -C_{2005}^{2006} = -2006$, as we have previously worked out. It should be noted that the right hand side of the above formula is in fact $(-1)^r H_r^n$.

With the extended binomial theorem, computation of coefficients for generating functions like $\frac{1}{(1+x)^5}$ and $\sqrt{1+x}$ becomes easy. Sometimes we also need the technique of partial fraction decomposition in the computation, as the following example shows.

Example 2.3.

What is the coefficient of x^{2005} in the generating function $G(x)=\frac{1}{(1-x)^2(1+x)^2}$?

Solution.

Let $\frac{1}{(1-x)^2(1+x)^2}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{1+x}+\frac{D}{(1+x)^2}$.

Upon simplification, the right hand side becomes

$$\frac{(C-A)x^3+(B+D-A-C)x^2+(A+2B-C-2D)x+(A+B+C+D)}{(1-x)^2(1+x)^2}.$$

Comparing coefficients, we have

$$\begin{cases} C-A=0 \\ B+D-A-C=0 \\ A+2B-C-2D=0 \\ A+B+C+D=1 \end{cases}.$$

Solving, we get $A=B=C=D=\frac{1}{4}$. It follows that

$$G(x)=\frac{1}{4}\left[(1-x)^{-1}+(1-x)^{-2}+(1+x)^{-1}+(1+x)^{-2}\right].$$

Thus the coefficient of x^{2005} is $\frac{1}{4}(-C_{2005}^{-1}-C_{2005}^{-2}+C_{2005}^{-1}+C_{2005}^{-2})=0$.

Finally, for those who know calculus, the following two examples illustrate some further computation techniques in dealing with generating functions.

Example 2.4.

Find the generating functions of the following sequences in closed form.

(a) 1, 2, 3, 4, 5, 6, 7

(b) 0, 1, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{4}$, …

Solution.

Formally, we can differentiate and integrate a power series term by term. In other words, if

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

then

$$G'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots.$$

The same is true for integration in place of differentiation.

(a) The generating function is

$$\begin{aligned} G(x) &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \frac{d}{dx}(x + x^2 + x^3 + x^4 + \cdots) \\ &= \frac{d}{dx}\left(\frac{x}{1-x}\right) \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

We can verify the answer by expanding $\frac{1}{(1-x)^2}$ using the extended binomial theorem.

(b) The generating function is

$$\begin{aligned} G(x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\ &= \int (1 - x + x^2 - x^3 + \cdots)\, dx \\ &= \int \frac{dx}{1+x} \\ &= \ln(1+x) + C \end{aligned}$$

To find the constant *C*, we put in $x = 0$ to get $C = G(0)$. If we write

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots,$$

then $G(0)$ is simply equal to a_0, which is 0 in this case. Hence the answer is

$$G(x) = \ln(1+x).$$

Example 2.5.

Find the coefficient of x^{2005} for each of the following generating functions.

(a) $G(x) = \ln(1-x)$

(b) $G(x) = \sin x$

Solution.

(a) We have $G(x) = \int \left(-\frac{1}{1-x}\right) dx$

$$= \int (-1 - x - x^2 - x^3 - \cdots) dx$$

$$= C - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

where $C = G(0) = 0$. From the above expression, we see that the coefficient of x^{2005} is $-\frac{1}{2005}$.

(b) Let $\sin x = G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Setting $x = 0$, we have

$$0 = a_0.$$

Differentiating both sides of $G(x)$ with respect to *x*, we have

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots.$$

Setting $x = 0$ again, we have $1 = a_1$. Differentiating the above equality with respect to *x*, we have

$$-\sin x = 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \cdots.$$

Again, setting $x = 0$ gives $0 = 2a_2$. Continuing this process, we see that

$$\begin{aligned} -1 &= 3!a_3 \\ 0 &= 4!a_4 \\ 1 &= 5!a_5 \\ 0 &= 6!a_6 \\ -1 &= 7!a_7 \\ 0 &= 8!a_8 \\ &\vdots \end{aligned}$$

Since $2005 \equiv 1$ (mod 4), the coefficient of x^{2005} is $\frac{1}{2005!}$.

For those who know about Taylor series, it should be noted that exactly the same idea is employed in part (b) above. In fact, given a generating function $G(x)$ in closed form, the coefficient of x^n is given by $\frac{1}{n!}a_n G^{(n)}(0)$.

3. Applications of Generating Functions

A. Basic Counting Problems

Some well-known results in combinatorics can be reproduced by means of generating functions, as the following examples show.

Example 3.1.

There are 50 students in the International Mathematical Olympiad (IMO) training programme. 6 of them are to be selected to represent Hong Kong in the IMO. How many ways are there to select 6 students?

Solution.

Each student is either selected or not selected. Hence each student contributes a factor of $1+x$ to the generating function, where the term 1 (i.e. x^0) refers to the case when the student is not selected (i.e. the student occupies 0 place) while the term x (i.e. x^1) refers to the case when the student is selected (i.e. the student occupies 1 place). Since there are 50 students, the generating function is

$$G(x) = (1+x)^{50}.$$

Since 6 students are to be selected, the answer is the coefficient of x^6 in $G(x)$, which, according to the binomial theorem, is C_6^{50}. This, of course, agreed with what we would have obtained without using generating functions.

It is worth noting how the generating function is formed. Basically, it is formed by a sequence of '+'s and '×'s, corresponding to a sequence of 'OR's and 'AND's, very much like how counting problems are typically formulated. For each student, he is either selected OR not selected, so each student contributes a factor of $1+x$. Now we need to do the same for the 1st student AND the 2nd student AND the 3rd student AND so on. That's why we multiply 50 copies of $1+x$ together to form the generating function.

Example 3.2.

There are 30 identical souvenirs, to be distributed among the 50 IMO trainees, and each trainee may get more than one souvenir. How many ways are there to distribute the 30 souvenirs among the 50 trainees?

Solution.

Each student may get 0 OR 1 OR 2 OR … souvenirs, thus contributing a factor of $1+x+x^2+\cdots$. Since there are 50 students, the generating function is

$$G(x)=(1+x+x^2+\cdots)^{50}=\left(\frac{1}{1-x}\right)^{50}=(1-x)^{-50}.$$

As there are 30 souvenirs, the answer is the coefficient of x^{30} in $G(x)$, which, according to the extended binomial theorem, is equal to $C_{30}^{-50}=C_{30}^{79}$. Of course, we note that this is indeed H_{30}^{50}.

One may argue that the term $1+x+x^2+\cdots$ should be replaced by $1+x+x^2+\cdots+x^{30}$ in the above generating function because each student may get at most 30 souvenirs. It turns out that this modification will not affect the final outcome, and the details are left as an exercise. In view of this, we will simply use $1+x+x^2+\cdots$ most of the time because it is easier to handle.

B. More Complicated Counting Problems

Using the same idea employed in the previous examples, we can solve more complicated counting problems using generating functions, as can be seen in the following examples.

Example 3.3.

How many integer solutions to the equation $a+b+c=6$ satisfy $-1\le a\le 2$ and $1\le b,c\le 4$?

Solution.

Since $-1\le a\le 2$, the variable a contributes a term $x^{-1}+x^0+x^1+x^2$ to the generating function. Similarly, each of b and c contributes a term $x^1+x^2+x^3+x^4$. Hence the generating function is

$$\begin{aligned}G(x)&=(x^{-1}+x^0+x^1+x^2)(x^1+x^2+x^3+x^4)^2\\&=x(1+x+x^2+x^3)^3\\&=x\left(\frac{1-x^4}{1-x}\right)^3\\&=x(1-3x^4+3x^8-x^{12})(1-x)^{-3}\\&=(x-3x^5+3x^9-x^{13})(1-x)^{-3}\end{aligned}$$

The answer is the coefficient of x^6 in $G(x)$. To get an x^6 term, we can multiply x with the x^5 term in $(1-x)^{-3}$, as well as multiply $-3x^5$ with the x term in $(1-x)^{-3}$. By the extended binomial theorem, the coefficient of x^5 and x in $(1-x)^{-3}$ are $-C_5^{-3}$ and $-C_1^{-3}$ respectively. Hence the answer is $-C_5^{-3}-3(-C_1^{-3})=C_5^7-3C_1^3=12$.

The reader may try to make a comparison by solving the above problem using traditional ways without the help of generating functions.

Example 3.4.

In a country there are coins of denominations \$2, \$3, \$5 and \$7. How many different ways are there to pay exactly \$10?

Solution.

The \$2 coins may contribute a sum of \$0, \$2, \$4, …, thus leading to the factor $1+x^2+x^4+\cdots$. Using the same idea for the \$3, \$5 and \$7 coins, the generating function is given by

$$\begin{aligned}G(x)&=(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)(1+x^7+x^{14}+\cdots)\\&=\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\end{aligned}$$

The answer is the coefficient of x^{10} in $G(x)$. This is hard to compute by hand (it would be easier to count the number of ways directly), but this can easily be done with the help of suitable computer software (by contrast, the computer will not be able to count the number of ways directly).

Example 3.5.

In the USA, there are coins of denominations 1, 5, 10, 25, 50 cents and \$1. In how many different ways can we make up \$1 (i.e. 100 cents) from these coins?

Solution.

Similar to the previous example, the generating function is

$$G(x)=\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}.$$

The answer is the coefficient of x^{100} in $G(x)$. At first glance it may seem hopelessly complicated to compute this by hand, but if we take a second look, we see that the powers of x in all but one parentheses are divisible by 5. In view of this, we replace $\frac{1}{1-x}$ by $\frac{1+x+x^2+x^3+x^4}{1-x^5}$ so that the generating function becomes

$$G(x)=(1+x+x^2+x^3+x^4)\left[\frac{1}{(1-x^5)^2(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}\right].$$

Now all powers of x in the second factor are divisible by 5, so the only way to form an x^{100} term is to multiply the term 1 in the first factor with an x^{100} term in the second factor. Setting $y=x^5$, we therefore have to compute the coefficient of y^{20} in the expansion of

$$\frac{1}{(1-y)^2(1-y^2)(1-y^5)(1-y^{10})(1-y^{20})}.$$

In the denominator, all powers of y are divisors of 20. Using the same technique as we replaced $\frac{1}{1-x}$ by $\frac{1+x+x^2+x^3+x^4}{1-x^5}$, the above expression can be rewritten as

$$\frac{(1+y+\cdots+y^{19})^2(1+y^2+y^4+\cdots+y^{18})(1+y^5+y^{10}+y^{15})(1+y^{10})}{(1-y^{20})^6}.$$

Finally, to extract the coefficient of y^{20}, we note that $(1-y^{20})^{-6}=1+6y^{20}+\cdots$ (here $6=-C_1^{-6}$) and so we need only find the coefficient of y^{20} and the constant term in the numerator. The constant term is clearly 1. The coefficient of y^{20} is, unfortunately, complicated to compute. Nevertheless one can consider the last two factors first and deal with the terms case by case. For instance, if the term y^5 is taken in the second last factor while the term y^{10} is taken in the last factor, there are 12 ways to get y^5 from the first 3 factors ($0+5+0$, $1+4+0$, $2+3+0$, $3+2+0$, $4+1+0$, $5+0+0$, $0+3+2$, $1+2+2$, $2+1+2$, $3+0+2$, $0+1+4$ and $1+0+4$). We shall omit the remaining details, but anyway it can be found that the coefficient of y^{20} in the numerator is 287. It follows that the answer is $287+6=293$.

C. Solving Recurrence Relations

Given a recurrence relation, one of the common ways to solve for the general term of a sequence is to use the method of characteristic equations. Here we will see how generating functions may be employed as an alternative way in solving recurrence relations.

For each sequence, we can form a generating function which may be regarded as an 'infinite polynomial'. If a recurrence relation is given, we can possibly reduce this 'infinite polynomial' to a finite one, so that we can get the generating function in closed form. Consequently the coefficient of a general term (i.e. the general term of the sequence) can be found. The idea is more or less the same as that employed in deriving the sum of a geometric series.

Example 3.6.

Using generating functions, find a_n in terms of n in each of the following cases.

(a) $a_0=2$ and $a_{n+1}=3a_n$ for $n\geq 0$

(b) $a_0=1$, $a_1=2$ and $a_{n+2}=5a_{n+1}-4a_n$ for $n\geq 0$

Solution.

In each case, we let $G(x)$ be the generating function for the given sequence $\{a_n\}$.

(a) We have

$$\begin{aligned} & G(x) = a_0 + a_1x + a_2x^2 + \cdots && (1) \\ (1)\times 3x: \quad & 3xG(x) = \quad 3a_0x + 3a_1x^2 + \cdots && (2) \\ (1)-(2): \quad & (1-3x)G(x) = a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1)x^2 + \cdots \end{aligned}$$

Since $a_0 = 2$ and $a_{n+1} = 3a_n$ for $n \ge 0$, we have $(1-3x)G(x) = 2$, i.e.

$$\begin{aligned} G(x) &= \frac{2}{1-3x} \\ &= 2\left[1 + (3x) + (3x)^2 + (3x)^3 + \cdots\right] \end{aligned}$$

In this way, we see that the coefficient of x^n in $G(x)$ is $2\cdot 3^n$, so that $a_n = 2\cdot 3^n$ for all *n*, as we would expect since $\{a_n\}$ is in fact a geometric sequence with first term 2 and common ratio 3.

(b) Similar to (a), we have

$$\begin{aligned} G(x) &= a_0 \quad + a_1x \quad + a_2x^2 \quad + a_3x^3 + \cdots \\ -5xG(x) &= \quad -5a_0x - 5a_1x^2 - 5a_2x^3 - \cdots \\ 4x^2G(x) &= \qquad\qquad + 4a_0x^2 + 4a_1x^2 + \cdots \end{aligned}$$

Adding these three equations, and using the initial conditions as well as the given recurrence relation, we get

$$(1-5x+4x^2)G(x) = 1-3x.$$

Applying partial fraction decomposition, we have

$$\begin{aligned} G(x) &= \frac{1-3x}{1-5x+4x^2} \\ &= \frac{2}{3}\left(\frac{1}{1-x}\right) + \frac{1}{3}\left(\frac{1}{1-4x}\right) \\ &= \frac{2}{3}\left[1 + x + x^2 + x^3 + \cdots\right] + \frac{1}{3}\left[1 + (4x) + (4x)^2 + (4x)^3 + \cdots\right] \end{aligned}$$

Thus the coefficient of x^n in $G(x)$ is $\frac{2}{3} + \frac{1}{3}\cdot 4^n$, so that $a_n = \frac{4^n + 2}{3}$.

4. Further Examples

In this section we shall look at a couple of problems from mathematical olympiads in which generating functions serve as a powerful tool.

Example 4.1.

(CGMO 2004) A set of poker has 32 cards. 30 of them are in red, yellow and blue, with 10 cards in each colour, given the numbers 1, 2, …, 10 respectively. 2 of them are different jokers, both with the number 0. A card with the number k counts for 2^k points. Call a group of cards a ‘good group’ if the sum of their points is 2004. Calculate the number of ‘good groups’.

Solution.

The generating function is given by

$$\begin{aligned}
G(x) &= (1+x)^2(1+x^2)^3(1+x^4)^3\cdots(1+x^{1024})^3 \\
&= \frac{(1+x)^2\left[(1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{1024})\right]^3}{(1-x^2)^3} \\
&= \frac{(1+x)^2(1-x^{2048})^3}{(1-x^2)^3} \\
&= (1+2x+x^2)(1-3x^{2048}+3x^{4096}-x^{6144})(1-x^2)^{-3} \\
&= (1+2x+x^2-3x^{2048}-6x^{2049}-\cdots)(1-x^2)^{-3}
\end{aligned}$$

The answer is the coefficient of x^{2004} in $G(x)$. Note that in the expansion of $(1-x^2)^{-3}$, there are only even powers of x. To form an x^{2004} term, one can multiply the 1 in the first factor with the x^{2004} term in $(1-x^2)^{-3}$, as well as multiply the x^2 term in the first factor with the x^{2002} term in $(1-x^2)^{-3}$. By the extended binomial theorem, the coefficients of x^{2004} and x^{2002} in $(1-x^2)^{-3}$ are C_{1002}^{-3} and $-C_{1001}^{-3}$ respectively. It follows that the answer is

$$C_{1002}^{-3}-C_{1001}^{-3}=C_{1002}^{1004}+C_{1001}^{1003}=C_2^{1004}+C_2^{1003}=1006009.$$

Example 4.2.

(IMO 1995) Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \ldots, 2p\}$ such that

(i) A has exactly p elements, and

(ii) the sum of all the elements in A is divisible by p.

Solution.

Since we need to count two things, namely, the number of elements of A as well as the sum of the elements of A, we form a bivariate generating function

$$G(t,x)=(t+x)(t+x^2)\cdots(t+x^{2p})$$

where the power of t counts the number of elements of A and the power of x counts the sum of the elements of A. Let $a(k,m)$ denote the coefficient of $t^k x^m$. Then the problem asks for nothing but the sum

$$a(p,0)+a(p,p)+a(p,2p)+\cdots.$$

Let E denote the set of p-th roots of unity, i.e. $E=\{1,\omega,\omega^2,\ldots,\omega^{p-1}\}$, where $\omega=e^{\frac{2\pi i}{p}}$. We shall compute the sum

$$S=\sum_{t\in E}\sum_{x\in E}G(t,x)$$

by two different methods. We shall give the outline below and the details are left as an exercise.

Method 1

(1) Establish the fact that $\sum_{z\in E} z^n=\begin{cases} p & \text{if } p\mid n \text{ or } z=1 \\ 0 & \text{otherwise}\end{cases}$.

(2) Replace $G(t,x)$ by $\sum_{k,m} a(k,m)t^k x^m$ in S.

(3) Show that $S=p^2\sum_{\substack{p|m\\p|k}} a(k,m)$ by first summing x and then t.

Method 2

(1) Replace $G(t,x)$ by $(t+x)(t+x^2)\cdots(t+x^{2p})$ in S.

(2) First fix t and sum x, treating the cases $x=1$ and $x\neq 1$ separately.

(3) Then sum t to show that $S=\frac{1}{p}\left(C_p^{2p}+4p-2\right)$.

Now we can equate the results of the two methods to get

$$\sum_{\substack{p|m\\p|k}} a(k,m)=\frac{1}{p}\left(C_p^{2p}+4p-2\right).$$

Bearing in mind that $a(0,0)=1$ and $a(2p,2p^2-p)=1$ (i.e. coefficients corresponding to the empty set and the whole set of $2p$ integers respectively) are included in the left-hand summation but are not wanted in our final answer, we subtract 2 from the right hand side to get the answer

$$\frac{1}{p}\left(C_p^{2p}+2p-2\right).$$

5. Exercises

1. Find the generating functions of the following sequences in closed form.

    (a) 1, 0, 1, 0, 1, 0, …

    (b) 2, –4, 6, –8, 10, –12, …

    (c) C_2^0, C_2^1, C_2^2, C_2^3, C_2^4, C_2^5, …

2. Find the coefficient of x^{2005} in each of the following generating functions.

    (a) $(1-2x)^{2006}$

    (b) $\left(\frac{1}{1+3x}\right)^4$

    (c) $\frac{1}{2x^2+5x+2}$

    (d) $\ln\left[2(1+x)\right]$

3. Using generating functions, find a_n in terms of n in each of the following cases.

    (a) $a_0 = 2$ and $a_{n+1} = 3a_n + 1$ for $n \geq 0$

    (a) $a_0 = 1$, $a_1 = 2$ and $a_{n+2} = 4a_{n+1} - 3a_n$ for $n \geq 0$

    (a) $a_0 = a_1 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$

4. Find the number of solutions to the equation $a+b+c+d=50$ if each variable is

    (a) a non-negative integer

    (b) a positive integer

    (c) an odd positive integer

    (d) an integer between 4 and 10 (inclusive)

5. Prove the formula $C_r^{-n} = (-1)^r C_r^{n+r-1}$ for positive integers n and r.

6. Solve Example 3.3 without using generating functions and compare the two methods.

7. Complete Example 3.5 by filling in the missing details in the last paragraph of the solution.

8. Complete the solution to Example 4.2 by following the steps outlined.

9. There are 10000 identical red balls, 10000 identical yellow balls and 10000 identical green balls. In how many different ways can we select 2005 balls so that the number of red balls is even or the number of yellow balls is odd?

10. Amy and Betty each picks n different positive integers, and Amy's numbers are not all the same as Betty's numbers. For each set of n numbers, there are H_2^n ways to add up two of them (adding a number with itself is allowed) and hence H_2^n sums can be obtained (some of them may be repeated). It turns out that the H_2^n sums obtained by each of Amy and Betty are exactly the same (same sums to the same multiplicities). Show that n is a power of 2.

11. For positive integer $n > 1$, let C_n be the number of ways of dissecting a regular ($n+2$)-sided polygon into triangles by $n-1$ diagonals which do not intersect inside the polygon. For convenience we also define $C_0 = C_1 = 1$.
    (a) Compute C_2, C_3 and C_4.
    (b) Show that for $n > 1$, C_n satisfies the recurrence relation
    $$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0 .$$
    (c) Show that the generating function for the sequence $\{C_n\}$ is given by
    $$G(x) = \frac{1-\sqrt{1-4x}}{2x} .$$
    (d) Show that $C_n = \dfrac{1}{n+1} C_n^{2n}$.

    Remark. The numbers C_n are known as **Catalan numbers**. In Richard Stanley's book *Enumerative Combinatorics*, he listed 66 different combinatorial interpretations of the Catalan numbers!