Assignment 6

Some of the problems are classic well known problems, so relevant links are provided. Best of Luck for Mid Sems !

(1) A 2xn grid is given. In how many ways you can cover it using 1x2 dominoes ?
This is simply the Fibonaci Recursion. if the number of ways is F(n). Then case 1: the first domino is vertical and we have F(n-1) possibilities. case 2: If We keep the initial domino horizontally then to cover the board we need to place another one exactly below it. So the number of ways is F(n-2). Hence total answer F(n) = F(n-1) + F(n-2). Now use characteristic equations to solve the problem.

(2) Show that every rational number written in decimal from will either have terminating or periodic decimal digits.
Say the number is a/b where a,b are co prime. Now lets start standard decimal conversion process starting by dividing a by b. Now the remainder is one of [0,1…,b-1]. If it is 0 then its a terminating. Otherwise by PHP it will repeat and hence it will be periodic.

(3) In a party n people were there. Some of these people did handshakes with each others. Prove that there are at least 2 people who did handshakes same number of times.
The number of handshakes can be [0,….,n-2] or [1,…,n-1] as 0 and n-1 can not stay together. Hence by PHP there is a duplicate.

(4) Prove the binomial theorem for natural index using only probability theory. (Hint: tossing a biased coin)
Let the probability of getting a head is a/(a+b) . Hence the probability of getting a tail is b/(a+b). Suppose we toss it n times. Now probability of getting r heads (0<=r<=n) is a^rb^(n-r) / (a+b)^n * nCr. Hence by summing the sample space and equating to 1 we get the binomial theorem for natural index.

(5) Using generating functions solve the following problem:
In how many ways you can distribute m coins among n persons ?
let f(x) = 1 + x + x^2 + …
now the answer of coefficient of x^m in f(x)^n. Now it can be found using binomial theorem for negative index.

(6) Prove the following identity in the most elegant method(no calculus or algebra)
nC1*1 + nC2*2 + … nCn * n = n *2^(n-1)
we will use double counting. say from n people we need to form a party of size at least 1 with a leader (if the party size is 1 , then the sole member is the leader).
LHS: we can select r members (1<=r <=n) and among them we can
RHS: select the leader first , then from the rest n-1 persons select 0 to n-1 members. Its trivial to show that both the process are identical.

(7) How many strings of [a,b,c] are there such that it contains no two consecutive pairs of same character ?
Let the number be S(n). If the string is starting with a, then count of such string is A(n), similar for B(n) and C(n). Now S(n) = A(n) + B(n) + C(n). Now we can write A(n) = B(n-1) + C(n-1). B(n) = A(n-1) + C(n-1) and C(n) = B(n-1) + A(n-1). Hence S(n) = 2(A(n-1) + B(n-1) + C(n-1)) = 2S(n-1).

(8) If p is a prime number then prove that p divides a^p - a (*trust me It can be solved by combinatorics, try at home or check from net if unable to do*)
https://en.wikipedia.org/wiki/Proofs_of_Fermat%27s_little_theorem

(9) How many n-ary trees (every node is either a leaf or has exactly n children) are there of height d ?
This does not have a closed from solution. Let the number be D(m). Now it has either 0 to n-1 leaves. so if it has 0 leaves then we have D(m-1)^n trees. If we have r leaves then we have D(m-1)^n-r trees. Hence the total number of trees comes from the recursion, D(m) = D(m-1)^1 + … + D(m-1)^n. Hence It can be solved by a simple Dynamic Programming.

(10) For any given integer prove that given a sequence of n^2+1 numbers, there exist a sub sequence of monotonically increasing numbers or monotonically decreasing numbers of length

n+1.
https://en.wikipedia.org/wiki/Erd%C5%91s%E2%80%93Szekeres_theorem

(11) Prove that among any set of 51 positive integers less than 100, there is a pair whose sum is 100

**** distinct integers needs to be mentioned.

(1,99) (2,98) ….. (49,51) (50) — 50 such buckets. Now the last bucket contains only one element. So By PHP there are two elements belonging to same bucket.

(12) A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.

similar problem:
http://math.stackexchange.com/questions/15903/chess-master-problem