Solutions of Tutorial V
Discrete Structures (CS21001)

Autumn Semester 2014

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1. (a) $F_n = F_0 \cdot F_1 \cdots F_{n-1} + 2$. We will prove this by induction.
   When $n = 1$, $F_0 + 2 = 3 + 2 = 5 = F_1$.
   Now, assume $F_n = F_0 \cdot F_1 \cdots F_{n-1} + 2$.
   Then,

$$\begin{aligned}
F_0 \cdot F_1 \cdots F_n + 2 &= F_0 \cdot F_1 \cdots F_{n-1} \cdot F_n + 2 \\
&= (F_n - 2) \cdot F_n + 2 \\
&= (2^{2^n} - 1) \cdot (2^{2^n} + 1) + 2 \\
&= 2^{2^{n+1}} + 1 \\
&= F_{n+1}
\end{aligned}$$

   (b) Assume for contradiction that there exist F_i and F_j such that $a > 1$ divides both of them. Also, without loss of generality, assume that $F_j > F_i$.
   Thus a divides both $F_0 \cdot F_1 \cdots F_{j-1}$ and F_j; hence a divides their difference, 2. Since $a > 1$, this forces $a = 2$. This is a contradiction, because each Fermat number is clearly odd.

2. Let $(S, \preccurlyeq)$ be a finite poset, and let A be a maximal chain. Because, $(A, \preccurlyeq)$ is also a poset it must have a minimal element m. Suppose that m is not minimal in S. Then there would be an element a of S with $a \prec m$. However, this would make the set $A \cup a$ a larger chain than A. To show this, we must show that a is comparable with every element of A. Because m is comparable with every element of A and m is minimal it follows that $m \prec x$ when x is in A and $x \neq m$. Because $a \prec m$ and $m \prec x$, the transitive law shows that $s \prec x$ for every element of A.

3. Suppose that there are finitely many primes $p_1, p_2 \cdots p_k$. Let us take the number $P = p_1 \cdot p_2 \cdots p_k + 1$. If P is a prime number then we are done.

   Suppose, there is p which divides P, but p can't be any of $p_1, p_2 \cdots p_k$, otherwise p would divide the difference $P - p_1 \cdot p_2 \cdots p_k = 1$, which is impossible. Thus p is a new prime $\Rightarrow$ a contradiction.

4. Let us prove this by contrapositive version.

   Suppose $n = k^2$ for some k.

   Case 1 : If $k (\text{mod } 4) = 0$, then $k = 4q$, for some integer q. Then, $n = k^2 = 16q^2 = 4(4q^2),$ i.e. $n (\text{mod } 4) = 0$.

   Case 2 : If $k (\text{mod } 4) = 1$, then $k = 4q + 1$, for some integer q. Then, $n = k^2 = 16q^2 + 8q + 1 = 4(4q^2 + 2q) + 1,$ i.e. $n (\text{mod } 4) = 1$.

   Case 3 : If $k (\text{mod } 4) = 2$, then $k = 4q + 2$, for some integer q. Then, $n = k^2 = 16q^2 + 16q + 4 = 4(4q^2 + 4q + 1),$ i.e. $n (\text{mod } 4) = 0$.

   Case 4 : If $k (\text{mod } 4) = 3$, then $k = 4q + 3$, for some integer q. Then, $n = k^2 = 16q^2 + 24q + 9 = 4(4q^2 + 6q + 2) + 1,$ i.e. $n (\text{mod } 4) = 1$.

5. Suppose g, f are two one-to-one functions and there composition is $h = g \circ f$. Now, suppose for a, b $h(a) = h(b)$. Therefore, $g(f(a)) = g(f(b))$. Since, g is one-to-one it follows $f(a) = f(b)$. But, as f is also one-to-one $a = b$ i.e. h is also one-to-one.