Assignment 4 Solutions

1. Let P(n) be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$

   P(0) is true as $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$

   The inductive hypothesis is the statement that P(k) is true that is $H_{2^k} \geq 1 + \frac{k}{2}$ ($k \in \mathbb{N}$)

   We must show that if P(k) $\Rightarrow P(k+1)$ i.e $H_{2^{(k+1)}} \geq 1 + \frac{k+1}{2}$

   $H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ 1/2^k + 1/(2^k + 1) + \ldots\ldots\ldots\ldots\ 1/2^{(k+1)}$

   $= H_{2^k} + 1/(2^k + 1) + \ldots\ldots\ldots\ldots\ 1/2^{(k+1)}$

   $\geq (\ 1 + \frac{k}{2}\ ) + 1/(2^k + 1) + \ldots\ldots\ldots\ldots\ 1/2^{(k+1)}$

   $\geq (1 + \frac{k}{2}) + 2^k \cdot \frac{1}{2^{(k+1)}}$ (there are 2^k terms)

   $\geq (1 + \frac{k}{2}) + \frac{1}{2}$

   $\geq 1 + (k+1)/2$

   This establishes the proof

2. Lets formalise the problem in the following way

   a $2^n \times 2^n$ board has 4^n squares. A domino has 3 squares in it. A

   2×2 *board* with 1 square removed can be tiled by a single domino.

   So the entire problem boils down to whether a $4^n - 1$ square board can be tiled by 3n squares i.e whether $4^n - 1$ is divisible by 3

   base case $4^1 - 1 = 3$ *is divisible by* 3

   let $4^k - 1$ is divisible by 3 is given by P(k) we need to show whether we can imply $4^{k+1} - 1$ *is divisible by* 3

$4^{(k+1)} - 1 = 4^k.4 - 4 + 3 = 4(4^k - 1) + 3 = 4P(k) + 3$ which is clearly divisible by 3 . This establishes the proof

3. We attempt proof by contradiction

lets assume $\sqrt{5}$ *is rational* hence it can be expressed as fraction $\frac{m}{n}$ where m,n $\in \mathbb{Z}$

$m^2 / n^2 = 5 \Rightarrow m^2 = 5n^2$ Now LHS m2 must have even no of prime factors (counting each prime factors as many times as it occurs) but RHS has odd number of prime factors . Thus a product of even prime factors is equal to a product of odd prime factors which is a contradiction . hence $\sqrt{5}$ is not rational.

4. n is a positive integer

to prove $n^2 > 100 \Rightarrow n > 10$

Let n not greater than 10 hence $1 \leq n \leq 10$

we know if $x \leq y$ and $c \geq 0$ then $cx \leq cy$

$n.n <= n.10 <= 10.10 = 100$

hence max value of n^2 is 100 thus $\neg n > 10 \Rightarrow \neg n^2 > 100$

hence the premise is proved by contraposition

5. The statement P(n) is $n^2 + 3n + 2$ is even

P(1) = 6 which is true

Consider P(k) is true

P(k+1) = $(k+1)^2 + 3(k+1) + 2 \Rightarrow (k^2 + 3k + 2) + (2k+4)$

$\Rightarrow P(k) + 2(k+2)$

P(k+1) sum of two even terms and clearly is even.

6. P(1) i.e base case is true because a set of 1 element has 2^1 subsets which are empty set and the set itself

Suppose P(n) is true . Now consider a set A with n+1 elements which is given by $A_{(n+1)} = A_n \cup (n+1)^{th}$ element . Now A_n has 2^n subsets which are also subsets of $A_{(n+1)}$. The additional subsets are obtained by adjoining $(n+1)^{th}$ element to each of the subsets of A_n which gives us additional 2^n elements . hence total number of subets is $2^n + 2^n = 2^{n+1}$. This completes the inductive step.