**TUTORIAL 4**

1) Given the poset ({1, 2, 3, 5, 6, 7, 10, 20, 30, 60, 70}, |)
   a) Draw the Hasse Diagram for this poset.
   b) Find the maximal elements.
   c) Find the minimal elements.
   d) Find the greatest element
   e) Find the least element
   f) Find all upper bounds of {2, 5}
   g) Find the least upper bound of {2, 5} (if it exists)
   h) Find all lower bounds of {6, 10}
   i) Find the greatest lower bound of {6, 10} (if it exists)
   j) is this poset a lattice? Justify your answer

   Solution:-
   ques14. http://web.mnstate.edu/jamesju/Fall2011/Content/M310Exam3F11PracSoln.pdf

2) Let S = {x, y, z}, and consider the set P(S) with relation R given by set inclusion. Is R a partial order?
   Solution:
   R is a partial order (i.e., R is reflexive, antisymmetric, and transitive). Recall elements A, B $\in$ P(S) are subsets A, B $\subseteq$ S. Then, ARB if and only if A $\subseteq$ B. R is reflexive since A $\subseteq$ A always. R is antisymmetric since if A $\subseteq$ B and B $\subseteq$ A, then A = B. R is transitive since if A $\subseteq$ B and B $\subseteq$ C, then A $\subseteq$ C.

3) Prove that the direct product of any two distributive lattice is a distributive.
   Proof : Question 58 http://iete-elan.ac.in/SolQP/soln/AC10_sol.pdf

4) Prove that if $l_1$ and $l_2$ are elements of a lattice < L;$\vee$ , $\wedge$ > then
   $(l_1 \vee l_2 = l_1) \leftrightarrow (l_1 \wedge l_2 = l_2) \leftrightarrow (l_1 \leq l_2)$

Proof: Let us use b = $l_1$ and a = $l_2$ . It is given that a$\vee$b = b. Since a$\vee$b is an upper bound of a, a $\leq$ a$\vee$b. This implies a $\leq$ b.
Next, let a $\leq$ b. Since $\leq$ is a partial order relation, b $\leq$ b. Thus, a$\leq$b and b$\leq$b together implies that b is an upper bound of a and b. We know that a$\vee$b is least upper bound of a and b, so a$\vee$b $\leq$ b. Also b $\leq$ a$\vee$b because a$\vee$b is an upper bound of b. Therefore, a$\vee$b $\leq$ b and b $\leq$ a$\vee$b $\Leftrightarrow$ a$\vee$b = b by the anti-symmetry property of partial order relation $\leq$. Hence, it is proved that a$\vee$b = b if and only if a $\leq$ b.

5) If [L, $\wedge$ , $\vee$] is a complemented and distributive lattice, then the complement of any element a $\in$ L is unique.

Proof :
Ans: Let I and 0 are the unit and zero elements of L respectively. Let b and c be two complements of an element a $\in$ L. Then from the definition, we have

$$a \wedge b = 0 = a \wedge c \text{ and}$$
$$a \vee b = I = a \vee c$$

We can write b = b $\vee$ 0 = b $\vee$ (a $\wedge$ c )
   = (b $\vee$ a) $\wedge$ (b $\vee$ c ) [since lattice is distributive ]

$= I \wedge (b \vee c)$
$= (b \vee c)$
Similarly, $c = c \vee 0 = c \vee (a \wedge b)$
$= (c \vee a) \wedge (c \vee b)$ [since lattice is distributive ]
$= I \wedge (b \vee c)$ [since $\vee$ is a commutative operation]
$= (b \vee c)$
The above two results show that $b = c$