Solution Assignment – 2

1) $2x+x = 3x$ is divisible by 3 , (x,x) is in R, hence the relation is reflexive

if $2x + y = 3t \rightarrow y = 3t - 2x \rightarrow 2y + x = 2(3t-2x) + x = 6t - 4x + x = 6t - 3x = 3(2t-x)$ is divisible by 3

Hence the relation is symmetric

Now if $2x+y = 3a$, $2y + z = 3c$ then $z = 3c - 2y \rightarrow 2x+z = 2x + 3c - 2y = 2x + 3c - 2(3a - 2x) = 3c - 6a + 6x = 3(c - 2a + 2x)$ is divisible by 3, hence the relation is transitive

Hence the relation R is an equivalence relation

2) Its not an equivalence relation as (x,x) is not in R for any integer x. Hence the relation is not symmetric

3) We will prove this by induction

For n = 1

Now there is a path of length 1 from a to b ↔ there is an directed edge from a to b ↔ (a,b) is in R1 by the definition of the graph representation of the relation R

Hence the statement is true for n = 1 (Induction base)

(Now proving the induction hypothesis)

Now suppose the statement is true for n = k (weak induction)

Now suppose there is a path of length k+1 from a to b ↔ there exist c such that there is a path of length k from a to c and there is a path of length 1 from c to b ↔ hence (a,c) is in Rk and (c,b) is in R. Now R(k+1) = Rk * R

So (a,b) is in R(k+1) by the definition of relation composition

4) Reflexive closure = R U {(x,x) : x in A} [straight forward]

Symmetric closure = R U {(y,x) : (x,y) in R} [straight forward]

Transitive Closure = R U R1 U R2 U … R(n-1) (By Floyd Warshal Algorithm)

5) part 1: this can be trivially solved by using relation. A directed graph can be represented by a relation. Use problem 3. If there is a path of length k from a to b then (a,b) is in Rk. Now if we can reach b from a then there definitely should be a path of length 1 or 2 … n-1 . Where n is the number of nodes in the graph ↔ if b can be reachable from a then (a,b) is in at least one of R,R2,.. R(n-1). Now we can find X = RUR1U.. R(n-1) [exactly same as floyd warshal algorithm] and see whether (a,b) is in X or not

part 2:

Read about all pair shortest path problem from the Internet (this will be covered in your Algorithm 1 course while studying graph algorithms)

6) If R and S are reflexive then so is RUS

Let both R and S are defined on the set A

Now for all x in A, (x,x) in R and (x,x) in S → for all x in A , (x,x) is in R U S. Hence it is reflexive

If R and S are symmetric then so is RUS

If (x,y) R US → (x,y) is in R or S → (WLOG assume its in R) → (y,x) is in R (as R is symmetric) → (y,x) is in R US . Hence RUS is symmetric.

R is reflexive then R2 is also reflexive
(x,x) is in R → (x,x) is in R and (x,x) is in R → (x,x) is in R2 for all x in A

R is symmetric then R2 is also symmetric
(x,y) is in R2 → (x,z) is in R and (z,y) is in R → (z,x) is in R and (y,z) is in R (as R is symmetric) → (y,x) is in R2 → hence R2 is also symmetric.

Now RUS has both symmetric + reflexive property. But Now we want the smallest equivalence relation containing both R and S. Let it Z. Now both R and S is contained in Z. So R US is also contained in Z.

So Z must contain RUS and must be reflexive, symmetric and transitive. And as RUS is already reflexive + symmetric, we just need to find the smallest set containing RUS which is transitive. So the smallest such transitive relation is the transitive Closure of RUS. Now TC(RUS) = (RUS) U (RUS)2 U …. (RUS)(n-1)
Now we already know that symmetry and reflexivity holds for finite union and composition.

Hence Z = TC(RUS) (proved)