Solutions of Tutorial II
Discrete Structures (CS21001)

Autumn Semester 2014

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1. (a) 2^{n^2}. Number of different relations is the cardinality of the powerset of Cartesian Product $(A \times A)$.
   (b) $2^{n(n+1)/2}$ Consider matrix representation of a relation. A symmetric relation can be formed by combining any number of elements in the upper-diagonal matrix and the the principal diagonal. Total number of elements $= n(n-1)/2 + n = n(n+1)/2$.
   (c) $2^{n(n-1)/2}$. If reflexive then $(a, a) \in R \quad \forall a \in A$. Consider graph representation of the relation R. As it is symmetric we consider only undirected graph. Total number of edges in the graph is $n(n-1)/2$.

2. We know that if $a \equiv b(\text{mod } m)$ then $m|(a-b)$.

   Reflexive : Note $(a-a) = 0$ is divisible be m i.e. $a \equiv a(\text{mod}m)$.

   Symmetric : Suppose, $a \equiv b(\text{mod}m)$ i.e. $m|(a-b)$ i.e. $a - b = k \cdot m$. So, $(b-a) = (-k) \cdot m$ i.e. $m|(b-a)$ i.e. $b \equiv a(\text{mod}m)$.

   Transitive : Suppose, $a \equiv b(\text{mod}m)$ and $b \equiv c(\text{mod}m)$. So, we can write $a - b = k \cdot m$ and $b - c = l \cdot m$. Adding both of them we get, $a - c = k \cdot m + l \cdot m$ i.e. $a - c = (k + l) \cdot m$. Therefore, $a \equiv c(\text{mod}m)$.

   The equivalence classes are $[0], [1], \cdots, [m-1]$.

3. **Necessity :** If R is symmetric then, $(a, b) \in R$ and $(b, a) \in R$ i.e. $(a, b) \in R^{-1}$. Therefore, $R \subseteq R^{-1}$. Similarly, if $(a, b) \in R^{-1}$, then $(b, a) \in R$ and as R is symmetric, so $(a, b) \in R$. Therefore $R^{-1} \subseteq R$.

   Sufficiency : Now consider $R = R^{-1}$. Suppose $(a, b) \in R$, then also $(a, b) \in R^{-1}$ i.e. $(b, a) \in R$. Therefore, R is symmetric.

4. (a) Use mathematical induction. The result is trivially true for $n = 1$. Assume, R^n is symmetric. Suppose, $(a, b) \in R^{n+1}$ for $a, b \in A$. Now, $\exists c \in A$ such that $(a, c) \in R^n$ and $(c, b) \in R$. As R^n is symmetric, $(c, a) \in R^n$ and also $(b, c) \in R$. Therefore, $(b, a) \in R^n \circ R = R^{n+1}$.

(b) Consider any $(a, b), (b, c) \in R^n$. Thus in the graph representation, there is a path of length n from a to b and from b to c in R. Then we have a path of length $2n$ from a to c in R.

Since R is transitive, we can replace the edges in this path with their two-step counterparts (they must exist since R is transitive). The $n = 3$ case would look like this:

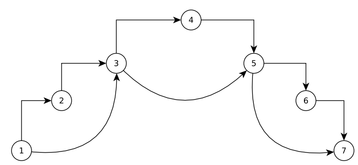

Figure 1: Graph

This constructs a path of length n from a to c. Thus, $(a, c) \in R^n$ and R^n is transitive.

5. The result follows from $(R^*)^{-1} = (\bigcup_{n=1}^{\infty} R^n)^{-1} = \bigcup_{n=1}^{\infty} (R^n)^{-1} = \bigcup_{n=1}^{\infty} R^n = R^*$.