Solution Tutorial 10

1. Each edge ends at two vertices. If we begin with just the vertices and no edges, every vertex has degree zero, so the sum of those degrees is zero, an even number. Now add edges one at a time, each of which connects one vertex to another, or connects a vertex to itself (if you allow that). Either the degree of two vertices is increased by one (for a total of two) or one vertex's degree is increased by two. In either case, the sum of the degrees is increased by two, so the sum remains even.
2. This is easy to prove by induction. If $n = 1$, zero edges are required, and $1(1 - 0)/2 = 0$. Assume that a complete graph with k vertices has $k(k - 1)/2$. When we add the $(k + 1)$st vertex, we need to connect it to the k original vertices, requiring k additional edges. We will then have $k(k - 1)/2 + k = (k + 1)((k + 1) - 1)/2$ vertices, and we are done
3. To show that a graph is bipartite, we need to show that we can divide its vertices into two subsets A and B such that every edge in the graph connects a vertex in set A to a vertex in set B. Proceed as follows: Choose any vertex from the graph and put it in set A. Follow every edge from that vertex and put all vertices at the other end in set B. Erase all the vertices you used. Now for every vertex in B, follow all edges from each and put the vertices on the other end in A, erasing all the vertices you used. Alternate back and forth in this manner until you cannot proceed. This process cannot encounter a vertex that is already in one set that needs to be moved to the other, since if it did, that would represent an odd number of steps from it to itself, so there would be a cycle of odd length.

   If the graph is not connected, there may still be vertices that have not been assigned. Repeat the process in the previous paragraph until all vertices are assigned either to set A or to set B.

   There is no reason that the graph has to be finite for this argument to work, but the proof does have to be modified slightly, and probably requires the axiom of choice to prove: divide the graph into connected components and select a vertex from each component and put it in set A. Then use the same process as above. The "select a vertex from each component" requires the axiom of choice.
4. We will build a path that does not reuse any edges. As we build the path, imagine erasing the edge we used to leave so that we will not use it again. Begin at vertex u and select an arbitrary path away from it. This will be the first component of the path. If, at any point, the path reaches a vertex of odd degree, we will be done, but each time we arrive at a vertex of even degree, we are guaranteed that there is another vertex out, and, having left, we effectively erase two edges from that meet at the vertex. Since the vertex originally was of even degree, coming in and going out reduces its degree by two, so it remains even. In this way, there is always a way to

continue when we arrive at a vertex of even degree. Since there are only a finite number of edges, the tour must end eventually, and the only way it can end is if we arrive at a vertex of odd degree.

5. One way to prove this is by induction on the number of vertices. We will first solve the problem in the case that there are two vertices of odd degree. (If all vertices have even degree, temporarily remove some edge in the graph between vertices a and b and then a and b will have odd degree. Find the path from a to b which we will show how to do below, and then follow the removed edge from b back to a to make a cycle.)

   Suppose the odd-degree vertices are a and b. Begin at a and follow edges from one vertex to the next, crossing off edges so that you won't use them again until you arrive at vertex b and you have used all the vertices into b. Why is it certain that you will eventually arrive at b? Well, suppose that you don't. How could this happen? After you leave a, if you arrive at a vertex that is not b, there were, before you arrived, an even number of unused edges leading into it. That means that when you arrive, there is guaranteed to be an unused path away from that vertex, so you can continue your route. After entering and leaving a vertex, you reduce the number of edges by 2, so the vertex remains one with an even number (possibly zero) of unused paths. So if you have not yet arrived at vertex b, you can never get stuck at any other vertex, since there's always a way out. Since the graph is finite, you cannot continue forever, so eventually you will have to arrive at vertex b. (And it has to be possible to get to vertex b since the graph is connected.)

   Note that a similar argument can be used to show that you can wait until you have used all the edges connecting to b. If b has more than one edge, leave each time you arrive until you get stuck at b. Now you have a path something like this: (a, a1, a2, . . . , an, b) leading from a to b. If all the edges are used in this path, you are done. If not, imagine that you have erased all the edges that you used. What remains will be a number of components of the graph (perhaps only one) where all the members of each component have even degree. Since b will not be in any of the components, all of them must have fewer vertices than the original graph.

6. The proof above works basically without change. Note that each time a vertex is visited, one incoming and one outgoing node is used, so the equality of incoming and outgoing edges is preserved.

7. The proof is easy, and can be done by induction. If $n = 1$, we simply need to visit each vertex of a two-vertex graph with an edge connecting them. Assume that it's true for $n = k$. To build a $(k + 1)$-cube, we take two copies of the k-cube and connect the corresponding edges. Take the Hamiltonian circuit on one cube and reverse it on the other. Then choose an edge on one that is part of the circuit and the corresponding edge on the other and delete them from the circuit. Finally, add to the path connections from the corresponding endpoints on the cubes which will produce a circuit on the $(k + 1)$-cube.

8. This is a trivial consequence of problem 3. A tree has no cycles, so it certainly does not contain any cycles of odd length.
9. Let T be a tree with $n \geq 2$ vertices. Consider any vertex v of T and assume T to be rooted at vertex v. Assign colour 1 to v. Then assign colour 2 to all vertices which are adjacent to v. Let v1, v2,..., vr be the vertices which have been assigned colour 2. Now assign colour 1 to all the vertices which are adjacent to v1, v2,..., vr . Continue this process till every vertex in T has been assigned the colour. We observe that in T all vertices at odd distances from v have colour 2, and v and vertices at even distances from v have colour 1. Therefore along any path in T , the vertices are of alternating colours. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same colour. Thus T is coloured with two colours. Hence T is 2-chromatic.
10. Let G be a connected graph with cycles of only even length and let T be a spanning tree in G. Then, by Problem 9, T can be coloured with two colours. Now add the chords to T one by one. As G contains cycles of even length only, the end vertices of every chord get different colours of T . Thus G is coloured with two colours and hence is 2-chromatic. Conversely, let G be bicolourable, that is, 2-chromatic. We prove G has even cycles only. Assume to the contrary that G has an odd cycle. Then by observation, G is 3-chromatic, a contradiction. Hence G has no odd cycles.