Discrete Structures Tutorial 7

1. Find the recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions

c) How many ways can this person climb a flight of 8 stairs

Solution:

$a_n = a_{n-1} + a_{n-2}$ for n>=2 ; $a_0 = a_1 = 1$; 34

2. Two persons A and B gamble dollars on the toss of a fair coin. A has $70 and B has $30. In each play either A wins $1 from B or loss $1 to B. The game is played without stop until one wins all the money of the other or goes forever. Find the probabilities of the following three possibilities (using recurrence relation formulation):

(a) A wins all the money of B.

(b) A loss all his money to B.

(c) The game continues forever.

Solution:

Either A or B can keep track of the game simply by counting their

own money. Their position n (number of dollars) can be one of the numbers

0, 1, 2, . . . , 100. Let p_n = probability that A reaches 100 at position n.

After one toss, A enters into either position n + 1 or position n − 1. The new

probability that A reaches 100 is either p_{n+1} or p_{n-1}. Since the probability of A

moving to position n + 1 or n − 1 from n is 1/2 . We obtain the recurrence relation

$p_n = 0.5\, p_{n+1} + 0.5 p_{n-1}$

$p_0 = 0$

$p_{100} = 1$

i.e., $p_{n+1} - p_n = p_n - p_{n-1}$.

Then

$p_{n+1} - p_n = p_n - p_{n-1} = \cdot\cdot\cdot = p_1 - p_0$.

Since $p_0 = 0$, we have $p_n = p_{n-1} + p_1$. Applying the recurrence relation again

and again, we obtain

$p_n = p_0 + np_1$.

Applying the conditions $p_0 = 0$ and $p_{100} = 1$, we have $p_n = n/100$.

3. A vending machine dispensing books of stamps accepts only dollar coins, $1 bills and $5 bills.

a) Find the recurrence relation for the number of ways to deposit n dollars, where the order in which the coins and bills are deposited matters.

b) Find initial conditions

c) How many ways are there to deposit $10?

Solution:

a) $a_n= 2a_{n-1} + a_{n-5}$ for n>=5

b) $a_0 =1$, $a_1=2$, $a_2=2$, $a_3=8$, $a_4=16$

c) 1217

4. Find the number of ways of arranging the numbers 1, 2, ..., 2004 in a row such that except for the leftmost number, each number differs from some number on its left by 1 [using recurrence relation]

Solution:

Let an denote the answer to the problem when 2004 is replaced by n. We first compute an for small n as follows:

n	Possible arrangement(s)	a_n
1	1	1
2	12, 21	2
3	123, 213, 231, 321	4
4	1234, 2134, 2314, 2341, 3214, 3241, 3421, 4321	8

From the above table, we could guess that $a_n = 2^{n-1}$ and hence the answer to the problem is 2^{2003} .

(Of course to 'play safe' you may want to try one or two more cases.) With $a_1 = 1$, to prove this rigorously all we need to do is to establish the recurrence relation $a_n = 2a_{n-1}$

We observe that the rightmost number in a valid arrangement of n numbers must be 1 or n. Once this fact is established, the rest would follow easily. We proceed as follows.

If the rightmost term is n, then it remains to arrange the numbers 1, 2, ..., n − 1 subject to the conditions. This can be done in a_{n-1} ways. Similarly, if the rightmost term is 1, we have a_{n-1} arrangements. As a result, we have an = $2a_n -1$, and so the answer to the problem is $a_{2004} = 2^{2003}$.

5. Find an explicit formula for the sequence given by the recurrence relation :

$x_n = 15x_{n-2} - 10x_{n-3} - 60x_{n-4} + 72x_{n-5}$

where $x_0 = 1$, $x_1 = 6$, $x_2 = 9$, $x_3 = -110$, $x_4 = -45$

Solution:

The characteristic equation

$r^5 = 15r^3 - 10r^2 - 60r + 72$

can be simplified as

$(r - 2)^3(r + 3)^2 = 0$.

There are roots $r_1 = 2$ with multiplicity 3 and $r_2 = -3$ with multiplicity 2. The general solution is given by

$x_n = c_1 2^n + c_2 n 2^n + c_3 n^2 2^n + c_4(-3)^n + c_5 n(-3)^n$.

The initial condition means that

$c_1+c_4=1$

$2c_1+2c_2+2c_3-3c_4-3c_5=1$

$4c_1+8c_2+16c_3+ 16c_4+ 9c_5=1$

$8c_1+24c_2+72c_3-27c_4-81c_5=1$

$16c_1+64c_2+256c_3+ 81c_4+ 324c_5=1$

Solving the linear system of equations, we have

$c_1 = 2$, $c_2 = 3$, $c_3 = -2$, $c_4 = -1$, $c_5 = 1$.

6. Solve the recurrence relations $a_{n+2} = 4a_{n+1} - 3a_n + 3^n$ subject to the initial conditions $a_1 = 1$ and $a_2 = 3$.

Solution:

We first solve the homogeneous part

$a_{n+2} = 4a_{n+1} - 3a_n$.

the general solution is

$a_n = A \cdot 3^n + B$

for some constants A and B. Now we want to find a particular solution to the original (non-homogenous) recurrence relation. Since the non-homogenous part is 3^n , naturally we would try $a_n = k \cdot 3^n$ for some constant k. However, since $a_n = k \cdot 3^n$ is a solution to the homogeneous part, we will not be able to find k. Therefore we consider $a_n = kn \cdot 3^n$ instead.

Putting into the recurrence relation, we have

$k (n + 2) \cdot 3^{n + 2} = 4k (n + 1)3^{n +1} - 3kn \cdot 3^n + 3^n$

$(9kn + 18k)3^n = 9kn + (12k + 1)3^n$

$k =6$

Therefore, we have

$a_n = A \cdot 3^n + B + 6n \cdot 3^n$.

Using the initial conditions, we obtain $A = - 44/3$ and $B = 27$, so

$a_n = 27 + (18n - 44) \cdot 3^{n -1}$