Discrete Structures - Tutorial 6

Inclusion-Exclusion Principle

1. Call a number "prime-looking" if it is *composite* but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000? [**BONUS**]

Soln: For any positive integers N and m, the number of integers divisible by m which are less than N is given by $\lfloor (N-1)/m \rfloor$. Turning to the problem, below 1000 there are $\lfloor 999/2 \rfloor = 499$ numbers divisible by 2,
$\lfloor 999/3 \rfloor = 333$ numbers divisible by 3,
$\lfloor 999/5 \rfloor = 199$ numbers divisible by 5,
$\lfloor 999/6 \rfloor = 166$ numbers divisible by $6 = 2\times 3$,
$\lfloor 999/10 \rfloor = 99$ numbers divisible by $10 = 2\times 5$,
$\lfloor 999/15 \rfloor = 66$ numbers divisible by $15 = 3\times 5$, and
$\lfloor 999/30 \rfloor = 33$ numbers divisible by $30 = 2\times 3\times 5$.
According to the Inclusion-Exclusion Principle, the amount of integers below 1000 that could not be prime-looking is $499 + 333 + 199 - 166 - 99 - 66 + 33 = 733$.

There are 733 numbers divisible by at least one of 2, 3, 5. Note that this count includes (prime) 2, 3, and 5. What can be said of the remaining $999 - 733 = 266$ numbers?
These are the numbers that are not divisible by either 2, 3, or 5. Are these prime-looking? No, not all of them. Some are really prime, not just appearing so. As was stated in the problem, there are 168 are primes below 1000. We have to exclude those. But number 2, 3, 5 have been discounted before, which leaves us with 165 primes extras. Subtracting gives $266 - 165 = 101$. Now, a final observation. Number 1 is not divisible by any greater number, 2, 3, 5 in particular. Thus it was not counted among the 733 composite numbers above. So, it is counted among the remaining 266 and, not being a prime, among the last group of 101 numbers. These are the numbers that are not divisible by either

2, 3, or 5, which trivially includes 1. But 1 is not composite and thus is not prime-looking. To get the total of prime-looking numbers we have to remove 1 from the count, leaving $101 - 1 = 100$.

2. In how many ways can a poker hand (5 cards) be selected from a regular deck (52 cards) such that the hand contains at least one card in each suit?

Soln: Let s_i be the property that the hand does not have any card of suit $i,\ i = \clubsuit, \diamondsuit, \heartsuit, \spadesuit$.
$N = {}^{52}C_5$
$N(s_i) = {}^{39}C_5$
$N(s_i s_j) = {}^{26}C_5$
$N(s_i s_j s_k) = {}^{13}C_5$
$N(s_i s_j s_k s_l) = 0$
$\overline{N} = N - ({}^4C_1)N(s_i) + ({}^4C_2)N(s_i s_j) - ({}^4C_3)N(s_i s_j s_k) + ({}^4C_4)N(s_i s_j s_k s_l)$
$\overline{N} = {}^{52}C_5 - 4 \times {}^{39}C_5 + 6 \times {}^{26}C_5 - 4 \times {}^{13}C_5 + 0 = 685,464$

3. How many integer solutions are there to the system $x_1 + x_2 + x_3 = 12;\ 0 \leq x_i \leq 5, i \in \{1, 2, 3\}$?

Soln: We know that $x_1 + x_2 + \cdots + x_k = n$ has ${}^{n+k-1}C_{k-1}$ solutions in non-negative integers. Let A be the set of integer solutions to the system $x_1 + x_2 + x_3 = 12;\ x_i \geq 0$, there are ${}^{14}C_2 = 91$ solutions. We want to throw out the ones with $x_i \geq 6$ for some i, so for $i = \{1, 2, 3\}$ let $A_i = \{(x_1, x_2, x_3) \in A : x_i \geq 6\}$. Thus, the question is now asking us for the size of $\overline{A_1 \cup A_2 \cup A_3}$.

The first step is to find the size of the sets A_i. Now, $|A_1|$ is the number of integer solutions to $x_1 + x_2 + x_3 = 12$ with $x_1 \geq 6$ and x_2, x_3 non-negative, which (letting $x_1' = x_1 - 6$) is the same as the number of solutions to $x_1' + x_2 + x_3 = 6$ in non-negative integers. We know from above that there are ${}^{6+2}C_2$ of these, so by symmetry $|A_1| = |A_2| = |A_3| = {}^8C_2 = 28$.
Next, we need to know the size of the pairwise intersections $A_i \cap A_j$. Of course, the only solution in A satisfying (for example) $x1, x2 \geq 6$ is $x1 = x2 = 6,\ x3 = 0$, so these intersections all have size 1. Lastly, we need to know the size of $A_1 \cap A_2 \cap A_3$. Clearly this set is empty (there are no solutions to our equation with each variable at least six), so finally the answer we want is $91 - 3 \times 28 + 3 \times 1 - 0 = 10$.

4. There are several people in the room. Some are acquaintances, others are not. (Being acquainted is a symmetric, non-reflexive relationship.) Show that some two people have the same number of acquaintances.

Soln: If there are N people in the room and each has a different number of acquaintances then one person (A, say) is bound to have $N-1$ and one person (B, say) 0 acquaintances. However, if A were to have $N-1$ acquaintances, then he must know every other person in the room including B; but then B cannot have 0 acquaintances (since being acquainted is a symmetric relationship). — This is a contradiction.

5. Show that among any $n+1$ numbers one can find 2 numbers so that their difference is divisible by n.

Soln: Since there are only n possible remainders on division by n, and we have $n+1$ numbers, by the pigeonhole principle some two of them have the same remainder on division by n. Thus we can write these two numbers as $n_1 = nk_1 + r$ and $n_2 = nk_2 + r$ where r is the remainder on division by n. Then, their difference is $n_1 - n_2 = (nk_1 + r) - (nk_2 + r) = nk_1 - nk_2 = n(k_1 - k_2)$ which is divisible by n.

6. Show that for any natural number n there is a number composed of digits 5 and 0 only and divisible by n. (Hint: Use the claim of the above problem.) [**BONUS**]

Soln: We will use the previous problem. We want to find a number divisible by n; the previous problem tells us that given any set of $n+1$ numbers, some two of them have a difference that is divisible by n. So we should try to find a set of $n+1$ numbers with the property that for any two of them, the difference is a number composed of digits 5 and 0 only. One possibility is the sequence of numbers $5, 55, 555, 5555, \ldots$, since the difference of any two of these will be some number of 5's followed by some number of 0's. So we can take the first $n+1$ numbers whose only digits are 5, and there must be some pair whose difference is composed of only 5's and 0's, and divisible by n.

7. Consider a chess board with two of the diagonally opposite corners removed. Is it possible to cover the board with pieces of domino whose size is exactly two board squares?

Soln: No, it's not possible. Two diagonally opposite squares on a chess board are of the same color. Therefore, when these are removed, the number of squares of one color exceeds by 2 the number of squares of another color. However, every piece of domino covers exactly two squares and these are of different colors. Every placement of domino pieces establishes a 1-1 correspondence between the set of white squares and the set of black squares. If the two sets have different number of elements, then, by the Pigeonhole Principle, no 1-1 correspondence between the two sets is possible.