Discrete Structures - Tutorial 5

1. Prove that if n is an integer and $n^3 + 5$ is odd, then n must be even.

Sol: (a) We prove the contrapositive. If n is odd, then n^3+5 is even. Assume that n is odd. Then we can write $n = 2k + 1$ for some integer k. Then $n^3 + 5 = (2k + 1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$. Thus $n^3 + 5$ is two times some integer, so it is even.

(b) Suppose that $n^3 + 5$ is odd and that n is odd. Since n is odd, and the product of odd numbers is odd, in two steps we see that n^3 is odd. But then subtracting we conclude that 5, being the difference of the two odd numbers $n^3 + 5$ and n^3 is even. This is not true. Therefore our supposition was wrong and the proof by contradiction is complete.

2. Prove that $\sqrt{2}$ is an irrational number.

Sol: Let us suppose $\sqrt{2}$ were a rational number. Then we can write $\sqrt{2} = a/b$ where a, b are whole numbers simplified to the lowest terms, b not zero. From the equality $\sqrt{2} = a/b$ it follows that $2 = a^2/b^2$, or $a^2 = 2b^2$. So the square of a is an even number since it is two times something, and thus a itself must also be an even number. Now, let $a = 2k$. If we substitute $a = 2k$ into the original equation $2 = a^2/b^2$, we get: $2 = (2k)^2/b^2$, i.e., $b^2 = 2k^2$. This means b^2 is even, from which it follows again that b itself is an even number. Our assumption was a/b is simplified to the lowest terms, and now it turns out that a and b would both be even — (contradiction). So, $\sqrt{2}$ cannot be rational.

3. Use induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$.

Sol: Let $P(n)$ be the identity for n sets.
Basis step: The statement $P(2)$ asserts that $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$ (Proof already covered in class).
Inductive step: The inductive hypothesis is: let $P(k), k \geq 2$ be true, i.e.,

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$$

holds whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U. Now we need to show that $P(k+1)$ must also hold.

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$$

(by the definition of intersection)

$$= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}}$$

(by DeMorgan's law where the two sets are $\bigcap_{j=1}^{k} A_j$ and A_{k+1})

$$= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}}$$

(by the inductive hypothesis)

$$= \bigcup_{j=1}^{k+1} \overline{A_j}$$

(by the definition of union)
This completes the inductive step.

4. An odd number of people stand in a yard at mutually distinct distances. At the same time each person throws a pie at his nearest neighbour, hitting this person. Use induction to show that there is at least one person who is not hit by a pie.
Can the same be said if there are even number of people?

Sol: Let $P(n)$ denote the statement 'there is a survivor (who is not hit) in the odd pie fight with $2n+1$ people'.
Basis step: $P(1)$, there are 3 people. Of the three people, suppose that

the closest pair is A and B, and C is the third person. Since distances between people are different, the distances between A and C, and B and C are greater than that between A and B. Therefore, A and B throws pies at each other, and C survives.
Inductive step: Suppose that $P(k)$ is true, that is, in the pie fight with $2k+1$ people there is a survivor. Consider the fight with $2(k+1)+1$ people. Let A and B be the closest pair of people in this group of $2k+3$ people. Then they throw pies at each other. If someone else throws a pie at one of them, then for the remaining $2k+1$ people there are only $2k$ pies, and one of them survives. Otherwise the remaining $2k+1$ people throw pies at each other, playing the pie fight with $2k+1$ people. By the inductive hypothesis, there is a survivor in such a fight.

For an even number of people there may *not* be any such survivor.

5. Prove by induction $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$

Sol: We know $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Therefore, $\displaystyle\left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2(n+1)^2}{4}$. So, we basically need to prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.
Basis step: $n = 1$, the basis step obviously holds since both LHS and RHS evaluate to 1.
Inductive step: Let for $n = k$, $1^3+2^3+3^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4}$ (inductive hypothesis). Now we need to show that the claim holds for $n = k+1$.
Thus, $\displaystyle\sum_{j=1}^{k+1} j^3 = 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$
$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$
$= \frac{(k+1)^2(k^2+4(k+1))}{4}$
$= \frac{(k+1)^2(k^2+4k+4)}{4}$
$= \frac{(k+1)^2(k+2)^2}{4}$
$= \displaystyle\left(\sum_{j=1}^{k+1} j\right)^2.$

6. Explain what is wrong with the reasoning of the following proof. Remember that saying the claim is false is not a justification.
For all x, y, n in $\{0, 1, 2, \ldots\}$, if $max(x, y) = n$, then $x = y$.
Proof (by induction):
Base case: Suppose that $n = 0$. If $max(x, y) = 0$ and x, y are in

$\{0, 1, 2, \ldots\}$, then $x = y = 0$.
Induction step: Assume that whenever we have $max(x, y) = k$, then $x = y$ must follow. Next, suppose x, y are such that $max(x, y) = k+1$. Then it follows that $max(x-1, y-1) = k$, so by the inductive hypothesis, $x - 1 = y - 1 \;\Rightarrow\; x = y$, completing the induction step.

Sol: If x and y are in $\mathbb{N} = \{0, 1, 2, \ldots\}$ it is not necessarily true that $x - 1$ and $y - 1$ are in $\mathbb{N}$. If you want to show $max(0, 1) = 1$ implies $0 = 1$, you are looking at $max(-1, 0) = 0$. That isn't covered in the $k = 0$ case.