1. **A function f : {1, 2, … , n} -> {1, 2, … , m} is called monotone non-decreasing if $1 \le i < j \le n \Rightarrow f(i) \le f(j)$.**
    a. **How many such functions are there?**
    b. **How many such functions are there that are surjective?**
    c. **How many such functions are there those are injective?**

Answer:

(a) There are $^{(n+m-1)}\mathbf{C}_{m}$ such functions. Consider the co-domain {1, 2, … , m} as bins and the domain {1, 2, …, n} as balls. If a bin (co-domain element) contains a ball, it means that one of the elements in the domain maps to it. Thus, if we represent this as bins and balls, any ordering of the bins and balls will give us a unique mapping from domain to co-domain since it has to be monotone non-decreasing(The bijection as follows: the smallest valued bin that has a ball must must map to the minimal domain element. Remove that ball and the new smallest valued bin that could be the same value as the bin in the previous step has a ball must map to the second minimal domain element, and so on).Now we can apply the canonical “unordered with repetition" formula.

(b) There are $^{((n-m)+m-1)}\mathbf{C}_{(n-m)} = {}^{(n-1)}\mathbf{C}_{(n-m)}$ such functions. Again, if we use the balls and bins analogy, we have to first allocate 1 ball for each bin, and then choose positions for the rest of the balls. Thus, n -m balls are left for us to put into bins, as in the canonical unordered with repetition problem.

(c) There are $^{m}\mathbf{C}_{n}$. Once we choose a set of n elements from m, we will know the exact mapping because the function must be monotone non-decreasing. Thus, we need to determine in how many ways can we choose n elements from m.

2. **Proof the following functions are primitive recursive:**
    a. **exp(x,y) = x^y**
    b. **Fac(x) = x!**
    c. **Pred(x)to be x−1 if x > 0 and Pred(0) to be 0**
    d. **x~y = max{x−y,0}**
    e. **|x−y|**

Answer:

a. exp(x,0)=1
exp(x,y+1) = mult(x,exp(x,y))

b. Fac(0)=1, Fac(x+1) = mult (x+1,Fac(x))

c. Pred(0)=0, Pred(x+1)=x

d. $x \sim 0 = x$, $x \sim (y+1) = \text{Pred}\,(x \sim y)$

e. $|x-y| = (x \sim y) + (y \sim x)$

3. Prove that if every point on a line is painted cardinal or white, there exists three points of the same color such that one is the midpoint of the line segment formed by the other two.

Answer:

Pick two points A; B of the same color. Let C be the midpoint of AB, and position D; E such that C is also the midpoint of DE and DA = AB = BE. If C; D; E are the same color then we're done; if not, then at least one of them is the same color as A; B and forms a trio with A; B.

4. Prove that there are an infinite number of integers.

Answer:

Proof:

1. Suppose that there are only finite many integers.
2. Then there must be a largest, say N.
   Then, $\forall\, n \in Z,\ n \leq N$.
   Now, N + 1 is an integer because N is an integer and 1 is an integer and Z is closed under addition.
   But $N + 1 > N \rightarrow\leftarrow$ [This contradicts our assumption that N is the largest integer.]
3. Hence our original statement must be true.

5. The sum of two integers is odd if and only if one of the integers is odd and the other is even. (Proof by Cases)

Answer:

Cases:

1. Both integers are even: $m + n = 2 * k + 2 * l = 2 * (k + l)$. Therefore the sum is even.
2. Both integers are odd: $m+n = 2*k+1+2*l+1 = 2*(k+l)+2 = 2*(k+l+1)$. Therefore the sum is even.
3. One integer is even and the other is odd: $m+n = 2 * k + 2 * l + 1 = 2 * (k +l) + 1$. Therefore the sum is odd.

6. Let the linear function given by $L(x) = (x_1 + 2x_2)x + (x_1 + 3x_2)$.

(a) Show that L is invertible with inverse L^{-1} given by $L^{-1}(ax + b) = [3a - 2b, -a + b]^T$

(b) Find x such that $L(x) = 5x + 8$.

Answer:

(a) For all $\mathbf{x}$ in $\mathbf{R}^2$, we have

$$L^{-1}(L(\mathbf{x})) = L^{-1}\big((x_1 + 2x_2)x + (x_1 + 3x_2)\big) = \begin{bmatrix} 3(x_1 + 2x_2) - 2(x_1 + 3x_2) \\ -(x_1 + 2x_2) + (x_1 + 3x_2) \end{bmatrix}$$
$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x},$$

and for all $ax + b \in \mathbf{P}_2$, we have

$$L\big(L^{-1}(ax + b)\big) = L\left(\begin{bmatrix} 3a - 2b \\ -a + b \end{bmatrix}\right) = \big((3a - 2b) + 2(-a + b)\big)x + \big((3a - 2b) + 3(-a + b)\big)$$
$$= ax + b.$$

(b) As was demonstrated in the introduction, the answer is

$$\mathbf{x} = L^{-1}(5x + 8) = \begin{bmatrix} 3(5) - 2(8) \\ -(5) + (8) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(Check: $L\big([-1, 3]^T\big) = \big((-1) + 2(3)\big)x + \big((-1) + 3(3)\big) = 5x + 8.$)