Discrete Structures Tutorial 3

1. Find the greatest lower bound and the least upper bound of the sets {1, 2, 3, 4, 5} and {1, 2, 4, 8, 16}, if they exist, in the poset (Z^+, |)

Solution:

{1, 2, 3, 4, 5} – GLB : 1

LUB : doesn't exist

{1, 2, 4, 8, 16} – GLB : 1

LUB : 16

2. a) Give an example of lattice that is not distributive

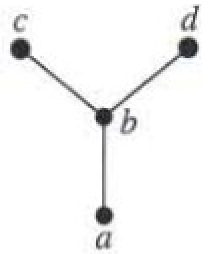
b) Prove or disprove: lattice (Z^+, |) is distributive.

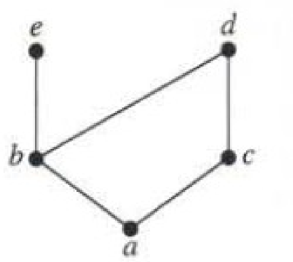
Solution:

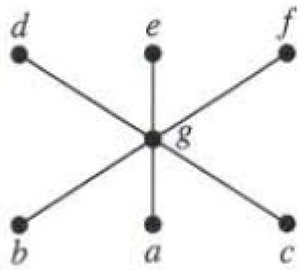
a) L = (S, ⊆) where S = {Ø, {1}, { 2}, {3}, { 1,2}, {2,3}, {1,2,3}}

b) It is distributive.

3. Find all antichains in these posets represented by the following Hasse diagram







Solution:

i) {c,d}

ii) {b,c} , {c,e}, {d,e}

iii) {a,b}, {a,c}, {b,c}, {a,b,c}, {d,e}, {d,f}, {e,f} and {d,e,f}

4. Show that every finite lattice has a least element and a greatest element

Solution:

We need to prove that “every subset with n element from a lattice has a least upper bound and a greatest lower bound”.

Prove this by induction.

Then it is very easy to say that these least upper bound and greatest lower bound are by definition the greatest and the least element.

5. Show that every finite poset can be partitoned into k chains, where k is the largest number of elements in an antichain in the poset.

Solution:

We use induction on the cardinality of poset say P. Let a be a maximal element of P, and let r be the size of a largest antichain in $P'=P\setminus\{a\}$; hence, $r\leq k$. By the induction hypothesis, P′ is the union of r disjoint chains $C_1,\dots,C_r$. Every r-element antichain in P′ consists of one element from each Ci. Let ai be the maximal element in C_i which belongs to some k-element antichain in P′.

It is not difficult to see that $A=\{a_1,\dots,a_r\}$ is an antichain in P′. Suppose towards a contradiction that $a_i<a_j$ for some i and j By definition, ai is in some r-antichain A_i and a_j is in some r-antichain Aj. Since Aj is a maximal antichain, it must intersect the chain Ci. Hence, there is an x in $C_i\cap A_j$. By definition of ai we have $x\leq a_i$, hence by transitivity $x<a_j$. But that's a contradiction because x and a_j are in the same antichain A_j.

If $A\cup\{a\}$ is an antichain in P, then $r\leq k-1$, and we are done: the chains C1,…,Cr and {a} give us a desired decomposition of P into $r+1\leq k$chains. Otherwise, we have $a> a_i$ for some i. Then $K=\{a\}\cup\{x\in C_i : x\leq a_i\}$ is a chain in P, and there are no r-element antichains in $P\setminus R$ (since ai was the maximal element of C_i participating in such an antichain). By the induction hypothesis, $P\setminus R$ is the union of $r-1$ disjoint chains. Together with the chain R, these chains cover entire P, as desired.

6. Show that in any group of *ab+1* people there is either a list of *a+1* people where a person in the list (except for the first person listed) is a descendant of the previous person in the list, or there are *b+1* people such that none of these people is a descendant of any of the other *b* people.

Solution:

Use result of prob 5.

Order the people according to the descendant relationship ... this is a partial order. If the maximal size of an antichain in this poset is *b +1* or larger, the second condition is met. On the other hand, if the maximal size of an antichain is at most b, then the people can be partitioned into *b* or fewer disjoint chains. Consequently, there must be a chain of length *a + 1* or larger (if *a* or less, then only *ab* total people).