1. Formulate each of the below as a single statement (proposition or predicate), using only mathematical and logical notation that has been defined in class. For example, the use of logical quantifiers and connectives, and arithmetic, number-theoretic, and set-theoretic operations is allowed, as is the use of operators like gcd or sets like Q, R, etc., but not the use of English-language words or informal shorthand like {1,2,…,n} [Use *Rule Method*]

   (a) If a and b are integers and $b \neq 0$, then there is a unique pair of integers q and r, such that $a = qb + r$ and $0 \leq r < |b|$

   (b) Two integers are co-prime if and only if every integer can be expressed as their linear combination.

Answer:

(b) $\forall a, b \in \mathbb{Z} : (b \neq 0) \rightarrow \Big(\forall q, r, q', r' \in \mathbb{Z} : \big((a = qb + r) \wedge (0 \leq r < |b|) \wedge (a = q'b + r') \wedge (0 \leq r < |b|)\big) \rightarrow (q = q') \wedge (r = r')\Big)$

(c) $\forall a, n \in \mathbb{Z} : \big(gcd(a, n) = 1\big) \rightarrow \Big(\forall b \in \mathbb{Z}, \forall x, x' \in \mathbb{Z}_n : \big((ax \equiv_n b) \wedge (ax' \equiv_n b)\big) \rightarrow (x = x')\Big)$

2. For each of the following relations, state whether they fulfill each of the 4 main properties - reflexive, symmetric, anti-symmetric, transitive. Briefly substantiate each of your answers.

   (a) The co-prime relation on Z. (Recall that a, b € Z are co-prime if and only if gcd(a, b) = 1.)
   (b) Divisibility on Z.
   (c) The relation T on R such that aTb if and only if a.b € Q.

Answer:

(a) It's definitely not reflexive, as no integer is co-prime with itself except -1 and 1. It is symmetric because gcd(a, b) = gcd(b, a), so gcd(a, b) = 1 iff gcd(b, a) = 1. Not anti-symmetric - every co-prime pair, such as (5,7)and (7,5), will show this. Not transitive - gcd(5,7) = 1, gcd(7,10) = 1, but gcd(5,10) ≠ 1.

(b) It's reflexive since any integer divides itself. Not symmetric, for example 2 divides 4 but 4 does not divide 2. It not anti-symmetric on Z, since a |-a and - a | a, although it would be anti-symmetric if restricted to N. It is transitive | if a | b then b = ka for some k € Z, and if b | c then c = lb for some l € Z, thus c = (lk)a and (lk) € Z so a | c.

(c) Not reflexive, for example $\sqrt[4]{2}\sqrt[4]{2} = \sqrt{2}$ which is definitely not in Q. Definitely symmetric since multiplication is commutative, ab = ba always. Not anti-symmetric, since $\sqrt{2}\sqrt{8} = \sqrt{8}\sqrt{2} =$ 4 but $\sqrt{2} \neq \sqrt{8}$. Also not transitive- consider a = π, b =$1/\pi$, and c = π. ab; bc € Q but ac =π^2 does not belong to Q.

3. In a partially ordered set (A (non-strict) **partial order** is a binary relation "≤" over a set *P* which is anti-symmetric, transitive, and reflexive) , a chain is a totally ordered subset. For example, in the set 1, 2, 3, 4, 5, 6, the divisibility relation is a partial order and 1, 2, 4 and 1, 3, 6 are chains.

   (a) What is the longest chain on the set (1, 2, . . . , n) using the divisibility relation? How many distinct chains have this length? For the second part, make sure to consider all positive values of n.

   (b) What is the longest chain on the power set of a set A with |A| = n with the ⊆ relation? How many distinct chains have this length?

Answer:

(a) Longest chain is powers of 2 as high as they can go, length is $\log_2 n + 1$. There is one chain of this length, except for n = 3 where there are two chains of length two.

(b) Each set in the chain must have distinct cardinality, so the longest chains are n + 1. The number of chains is the product of all binomial coefficients for n as they correspond to the number of sets of each cardinality.

4. Can a relation on a set be neither reflexive nor irreflexive [(a,a) does not belong to A, if a € A] ? Give reasons.

Answer:

Digraph: 

Neither Reflexive nor Irreflexive.

5. The relation R consisting of all pairs (x; y) is such that x and y are bit strings of length three or more that agree in their first three bits. Is R an equivalence relation on the set of all bit strings of length three or more?

Answer:

1) Reflexivity: any string agrees with itself everywhere.

2) Symmetricity: If a agrees with b on any character after 4 then b also agrees with a.

3) Transitivity: If a agrees with b and b agrees with c on some character then a agrees with c on this character, thus the property holds.

6. Give an example of a relation on a set that is (Use digraphs)
(a) symmetric and anti-symmetric
(b) neither symmetric nor anti-symmetric

Answer:

Draw your own digraphs.

a) Equality relation.

b) Suppose *aRb* and *bRc* and *cRb*. And that's as far as *R* goes. It's not symmetric since (not *bRa*) and it's not anti-symmetric since both *aRb* and *bRa*.

Or

The relation "divides" on the set ℤ; The relation "preys on" in biological sciences.

7. Let n be a positive integer and S be a set of strings. Rn is a relation on S such that sRnt for s, t € Rn, if and only if s = t or both s and t have at least n characters and the first n of them are the same. For instance, 01R301; 00111R300101 hold but 01R3010; 01011R301110 do not hold. Is Rn an equivalence relation on S?

Answer:

We show that the relation Rn is reflexive, symmetric, and transitive.

- **Reflexive:** The relation Rn is reflexive because s = s, so that sRn s whenever s is a string i n S.
- **Symmetric:** If sRn t, then either s = t or s and t are both at least n characters long that beg in with the same n characters. This means that tRn s. We conclude that Rn is symmetric.
- **Transitive:** Now suppose that sRn t and tRn u. Then either s = t or s and t are at least n characters long and s and t begin with the same n characters, and either t = u or t and u ar e at least n characters long and t and u begin with the same n characters. From this, we can deduce that either s = u or both s and u are n characters long and s and u begin with th e same n characters, i.e. s Rn u. Consequently, Rn is transitive.
- It follows that ***Rn is an equivalence relation***.