Discrete Structures Tutorial 1

1. Prove the following set identity using the element method (prove X = Y by proving X ⊆ Y and Y ⊆ X).
(A ∪ B)' ∩ C = C - (A ∪ B)

Solution:

Let X = (A ∪ B)' ∩ C and Y = C - (A ∪ B)

• Claim: X ⊆ Y .

Proof :

Let x ∈ X. x ∈ (A ∪ B)'∩ C.

By definition of set intersection, x ∈ (A ∪ B)' and x ∈ C.

By set complement, x not ∈ (A ∪ B).

By definition of set difference, x ∈ C - (A ∪ B).

Thus x ∈ Y , and X ⊆ Y .

• Claim: Y ⊆ X.

Proof :

Let y ∈ Y . y ∈ C - (A ∪ B)

By definition of set difference, y ∈ C and y not ∈ (A ∪ B).

By definition of set complement, y ∈ (A ∪ B)'.

Then, by definition of set intersection, y ∈ (A ∪ B)'∩ C.

Therefore y ∈ X, and Y ⊆ X.

Since X ⊆ Y and Y ⊆ X, X = Y .

2. Prove that (A x A) ∪ (B x C) = (A∪B) x (A∪C)

Solution:
take A = {0}, B = φ, C = {0,1}
LHS : {(0,0)}∪ φ
RHS : {0}x{0,1} = {(0,0),(0,1)}

3. Prove that (A x A) ∩ (B x C) = (A∩B) x (A∩C)

Solution:
If (x,y) Є (A x A) ∩ (B x C) then (x,y) Є (A x A) and (x,y) Є (B x C). Thus, x Є A, y Є A, x Є B, y Є C. So, x Є (A∩B) , y Є (A∩C)

similarly the reverse way

4. If $2^A \subseteq 2^B$, what is the relation between A and B?

Solution:

2^A is the set of all subsets of A, including A itself. The condition tells us that every subset of A is also a subset of B, and in particular A itself is a subset of B. So $A \subseteq B$.

5. If a(t), b(t) and c(t) are the lengths of the three sides of a triangle

t in non-decreasing order (i.e. $a(t) \leq b(t) \leq c(t)$), we define the sets:

• X := {Triangle t : a(t) = b(t)}

• Y := {Triangle t : b(t) = c(t)}

• T := the set of all triangles

Using only set operations on these three sets, define:

(a) The set of all equilateral triangles (all sides equal)

(b) The set of all isosceles triangles (at least two sides equal)

(c) The set of all scalene triangles (no two sides equal)

Solution:

(a) We require a(t) = b(t) and b(t) = c(t) (this obviously implies a(t) = c(t)), so the set is $X \cap Y$.

(b) An isosceles triangle t can have

i. a(t) = b(t), or

ii. b(t) = c(t), or

iii. a(t) = c(t).

Now we have assumed that a(t), b(t) and c(t) are in non-decreasing order, so the last condition holds if and only if both the first two do. So the required set is $X \cup Y \cup (X \cap Y)$, which simplifies to just $X \cup Y$.

(c) A scalene triangle has its two smaller sides a(t) and b(t) unequal (set $T \setminus X$) and its two larger sides b(t) and c(t) unequal (set $T \setminus Y$). Since the sides are listed in nondecreasing order, either of the above conditions guarantees $a(t) \neq c(t)$. So the required set is $(T \setminus X) \cap (T \setminus Y)$.

An alternative argument is: A triangle is scalene if and only if it is not isosceles. So using the result of the previous part, the set of scalene triangles is $T \setminus (X \cup Y)$.

6. Let us define pairwise addition: $A \oplus B := \{a + b : a \in A, b \in B\}$ and pairwise

multiplication: $A \otimes B := \{a \times b : a \in A, b \in B\}$. Under the definition of such set operations, answer the following questions:

a) If E is the set of all positive even numbers, what is the shortest way to write the set of all positive multiples of 4? Of 8?

b) Let $S := \{n^2 : n \in N^+\}$. A Pythagorean triple consists of three positive integers x, y and z such that $x^2 + y^2 = z^2$. Construct the set of all possible z^2 such that z is the last element of a Pythagorean triple using only the set S and the set operations you have been taught so far and the pairwise addition and multiplication operations defined above.

Solution:

a) Let F be the set of multiples of 4. We claim that $F = E \otimes E$. Every positive even number can be written as 2k for some $k \in N+$, so $E \otimes E$ consists of elements of the general form $2j \times 2k = 4jk$, for $j, k \in N+$. In other words, every element of $E \otimes E$ is a multiple of 4, so $E \otimes E \subseteq F$. Also, every multiple of 4 is of the form $4k = 2 \times 2k$, for $k \in N+$, so $F \subseteq \{2\} \otimes E \subseteq E \otimes E$. This proves the claim.

A virtually identical argument shows that T, the set of positive multiples of 8, is $E \otimes E \otimes E$.

b) The set of all possible numbers of the form $x^2 + y^2$, where x and y are positive integers, is $S \oplus S$. If such a number is also the square of a positive integer z, it must be in $(S \oplus S) \cap S$, which is the required set.