Discrete Structures - Quiz 1

Question:
Let S be a set and let Q and R be two equivalence relations on S. Let $P = Q \cup R$ (this means aPb holds if and only if aQb and/or aRb holds). Is P an equivalence relation?

Solution:
For P to be an equivalence relation, it must be reflexive, symmetric, and transitive.

- *Reflexive:* P is reflexive. Choose any $m \in S$. Q and R are equivalence relations, so they are both reflexive; this means that mQm and mRm. Thus, mPm because mQm and mRm hold.
- *Symmetric:* P is symmetric. Choose any $m, n \in S$. Let mPn. We must show that $mPn \Rightarrow nPm$. Since mPn holds, we know mQn and/or mRn hold. Since Q and R are equivalence relations, they are both symmetric.
  - If mQn, then nQm, which is sufficient for nPm.
  - If mRn, then nRm, which is also sufficient for nPm.

  At least one of mQn and mRn is true, so at least one of nQm and nRm is true. This means nPm holds, so P is symmetric.
- *Transitive:* P is not transitive. Counterexample: let $S = \mathbb{Z}$. Define Q and R be as follows: For all $m, n \in \mathbb{Z} : mQn$ if and only if m and n have the same parity (i.e., they are both odd or both even), and for all $m, n \in \mathbb{Z} : mRn$ if and only if m and n are both divisible by 3. Let $a = 1, b = 3$, and $c = 6$. Then aPb holds, since aQb, because 1 and 3 are both odd, and bPc, since bRc, because 3 and 6 are both divisible by 3, but aPc is false, because 1 and 6 do not have the same parity, and 1 is not divisible by 3. Thus, P is not transitive.

  Since P is not transitive, it is not an equivalence relation.