1.a. Assume A and P are as stated. To show $A \subseteq P$, we must show that $p \in A$ implies $p \in P$. Thus suppose $p \in A$. By definition of A, this means $p = 2^{n-1}(2^n - 1)$ for some $n \in N$ for which 2^{n-1} is prime. We want to show that $p \in P$, that is, we want to show p is perfect. Thus, we need to show that the sum of the positive divisors of p that are less than p add up to p. Notice that since 2^{n-1} is prime, any divisor of $p = 2^{n-1}(2^n - 1)$ must have the form 2^k or $2^k(2^n - 1)$ for $0 \leq k \leq n-1$. Thus the positive divisors of p are as follows:

1. $2^0, 2^1, 2^2, \ldots 2^{n-2}, 2^{n-1}$
2. $2^0(2^n - 1), 2^1(2^n - 1), 2^2(2^n - 1), \ldots 2^{n-2}(2^n - 1), 2^{n-1}(2^n - 1)$

Notice that this list starts with $2^0 = 1$ and ends with $2^{n-1}(2^n - 1) = p$.
If we add up all these divisors except for the last one (which equals p) we get the following:

$$\sum_{k=0}^{n-1} 2^k + \sum_{k=0}^{n-2} 2^k(2^n - 1) = (2^n - 1) + (2^n - 1)(2^{n-1} - 1) = p$$

This shows that the positive divisors of p that are less than p add up to p. Therefore p is perfect, by definition of a perfect number. Thus $p \in P$, by definition of P. We have shown that $p \in A$ implies $p \in P$, which means $A \subseteq P$.

1.b. $31 = 2^5 - 1$
Therefore, take $n = 5$.
$p = 2^{5-1}(2^5 - 1) = 16 * 31 = 496$

1.c. To show that $A = E$, we need to show $A \subseteq E$ and $E \subseteq A$.
First we will show that $A \subseteq E$. Suppose $p \in A$. This means p is even, because the definition of A shows that every element of A is a multiple of a power of 2. Also, p is a perfect number because part (a) states that every element of A is also an element of P, hence perfect. Thus p is an even perfect number, so $p \in E$. Therefore $A \subseteq E$.

Next we show that $E \subseteq A$. Suppose $p \in E$. This means p is an even perfect number. Write the prime factorization of p as $p = 2^k 3^a 5^b \ldots$, where some of the powers a,b, ... may be zero. But, as p is even, the power k must be greater than zero. It follows $p = 2^k q$ for some positive integer k and an odd integer q. Now,

our aim is to show that p ∈ A, which means we must show p has form
$p = 2^{n-1}(2^n - 1)$.
To get our current $p = 2^k q$ closer to this form, let n = k +1, so we now have $p = 2^{n-1}q$. List the positive divisors of q as $d_1, d_2, d_3, \ldots, d_m$. (Where $d_1 = 1$ and $d_m = q$.) Then the divisors of p are:

$2^0d_1, 2^0d_2, 2^0d_3 \ldots 2^0d_m$
$2^1d_1, 2^1d_2, 2^1d_3 \ldots 2^1d_m$
$2^2d_1, 2^2d_2, 2^2d_3 \ldots 2^2d_m$
...
$2^{n-1}d_1, 2^{n-1}d_2, 2^{n-1}d_3 \ldots 2^{n-1}d_m$

Since p is perfect, these divisors add up to 2p. Also, $2p = 2(2^{n-1}q) = 2^n q$. Adding the divisors column-by-column, we get:

$$\sum_{k=0}^{n-1} 2^k d1 + \sum_{k=0}^{n-1} 2^k d2 + \ldots + \sum_{k=0}^{n-1} 2^k dm = 2^n q$$

$$(2^n - 1)d1 + (2^n - 1)d2 + \ldots + (2^n - 1)dm = 2^n q$$

$$(2^n - 1)(d1 + d2 + \ldots + dm) = 2^n q$$

$$d1 + d2 + \ldots + dm = (2^n q)/(2^n - 1) = (2^n - 1 + 1)q/(2^n - 1) = q + q/(2^n - 1)$$

From this we see that $q/(2^n - 1)$ is an integer. It follows that both q and $q/(2^n - 1)$ are positive divisors of q. Since their sum equals the sum of all positive divisors of q, it follows that q has only two positive divisors, q and $q/(2^n - 1)$. Since one of its divisors must be 1, it must be that $q/(2^n - 1) = 1$, which means $q = 2^n - 1$. Now a number with just two positive divisors is prime, so $q = 2^n - 1$ is prime.
Using this gives $p = 2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. This means p ∈ A, by definition of A. We have now shown that p ∈ E implies p ∈ A, so E ⊆ A. Since A ⊆ E and E ⊆ A, it follows that A = E.

2. We need to show that isomorphism relations is Reflexive, Symmetric and Transitive.

a.Reflexive - That means there exists a bijection from the poset to itself that preserves the poset-relation. There might be many such bijections, but there is one that always works: identity, obviously

$$\forall x,y \in S1.\ (x,y) \in R1 \leftrightarrow (x,y) \in R1$$

and so function idS1:S1→S1 given by f(x)=x is a valid isomorphism between (S1,R1) and itself.

b.Symmetric - That means that if f:S1→S2 is a valid isomorphism between (S1,R1) and (S2,R2), then f^{-1}:S2→S1 is a valid isomorphism between (S2,R2) and (S1,R1). Note, that we don't change or inverse the R1 or R2 relations. Instead we inverse the isomorphism, which is one level higher. In particular we would like to show

$(\forall x,y \in S1.\ (x,y) \in R1 \leftrightarrow (f(x),f(y)) \in R2) \Leftrightarrow (\forall x',y' \in S2.\ (x',y') \in R2 \leftrightarrow (f^{-1}(x'),f-1(y')) \in R1)$

which is true because f and f^{-1} are both bijections and mutual inverses.

c.Transitive - That is, if f:S1→S2 and g:S2→S3 are valid isomorphisms, then so is $(g \circ f)$:S1→S3. Note that we don't compose the poset relations, but the isomorphisms. Then we would like to show that

$(\forall x,y \in S1.\ (x,y) \in R1 \leftrightarrow (f(x),f(y)) \in R2)$ and $(\forall x',y' \in S2.\ (x,y) \in R2 \leftrightarrow (g(x'),g(y')) \in R3)$

together imply $(\forall x'',y'' \in S1.\ (x'',y'') \in R1 \leftrightarrow ((g \circ f)(x''),(g \circ f)(y')) \in R3)$

which is true by transitivity of ↔.

3. Let $a \le a$. Since $a \le b$, it implies that $a \le a \wedge b$. But from definition of $\wedge$, $a \wedge b \le a$. Thus,

$a \le b \Rightarrow a \wedge b = a$.

On the other hand, let $a \wedge b = a$, which is possible only if $a \le b$ i.e.

$a \wedge b = a \Rightarrow a \le b$. Therefore, $a \le b \Leftrightarrow a \wedge b = a$.

Again, assume $a \wedge b = a$. Then,

$b \vee (a \wedge b) = b \vee (a \vee a) \vee b = b \vee a \vee b = (a \vee b) \vee b = a \vee (b \vee b) = a \vee b$

Also, $b \vee (a \wedge b) = b \vee (b \wedge a)$ [Commutative property]

$= b$ [Absorption property]

Hence, $a \vee b = b$.

Similarly, rest of the proof follows.