Monte Carlo Methods

CS60077: Reinforcement Learning

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Agenda

- Understand how to evaluate policies in model-free setting using Monte Carlo methods
- Understand Monte Carlo methods in model-free setting for control of Reinforcement Learning problems

Resources

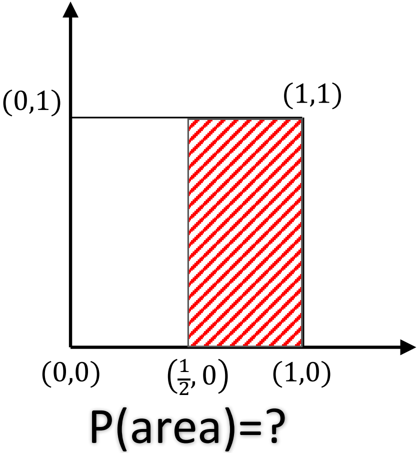
- Reinforcement Learning by David Silver [Link]
- Reinforcement Learning by Balaraman Ravindran [Link]
- Monte Carlo Simulation by Nando de Freitas [Link]
- SB: Chapter 5

Model Free Setting

- Like the previous few lectures, here also we will deal with prediction and control problems but this time it will be in a model-free setting
- In model-free setting we do not have the full knowledge of the MDP
- **Model-free prediction**: Estimate the value function of an unknown MDP
- **Model-free control**: Optimise the value function of an unknown MDP
- Model-free methods require only *experience* - sample sequences of states, actions, and rewards $(S_1, A_1, R_2, \cdots)$ from **actual** or **simulated** interaction with an environment.
- Actual experince requires no knowledge of the environment's dynamics.
- Simulated experience 'requires' models to generate samples only. No knowledge of the complete probability distributions of state transitions is required. In many cases this is easy to do.

Monte Carlo

- What is the probability that a dart thrown uniformly at random in the unit square will hit the red area?



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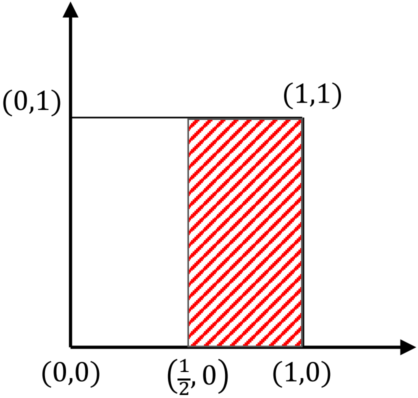

$P(\text{area}) = {}^{1}/_{2}$

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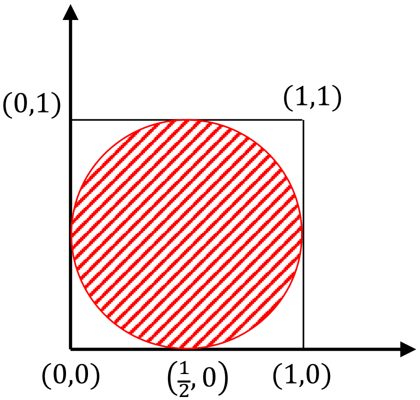

P(area)=?

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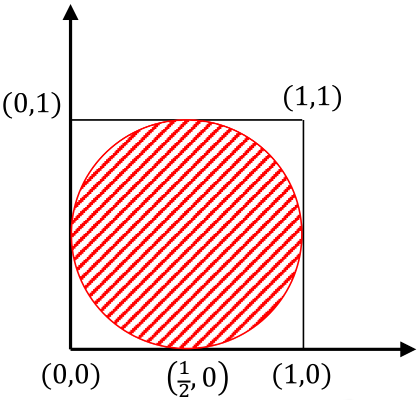

P(area)= $\pi\left({}^{1}/_{2}\right)^2$

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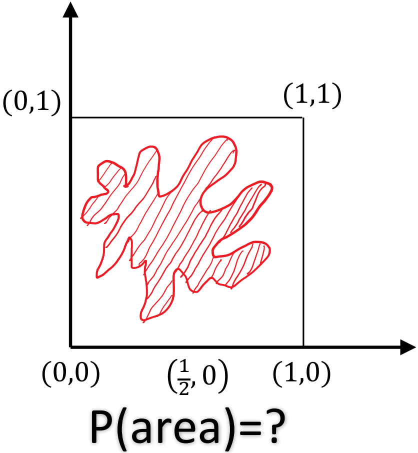

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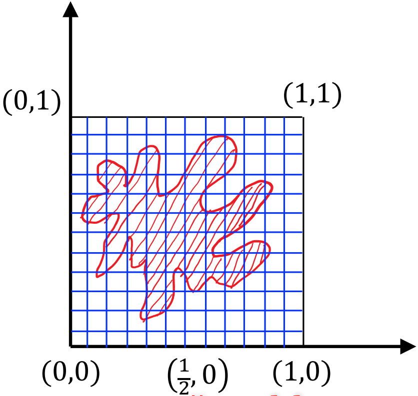

$$P(\text{area}) = \frac{\#\ \textit{red boxes}}{\#\ \textit{blue boxes}}$$

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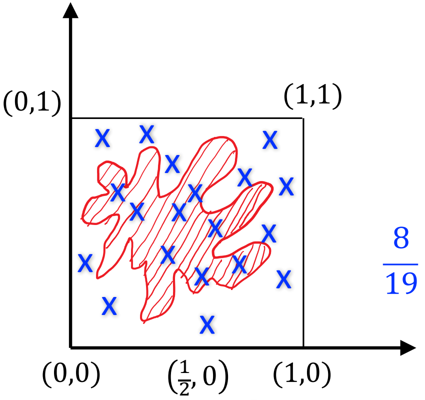


$$P(\text{area}) = \frac{\#\ darts\ in\ red\ area}{\#\ darts}$$

History of Monte Carlo

§ The bomb and ENIAC

Image taken from: *www.livescience.com*

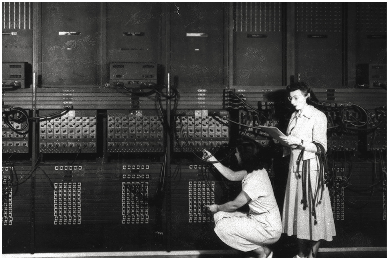


Image taken from: *www.digitaltrends.com*

Monte Carlo for Expectation Calculation

- Lets say we want to compute $\mathbb{E}[f(x)] = \int f(x)p(x)dx$
- Draw i.i.d. samples $\{x^{(i)}\}_{i=1}^{N}$ from the probability density $p(x)$

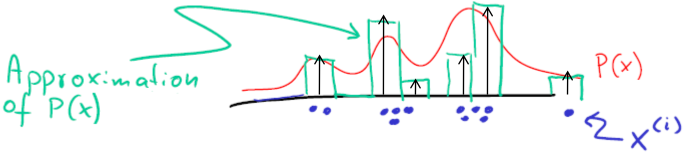


Image taken from: *Nando de Freitas: MLSS 08*

- Approximate $p(x) \approx \frac{1}{N}\sum_{i=1}^{N} \delta_{x^{(i)}}(x)$ [$\delta_{x^{(i)}}(x)$ is impulse at $x^{(i)}$ on x axis]
- $\mathbb{E}[f(x)] = \int f(x)p(x)dx \approx \int f(x)\frac{1}{N}\sum_{i=1}^{N} \delta_{x^{(i)}}(x)dx =$

$$\frac{1}{N}\sum_{i=1}^{N}\underbrace{\int f(x)\delta_{x^{(i)}}(x)dx}_{f\left(x^{(i)}\right)} = \frac{1}{N}\sum_{i=1}^{N} f\left(x^{(i)}\right)$$

Monte Carlo Policy Evaluation

- Learn v_π from episodes of experience under policy π

$$S_1, A_1, R_2, S_2, A_2, R_3, \cdots, S_k, A_k, R_k \sim \pi$$

- Recall that the *return* is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$$

- Recall that the value function is the expected return:

$$v_\pi(s) = \mathbb{E}\left[G_t | S_t = s\right]$$

- Monte-Carlo policy evaluation uses empirical mean return instead of expected return

First Visit Monte Carlo Policy Evaluation

- To evaluate state s i.e. to learn $v_\pi(s)$
- The first time-step t that state s is visited in an episode,
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total retun $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = S(s)/N(s)$
- By law of large number, $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

Every Visit Monte Carlo Policy Evaluation

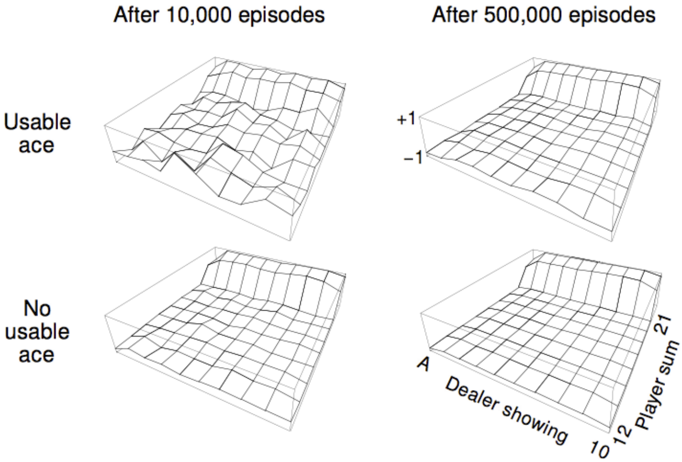
- To evaluate state s i.e. to learn $v_\pi(s)$
- Every time-step t that state s is visited in an episode,
- Increment counter $N(s) \leftarrow N(s) + 1$
- Increment total retun $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return $V(s) = S(s)/N(s)$
- By law of large number, $V(s) \rightarrow v_\pi(s)$ as $N(s) \rightarrow \infty$

Blackjack Example

- States (200 of them):
  - Current sum (12-21)
  - Dealer's showing card (ace-10)
  - Do I have a "useable" ace? (yes-no)
- Action stick: Stop receiving cards (and terminate)
- Action twist: Take another card (no replacement)
- Reward for stick:
  - +1 if sum of cards > sum of dealer cards
  - 0 if sum of cards = sum of dealer cards
  - -1 if sum of cards < sum of dealer cards
- Reward for twist:
  - -1 if sum of cards > 21 (and terminate)
  - 0 otherwise
- Transitions: automatically twist if sum of cards < 12

Slide courtesy: *David Silver [Deepmind]*

Blackjack Example

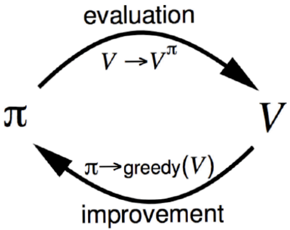


Policy: stick if sum of cards ≥ 20, otherwise twist

Slide courtesy: *David Silver [Deepmind]*

Monte Carlo Control

- We will now, see how Monte Carlo estimation can be used in *control*.
- This is mostly like the generalized policy iteration (GPI) where one maintains both an approximate policy and an approximate value function.



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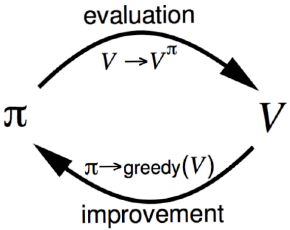

- Policy evaluation is done as Monte Carlo evaluation
- Then, we can do greedy policy improvement.

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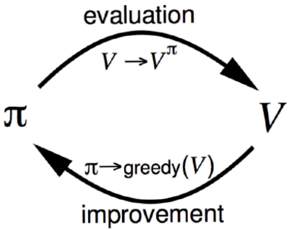

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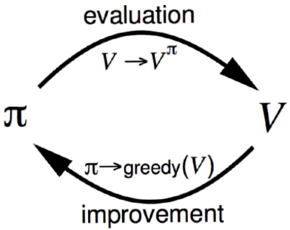

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- Then, we can do greedy policy improvement.
- What is the problem!!
- $\pi'(s) \doteq \arg\max_{a \in \mathcal{A}} \left\{ r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) v_{\pi}(s') \right\}$

Monte Carlo Control

§ Greedy policy improvement over $v(s)$ requires model of MDP

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- How can we do Monte Carlo policy evaluation for $q(s,a)$?
- Essentially the same as Monte Carlo evaluation for state values. Start at a state s, pick an action a and then follow the policy.
- After few such episodes average the returns to get an estimate of $q(s,a)$.

Monte Carlo Control

- What are some concerns?
- First visit/Every visit!!
- Suppose you start at a state s and take action a. You reach at state s_1 and then following the policy π at s, you take the action $a_1 = \pi(s_1)$. Can you take the rest of the trajectory as a sample to estimate $q(s_1, a_1)$?
- Practically you can, but convergence can not be guaranteed. The reason is that this strategy draws a disproportionately large number of actions corresponding to π. So, each sample is considered only for the starting s and a.

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- Practically you can, but convergence can not be guaranteed. The reason is that this strategy draws a disproportionately large number of actions corresponding to π. So, each sample is considered only for the starting s and a.
- How to make sure we have $q(s, a)$ estimates for all s and a? Especially because of the above the '*exploring starts*' becomes important.

Monte Carlo Control

- Many state-action pairs may never be visited.
- For deterministic policy, with no returns to average, the Monte Carlo estimates of many actions will not improve with experience.
- This is the general problem of maintaining exploration.
- One way to do this is by specifying that the episodes start in a state-action pair, and that every pair has a nonzero probability of being selected as the start.
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- This assumption is called '*exploring starts*'
- Monte Carlo Exploration Starts is an 'on policy' method. On policy methods evaluate or improve the policy by drawing samples from the same policy.
- Off-policy methods evaluate or improve a policy different from that used to generate the samples.

Monte Carlo Control

- Before going to off-policy methods let us look into an on policy Monte Carlo control method that does not use *exploring starts*.
- The assumption of exploring starts is sometimes useful, but it cannot be relied upon in general, particularly when learning directly from actual interaction with an environment.
- The easiest alternative is to consider stochastic policies with a nonzero probability of selecting all actions in each state.

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- The assumption of exploring starts is sometimes useful, but it cannot be relied upon in general, particularly when learning directly from actual interaction with an environment.
- The easiest alternative is to consider stochastic policies with a nonzero probability of selecting all actions in each state.
- Instead of getting a greedy policy in the policy improvement step, an ϵ-greedy policy is obtained.
- It means most of the time, the action corresponding to maximum estimated action value is chosen, but sometimes (with probability ϵ) an action at random is chosen.
- Probability of choosing nongreedy actions is $\frac{\epsilon}{|\mathcal{A}(s)|}$ whereas remaining bulk of the probability, $1 - \epsilon + \frac{\epsilon}{|\mathcal{A}(s)|}$, is given to the greedy action.

Monte Carlo Control

§ ϵ-greedy policy is an example of a bigger class of policies known as ϵ-soft policies where $\pi(a|s) \geq \frac{\epsilon}{|\mathcal{A}(s)|}$ for all states and actions, for some $\epsilon > 0$.

§ Among ϵ-soft policies, ϵ-greedy policy is, in some sense, closest to greedy.

§ By using ϵ-greedy policy improvement strategy, we achieve the best policy among ϵ-soft policies, but we eliminate the assumption of 'exploring starts'.

Off-policy Methods

§ All methods trying to learn control face a dilemma.

- They seek to learn action values conditional on subsequent optimal behavior.
- But they need to behave non-optimally in order to explore all actions (to find the optimal actions).

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§ Off-policy methods address this by using two policies for two different purposes.

- one that is learned about and that becomes the optimal policy - target policy.
- one that is more exploratory and is used to generate behavior - behavior policy.

Off-policy Prediction

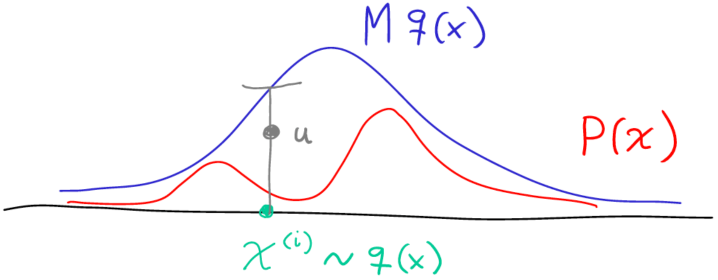
- Estimate v_π or q_π of the target policy π, but we have episodes from another policy μ, the behavior policy.
- Almost all off-policy methods utilize concepts from sampling theory for such operations.

Rejection Sampling

set $i = 1$

Repeat until $i = N$

1. Sample $x^{(i)} \sim q(x)$ and $u \sim \mathcal{U}_{(0,1)}$
2. If $u < \frac{p(x^{(i)})}{Mq(x^{(i)})}$, then accept $x^{(i)}$ and increment counter i by 1. Otherwise, reject.



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- What is bad about rejection sampling?
- Many wasted samples! why?
- Importance sampling is a classical way to address this. You keep all the samples from the proposal/behavior distribution, you just weigh them.
- Lets say we want to compute $\mathbb{E}_{x\sim p(.)}[f(x)] = \int f(x)p(x)dx$

$$\begin{aligned}\mathbb{E}_{x\sim p(.)}[f(x)] &= \int f(x)p(x)dx = \int f(x)\frac{p(x)}{q(x)}q(x)dx \\ &= \mathbb{E}_{x\sim q(.)}\left[f(x)\frac{p(x)}{q(x)}\right] \\ &\approx \frac{1}{N}\sum_{x^{(i)}\sim q(.),i=1}^{N} f(x^{(i)})\frac{p(x^{(i)})}{q(x^{(i)})}\end{aligned}$$

- $\frac{p(x^{(i)})}{q(x^{(i)})}$ is called the *importance weight*.

Normalized Importance Sampling

To avoid numerical instability, the denominator is changed in the following way

$$\mathbb{E}_{x\sim p(.)}[f(x)] \approx \frac{\sum\limits_{x^{(i)}\sim q(.)} f(x^{(i)})\frac{p(x^{(i)})}{q(x^{(i)})}}{\sum\limits_{x^{(i)}\sim q(.)} \frac{p(x^{(i)})}{q(x^{(i)})}}$$

MC Control with Importance Sampling

§ What are the samples $x^{(i)}$? What are the $p(.)$ and $q(.)$ in our case? and what is $f(x^{(i)})$?

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§ $f(x^{(i)})$ is the return.

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§ Refresher from the very first lecture.

- **Goal in RL Problem**:- to maximize the total reward "in expectation" over the long run.
- $\tau \stackrel{\text{def}}{=} (s_1, a_1, s_2, a_2, \dots), p(\tau) = p(s_1) \prod_t p(a_t|s_t) p(s_{t+1}|s_t, a_t)$
- $\max \mathbb{E}_{\tau \sim p(\tau)} [\sum_t R(s_t, a_t)]$

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§ $\frac{p(x^{(i)})}{q(x^{(i)})} = \frac{\cancel{p(s_1)}\pi(a_1|s_1)\cancel{p(s_2|s_1,a_1)}\pi(a_2|s_2)\cancel{p(s_3|s_2,a_2)}\cdots}{\cancel{p(s_1)}\mu(a_1|s_1)\cancel{p(s_2|s_1,a_1)}\mu(a_2|s_2)\cancel{p(s_3|s_2,a_2)}\cdots} = \frac{\pi(a_1|s_1)\pi(a_2|s_2)\cdots}{\mu(a_1|s_1)\mu(a_2|s_2)\cdots} = \prod_{t=1}^{T_i} \frac{\pi(a_t|s_t)}{\mu(a_t|s_t)}$

MC Control with Importance Sampling

$$\mathbb{E}_{x\sim\pi}[f(x)] \approx \frac{\sum\limits_{x^{(i)}\sim\mu} f(x^{(i)})\frac{p(x^{(i)})}{q(x^{(i)})}}{\sum\limits_{x^{(i)}\sim\mu} \frac{p(x^{(i)})}{q(x^{(i)})}}$$

$$\begin{aligned} v_\pi(s) &= \mathbb{E}\left[G|S_1 = s\right] \\ &\approx \frac{\sum\limits_{i=1}^{N} G^{(i)} \prod\limits_{t=1}^{T_i} \frac{\pi(a_t^{(i)}|s_t^{(i)})}{\mu(a_t^{(i)}|s_t^{(i)})}}{\sum\limits_{i=1}^{N} \prod\limits_{t=1}^{T_i} \frac{\pi(a_t^{(i)}|s_t^{(i)})}{\mu(a_t^{(i)}|s_t^{(i)})}} \end{aligned}$$

§ This was the evaluation step then do the greedy policy improvement.