Algebraic Curves

An Elementary Introduction

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Part I

Affine and Projective Curves

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Affine and Projective Curves

- Rational Points on Curves
- Polynomial and Rational Functions on Curves
- Divisors and Jacobians on Curves

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- Elliptic curves are hyperelliptic curves of genus 1.

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- Define a relation $\sim$ on $K^3 \setminus \{(0,0,0)\}$ as $(h,k,l) \sim (h',k',l')$ if $h' = \lambda h$, $k' = \lambda k$ and $l' = \lambda l$ for some non-zero $\lambda \in K$.

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- Through any two distinct points in $\mathbb{P}^2(K)$ passes a unique line.

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- The rational points of $C^{(h)}$ are all $[h, k, l]$ with $f^{(h)}(h, k, l) = 0$.

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- $f^{(h)}(X,Y,Z) = Z^d f(X/Z, Y/Z)$ is the **homogenization** of f.
- $C^{(h)} : f^{(h)}(X,Y,Z) = 0$ is the **projective curve** corresponding to C.
- For any non-zero $\lambda \in K$, we have $f^{(h)}(\lambda h, \lambda k, \lambda l) = \lambda^d f^{(h)}(h,k,l)$. So $f^{(h)}(\lambda h, \lambda k, \lambda l) = 0$ if and only if $f^{(h)}(h,k,l) = 0$.
- The rational points of $C^{(h)}$ are all $[h,k,l]$ with $f^{(h)}(h,k,l) = 0$.
- **Finite points on $C^{(h)}$:** Put $Z = 1$ to get $f^{(h)}(X,Y,1) = f(X,Y)$. These are the points on C.

Passage from Affine to Projective Curves

- A (multivariate) polynomial is called **homogeneous** if every non-zero term in the polynomial has the same degree.
- Example: $X^3 + 2XYZ - 3Z^3$ is homogeneous of degree 3. $X^3 + 2XY - 3Z$ is not homogeneous. The zero polynomial is homogeneous of any degree.
- Let $C : f(X, Y) = 0$ be an affine curve of degree d.
- $f^{(h)}(X, Y, Z) = Z^d f(X/Z, Y/Z)$ is the **homogenization** of f.
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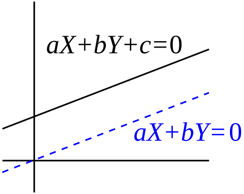
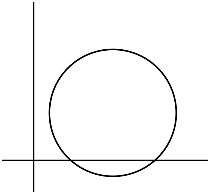
- **Finite points on $C^{(h)}$:** Put $Z = 1$ to get $f^{(h)}(X, Y, 1) = f(X, Y)$. These are the points on C.
- **Points at infinity on $C^{(h)}$:** Put $Z = 0$ and solve for $f^{(h)}(X, Y, 0) = 0$. These points do not belong to C.

Examples of Projective Curves

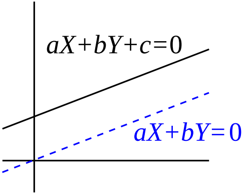


Straight Line

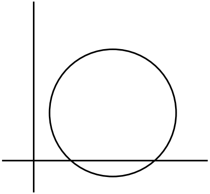


Circle

Examples of Projective Curves



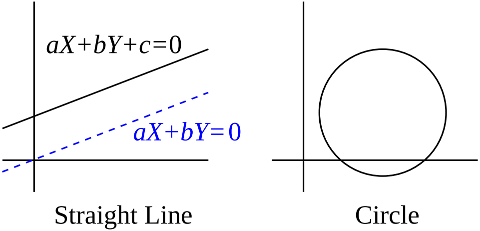
Straight Line



Circle

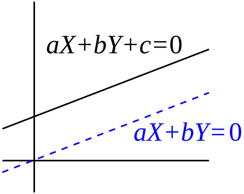
- **Straight line:** $aX + bY + cZ = 0.$

Examples of Projective Curves

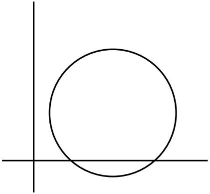


- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.

Examples of Projective Curves



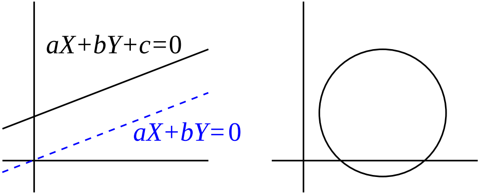
Straight Line



Circle

- **Straight line:** $aX + bY + cZ = 0.$
  - Finite points: Solutions of $aX + bY + c = 0.$
  - Points at infinity: Solve for $aX + bY = 0.$

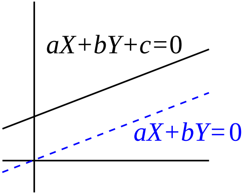
Examples of Projective Curves



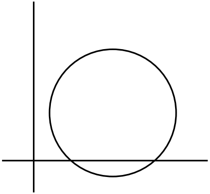
Straight Line　　　　Circle

- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.
  - Points at infinity: Solve for $aX + bY = 0$.
    If $b \neq 0$, we have $Y = -(a/b)X$. So $[1, -(a/b), 0]$ is the only point at infinity.

Examples of Projective Curves



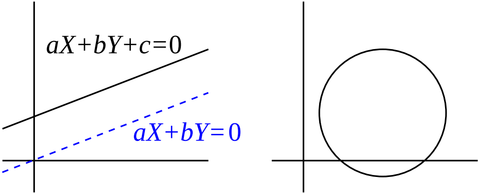
Straight Line



Circle

- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.
  - Points at infinity: Solve for $aX + bY = 0$.
    If $b \neq 0$, we have $Y = -(a/b)X$. So $[1, -(a/b), 0]$ is the only point at infinity.
    If $b = 0$, we have $aX = 0$, that is, $X = 0$. So $[0, 1, 0]$ is the only point at infinity.

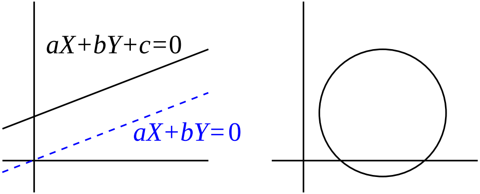
Examples of Projective Curves



Straight Line Circle

- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.
  - Points at infinity: Solve for $aX + bY = 0$.
    If $b \neq 0$, we have $Y = -(a/b)X$. So $[1, -(a/b), 0]$ is the only point at infinity.
    If $b = 0$, we have $aX = 0$, that is, $X = 0$. So $[0, 1, 0]$ is the only point at infinity.
- **Circle:** $(X - aZ)^2 + (Y - bZ)^2 = r^2Z^2$.

Examples of Projective Curves



Straight Line　　　　Circle

- **Straight line:** $aX + bY + cZ = 0$.
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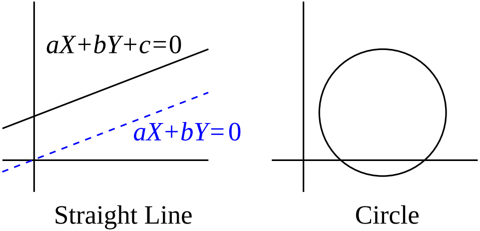
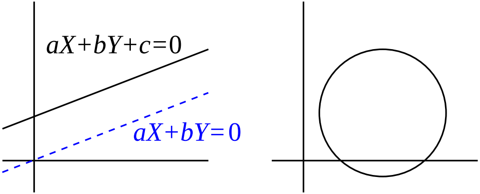
  - Finite points: Solutions of $(X - a)^2 + (Y - b)^2 = r^2$.

Examples of Projective Curves



- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.
  - Points at infinity: Solve for $aX + bY = 0$.
    If $b \neq 0$, we have $Y = -(a/b)X$. So $[1, -(a/b), 0]$ is the only point at infinity.
    If $b = 0$, we have $aX = 0$, that is, $X = 0$. So $[0, 1, 0]$ is the only point at infinity.
- **Circle:** $(X - aZ)^2 + (Y - bZ)^2 = r^2Z^2$.
  - Finite points: Solutions of $(X - a)^2 + (Y - b)^2 = r^2$.
  - Points at infinity: Solve for $X^2 + Y^2 = 0$.

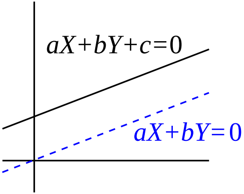
Examples of Projective Curves



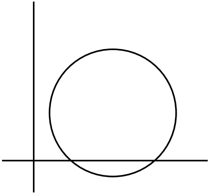
Straight Line | Circle

- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.
  - Points at infinity: Solve for $aX + bY = 0$.
    If $b \neq 0$, we have $Y = -(a/b)X$. So $[1, -(a/b), 0]$ is the only point at infinity.
    If $b = 0$, we have $aX = 0$, that is, $X = 0$. So $[0, 1, 0]$ is the only point at infinity.
- **Circle:** $(X - aZ)^2 + (Y - bZ)^2 = r^2Z^2$.
  - Finite points: Solutions of $(X - a)^2 + (Y - b)^2 = r^2$.
  - Points at infinity: Solve for $X^2 + Y^2 = 0$.
    For $K = \mathbb{R}$, the only solution is $X = Y = 0$, so there is no point at infinity.

Examples of Projective Curves



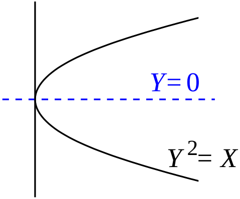
Straight Line



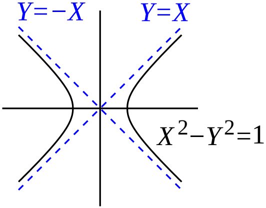
Circle

- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.
  - Points at infinity: Solve for $aX + bY = 0$.
    If $b \neq 0$, we have $Y = -(a/b)X$. So $[1, -(a/b), 0]$ is the only point at infinity.
    If $b = 0$, we have $aX = 0$, that is, $X = 0$. So $[0, 1, 0]$ is the only point at infinity.
- **Circle:** $(X - aZ)^2 + (Y - bZ)^2 = r^2Z^2$.
  - Finite points: Solutions of $(X - a)^2 + (Y - b)^2 = r^2$.
  - Points at infinity: Solve for $X^2 + Y^2 = 0$.
    For $K = \mathbb{R}$, the only solution is $X = Y = 0$, so there is no point at infinity.
    For $K = \mathbb{C}$, the solutions are $Y = \pm\mathrm{i}X$, so there are two points at infinity: $[1, \mathrm{i}, 0]$ and $[1, -\mathrm{i}, 0]$.

Examples of Projective Curves (contd.)

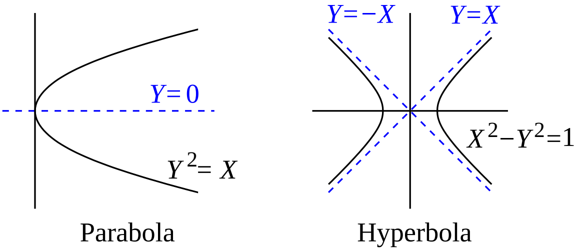


Parabola



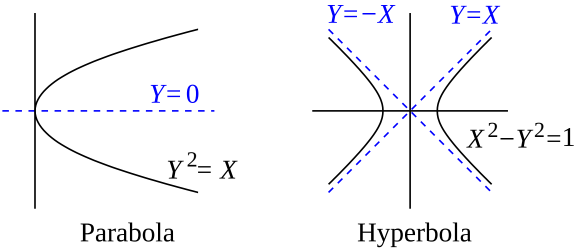
Hyperbola

Examples of Projective Curves (contd.)



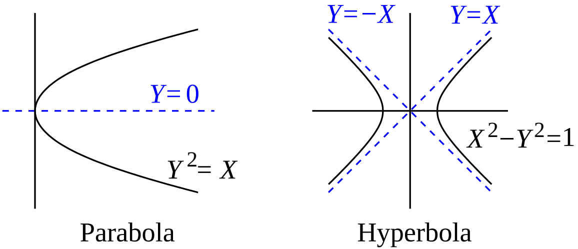
- **Parabola:** $Y^2 = XZ$.

Examples of Projective Curves (contd.)



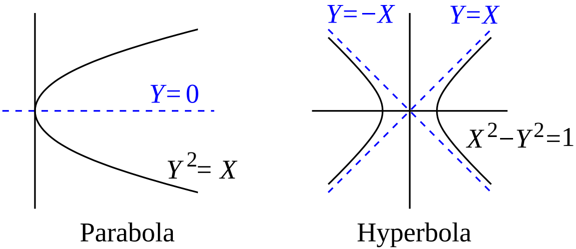
- **Parabola:** $Y^2 = XZ$.
- Finite points: Solutions of $Y^2 = X$.

Examples of Projective Curves (contd.)



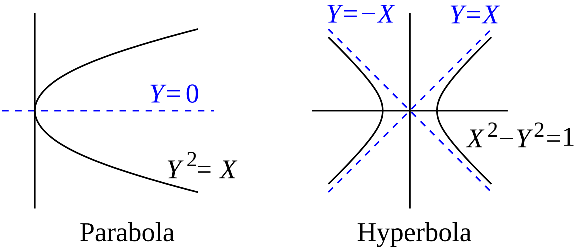
- **Parabola:** $Y^2 = XZ$.
  - Finite points: Solutions of $Y^2 = X$.
  - Points at infinity: Solve for $Y^2 = 0$.

Examples of Projective Curves (contd.)



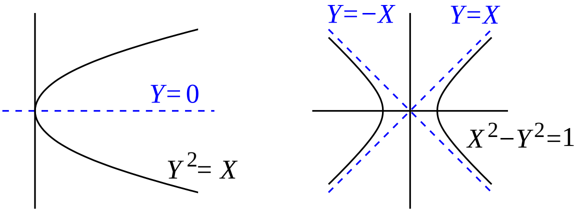
- **Parabola:** $Y^2 = XZ$.
  - Finite points: Solutions of $Y^2 = X$.
  - Points at infinity: Solve for $Y^2 = 0$.
    $Y = 0$, so $[1, 0, 0]$ is the only point at infinity.

Examples of Projective Curves (contd.)



- **Parabola:** $Y^2 = XZ$.
  - Finite points: Solutions of $Y^2 = X$.
  - Points at infinity: Solve for $Y^2 = 0$.
    $Y = 0$, so $[1, 0, 0]$ is the only point at infinity.
- **Hyperbola:** $X^2 - Y^2 = Z^2$.

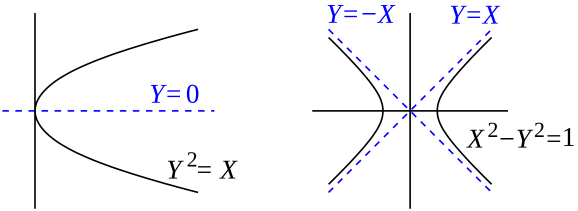
Examples of Projective Curves (contd.)



Parabola Hyperbola

- **Parabola:** $Y^2 = XZ$.
  - Finite points: Solutions of $Y^2 = X$.
  - Points at infinity: Solve for $Y^2 = 0$.
    $Y = 0$, so $[1, 0, 0]$ is the only point at infinity.
- **Hyperbola:** $X^2 - Y^2 = Z^2$.
  - Finite points: Solutions of $X^2 - Y^2 = 1$.

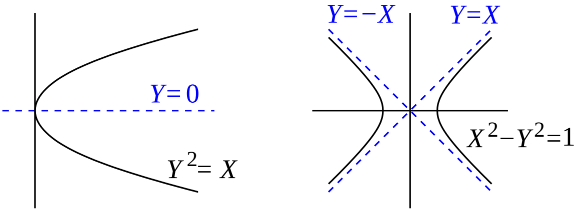
Examples of Projective Curves (contd.)



Parabola   Hyperbola

- **Parabola:** $Y^2 = XZ$.
  - Finite points: Solutions of $Y^2 = X$.
  - Points at infinity: Solve for $Y^2 = 0$.
    $Y = 0$, so $[1, 0, 0]$ is the only point at infinity.
- **Hyperbola:** $X^2 - Y^2 = Z^2$.
  - Finite points: Solutions of $X^2 - Y^2 = 1$.
  - Points at infinity: Solve for $X^2 - Y^2 = 0$.

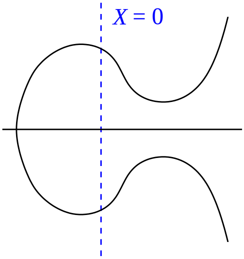
Examples of Projective Curves (contd.)



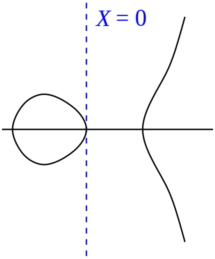
Parabola   Hyperbola

- **Parabola:** $Y^2 = XZ$.
  - Finite points: Solutions of $Y^2 = X$.
  - Points at infinity: Solve for $Y^2 = 0$.
    $Y = 0$, so $[1, 0, 0]$ is the only point at infinity.
- **Hyperbola:** $X^2 - Y^2 = Z^2$.
  - Finite points: Solutions of $X^2 - Y^2 = 1$.
  - Points at infinity: Solve for $X^2 - Y^2 = 0$.
    $Y = \pm X$, so there are two points at infinity: $[1, 1, 0]$ and $[1, -1, 0]$.

Examples of Projective Curves (contd.)

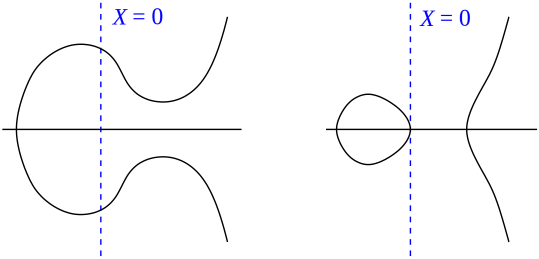


$Y^2 = X^3 - X + 1$



$Y^2 = X^3 - X$

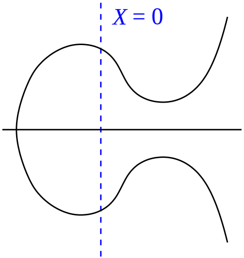
Examples of Projective Curves (contd.)



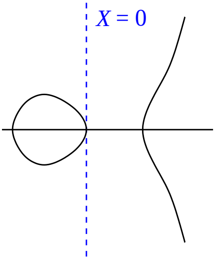
$Y^2 = X^3 - X + 1$ $\quad$ $Y^2 = X^3 - X$

- **Elliptic curve:** $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$.

Examples of Projective Curves (contd.)



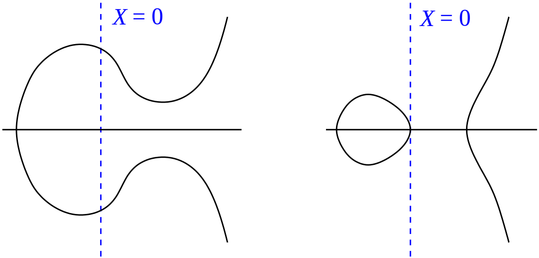
$Y^2 = X^3 - X + 1$



$Y^2 = X^3 - X$

- **Elliptic curve:** $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$.
  - Finite points: Solutions of $Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$.

Examples of Projective Curves (contd.)



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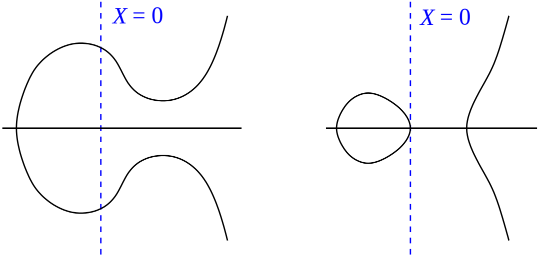
  - Finite points: Solutions of $Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$.
  - Points at infinity: Solve for $X^3 = 0$.

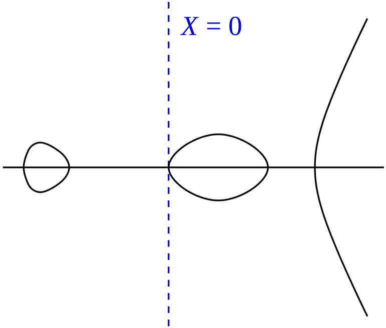
Examples of Projective Curves (contd.)



$Y^2 = X^3 - X + 1$ $\qquad$ $Y^2 = X^3 - X$

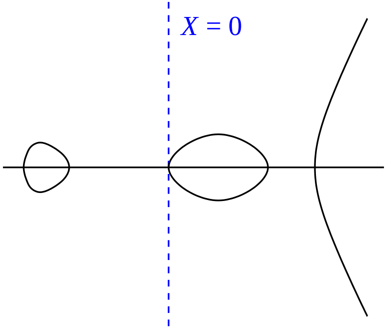
- **Elliptic curve:** $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$.
  - Finite points: Solutions of $Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$.
  - Points at infinity: Solve for $X^3 = 0$.
    $X = 0$, that is, $[0, 1, 0]$ is the only point at infinity.

Examples of Projective Curves (contd.)



A hyperelliptic curve of genus 2: $Y^2 = X(X^2 - 1)(X^2 - 2)$

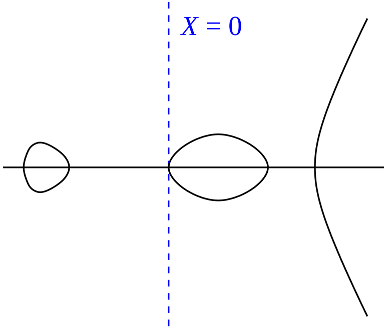
Examples of Projective Curves (contd.)



A hyperelliptic curve of genus 2: $Y^2 = X(X^2-1)(X^2-2)$

- **Hyperelliptic curve:** $Y^2Z^{2g-1} + Z^g u(X/Z)YZ^g = Z^{2g+1}v(X/Z)$.

Examples of Projective Curves (contd.)



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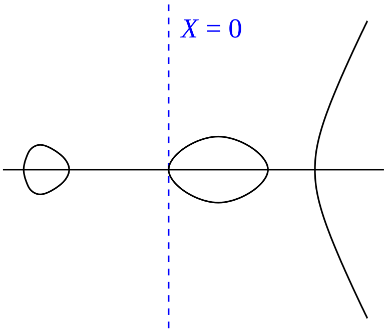
- **Hyperelliptic curve:** $Y^2Z^{2g-1} + Z^g u(X/Z)YZ^g = Z^{2g+1}v(X/Z)$.
  - Finite points: Solutions of $Y^2 + u(X)Y = v(X)$.

Examples of Projective Curves (contd.)



A hyperelliptic curve of genus 2: $Y^2 = X(X^2 - 1)(X^2 - 2)$

- **Hyperelliptic curve:** $Y^2Z^{2g-1} + Z^g u(X/Z)YZ^g = Z^{2g+1}v(X/Z)$.
  - Finite points: Solutions of $Y^2 + u(X)Y = v(X)$.
  - Points at infinity: The only Z-free term is X^{2g+1} (in $Z^{2g+1}v(X/Z)$). So $[0, 1, 0]$ is the only point at infinity.

Bézout’s Theorem

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- A curve of degree m and a curve of degree n intersect at exactly mn points.

Bézout's Theorem

- A curve of degree m and a curve of degree n intersect at exactly mn points.
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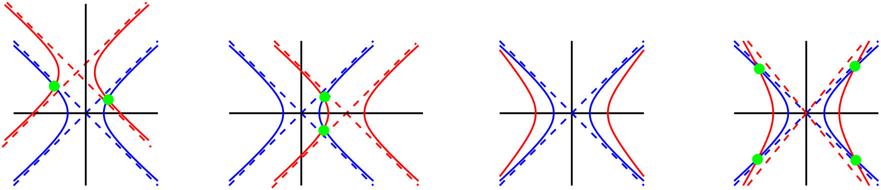

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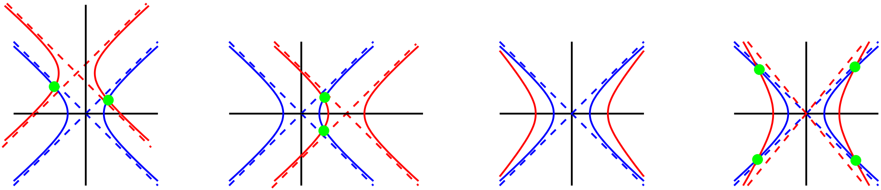

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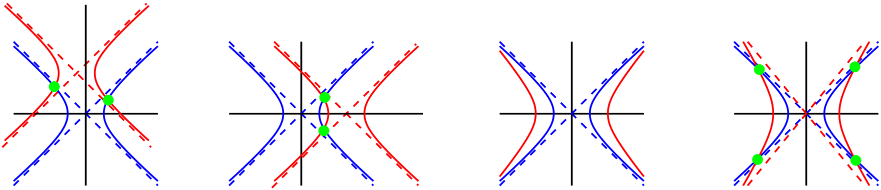

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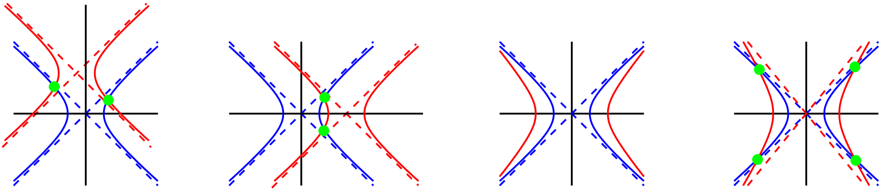

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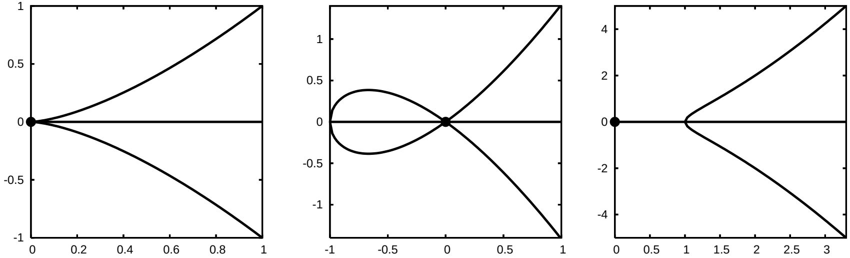
- C is a **smooth curve** if it is smooth at every rational point on it.

Types of Singularity



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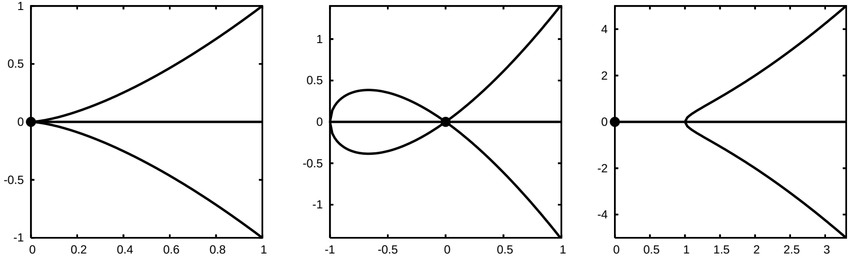

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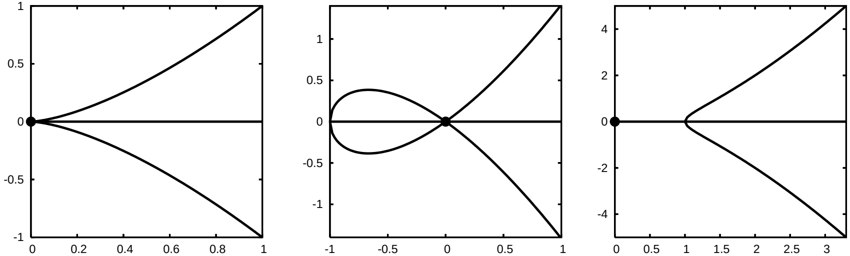

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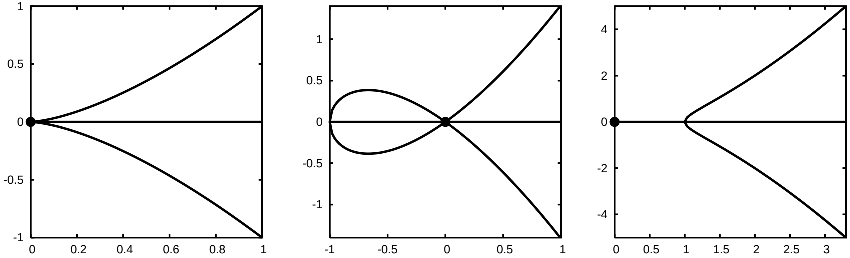

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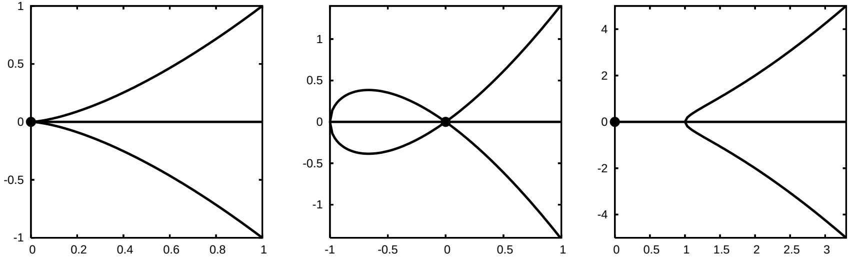

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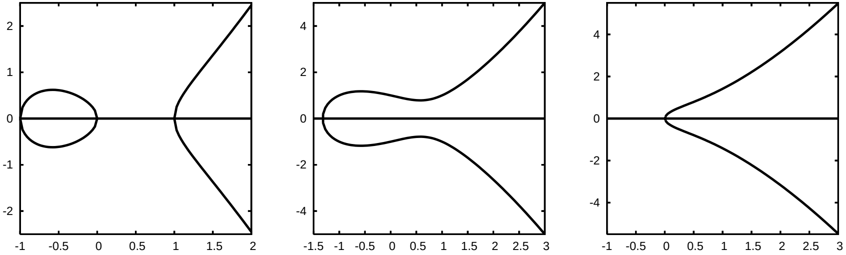
- For a *real* curve $f(X, Y) = 0$, the type of singularity is determined by the matrix $\text{Hessian}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$.

Examples of Smooth Curves



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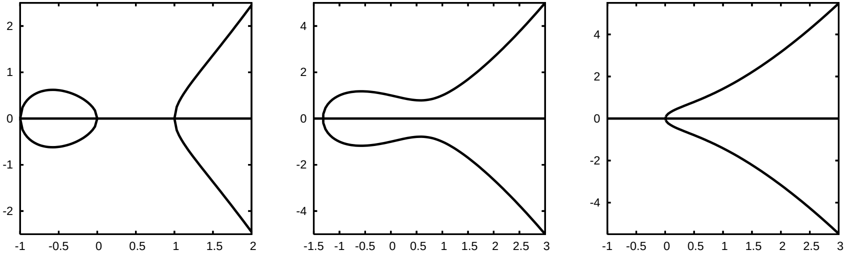

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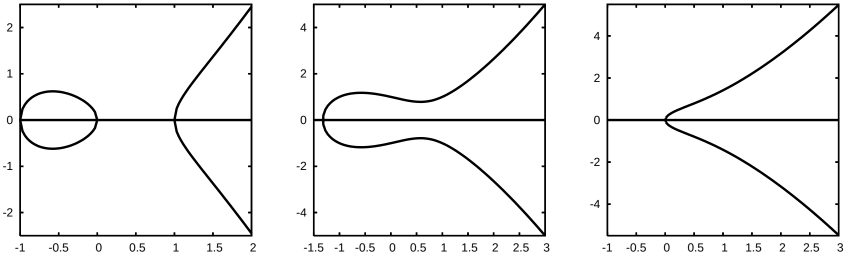

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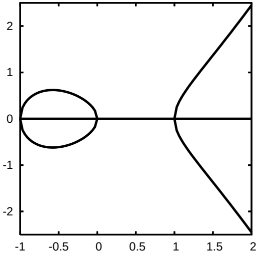

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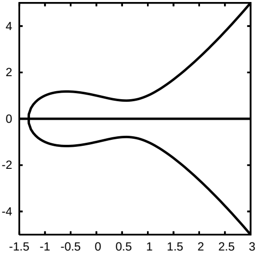

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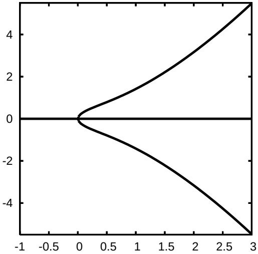

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- The point at infinity on an elliptic or hyperelliptic curve is never a point of singularity.

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- $K(C)$ is the quadratic extension of $K(X)$ obtained by adjoining a root of the irreducible polynomial $Y^2 + u(X)Y - v(X) \in K(X)[Y]$. The current notion of conjugacy coincides with the standard notion for field extensions.

Polynomial and Rational Functions on Elliptic and Hyperelliptic Curves

Consider the elliptic curve $Y^2 + u(X)Y = v(X)$, where $u(X) = a_1X + a_3$ and $v(X) = X^3 + a_2X^2 + a_4X + a_6$.

- $y^2 = -u(x)y + v(x)$.
- Every polynomial function on C can be represented uniquely as $a(x) + yb(x)$ with $a(x), b(x) \in K[x]$.
- For $G(x, y) = a(x) + yb(x) \in K[C]$, define:
  - **Conjugate of G:** $\hat{G}(x, y) = a(x) - b(x)(u(x) + y)$.
  - **Norm of G:** $\mathrm{N}(G) = G\hat{G}$.
- $\mathrm{N}(G) = a(x)^2 - a(x)b(x)u(x) - v(x)b(x)^2 \in K[x]$.
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- These results hold equally well for hyperelliptic curves too.

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- The notion of value of a rational function can be extended to the points at infinity on C.

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- For hyperelliptic curves, analogous results hold. Now, X and Y are given weights 2 and $2g+1$ respectively.

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- For a *projective* curve over an *algebraically closed* field, the sum of the orders of the poles and zeros of a (non-zero) rational function is 0.

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- For hyperelliptic curves, identical results hold. A uniformizer at $\mathcal{O}$ is x^g/y.

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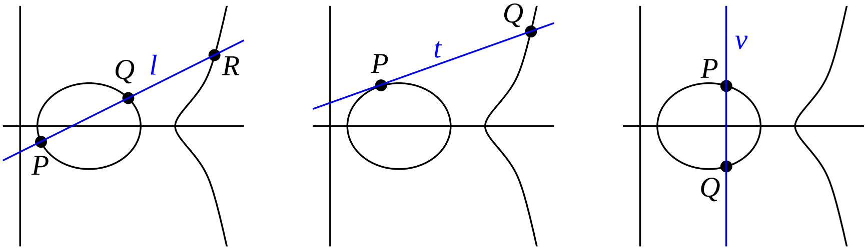

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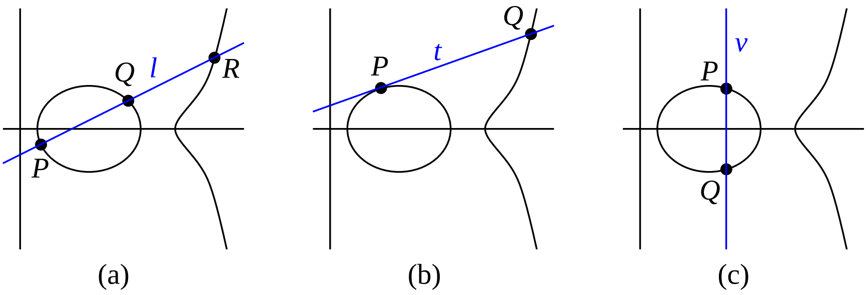

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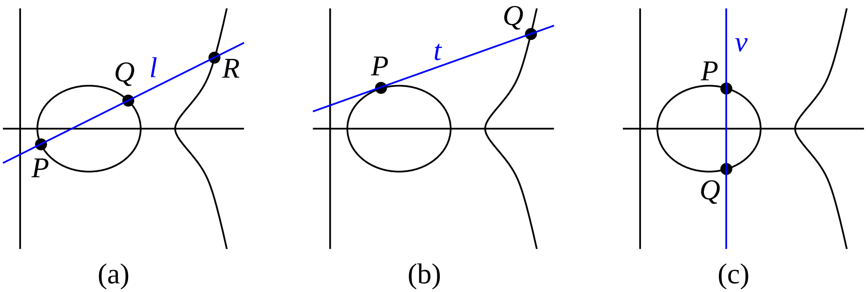

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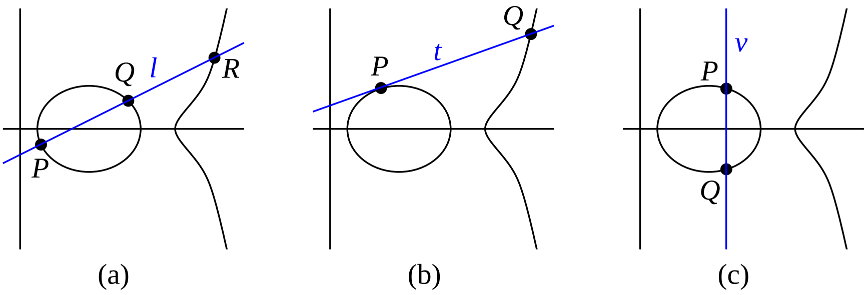

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- The set of all finite formal sums is an Abelian group called the **free Abelian group** generated by a_i, $i \in I$.

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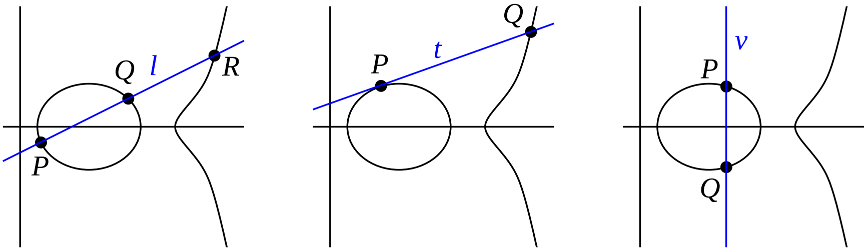
Divisor of a line: Example



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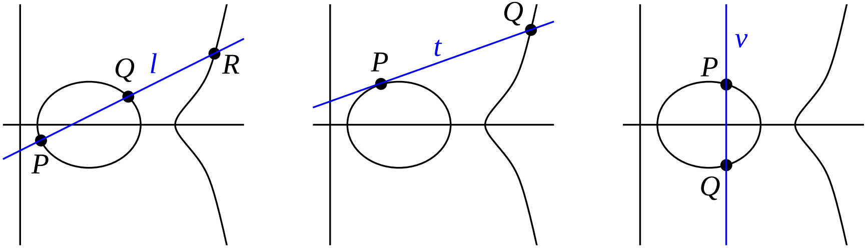

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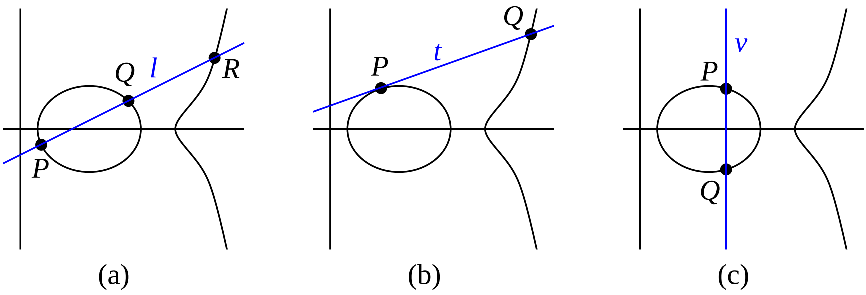

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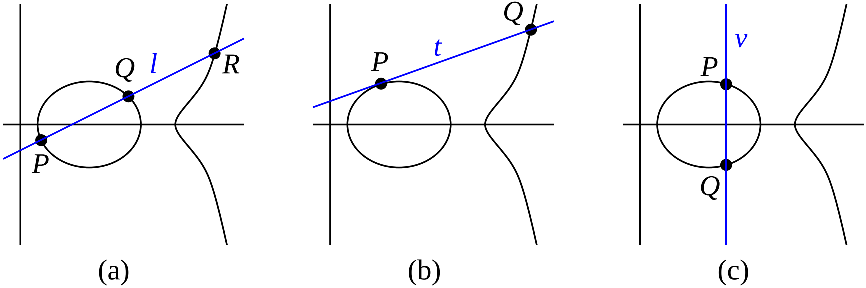

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- For elliptic curves, the Jacobian can be expressed by a more explicit **chord-and-tangent** rule.

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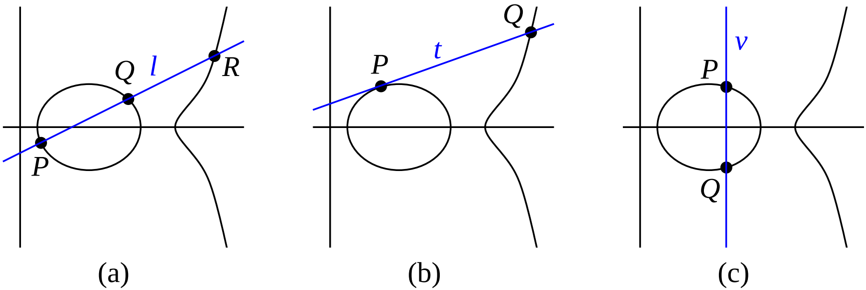
- $\sum_p m_P P = \mathcal{O}$ (sum under the chord-and-tangent rule).

Illustrations of the Chord-and-Tangent Rule



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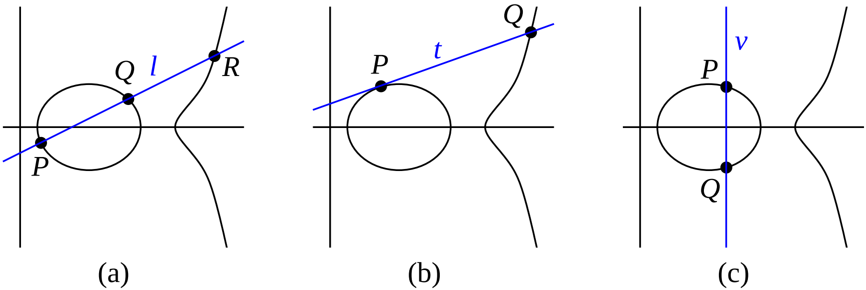

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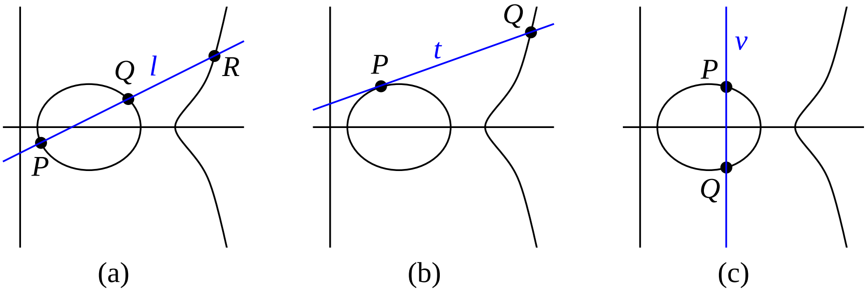

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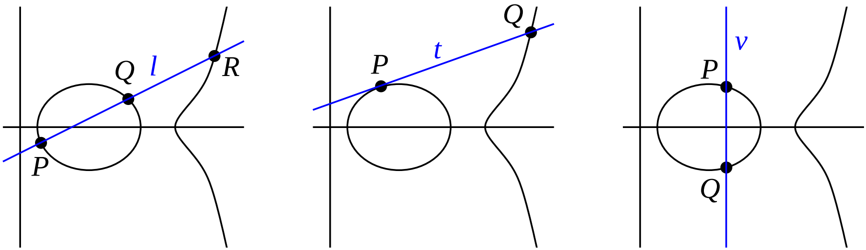

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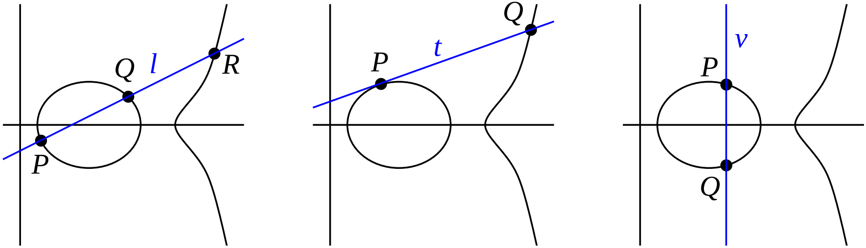

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References for Part I

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Part II

Elliptic Curves

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- Rational Maps and Endomorphisms on Elliptic Curves
- Multiplication-by-m Maps and Division Polynomials
- Weil and Tate Pairing

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- Since K is not algebraically closed, $\mathbb{J}_K(E)$ cannot be defined like $\mathbb{J}_{\bar{K}}(E)$.
- Thanks to the chord-and-tangent rule, we do not need to worry too much about $\mathbb{J}_K(E)$ (at least so long as computational issues are of only concern).

Discriminants and j-invariants

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- Define the following quantities for E:

$$\begin{aligned}
d_2 &= a_1^2 + 4a_2 \\
d_4 &= 2a_4 + a_1a_3 \\
d_6 &= a_3^2 + 4a_6 \\
d_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\
c_4 &= d_2^2 - 24d_4 \\
\Delta(E) &= -d_2^2d_8 - 8d_4^3 - 27d_6^2 + 9d_2d_4d_6 \\
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- $\Delta(E)$ is called the **discriminant** of E.
- $j(E)$ is called the ***j*-invariant** of E.
- E is smooth (that is, an elliptic curve) if and only if $\Delta(E) \neq 0$.

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d_6 &= a_3^2 + 4a_6 \\
d_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\
c_4 &= d_2^2 - 24d_4 \\
\Delta(E) &= -d_2^2d_8 - 8d_4^3 - 27d_6^2 + 9d_2d_4d_6 \\
j(E) &= c_4^3/\Delta(E), \ \ \text{if } \Delta(E) \neq 0\,.
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- $\Delta(E)$ is called the **discriminant** of E.
- $j(E)$ is called the ***j*-invariant** of E.
- E is smooth (that is, an elliptic curve) if and only if $\Delta(E) \neq 0$.
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Discriminants and j-invariants

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- For two elliptic curves E, E', we have $j(E) = j(E')$ if and only if E and E' are isomorphic.

Addition Formula for the General Weierstrass Equation

Let $P = (h_1, k_1)$ and $Q = (h_2, k_2)$ be points on E. Assume that $P, Q, P + Q$ are not $\mathcal{O}$. Let $R = (h_3, k_3) = P + Q$.

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The opposite of (h, k) is $(h, -k - a_1h - a_3)$.

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Let $E : Y^2 + (a_1X + a_3)Y = X^3 + a_2X^2 + a_4X + a_6$ be defined over K.
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- Use a root-finding algorithm to compute a root k.
- Output (h, k).

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  - Define $\varphi(h, k) = (h^q, k^q)$.

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- If $\mathrm{End}(E)$ contains a map other than the maps $[m]$, E is called a curve with **complex multiplication**.

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- $g_1 = x$, $h_1 = y$.
- $g_2 = -2x + \lambda^2 + a_1\lambda - a_2$ and
  $h_2 = -\lambda(g_2 - x) - a_1 g_2 - a_3 - y$,
  where $\lambda = \dfrac{3x^2 + 2a_2x + a_4 - a_1y}{2y + a_1x + a_3}$.
- For $m \geqslant 3$, we have the recursive definition:

$$g_m = -g_{m-1} - x + \lambda^2 + a_1\lambda - a_2 \text{ and}$$
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- If $p \in \{2, 3\}$, $\gcd(m, p) = 1$, and $n = p^\nu n'$ with $\nu \geqslant 1$ and $\gcd(n', p) = 1$, we have $\operatorname{Div}(g_m - g_n) = [E[m+n]] + [E[m-n]] - 2[E[m]] - 2\alpha^\nu[E[n]]$.

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- If we arrange the leading coefficient of ψ_m to be m, then ψ_m becomes unique and is called the ***m*-th division polynomial**.

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- **Weil's Theorem:** $|E_{q^k}| = q^k + 1 - (\alpha^k + \beta^k)$.
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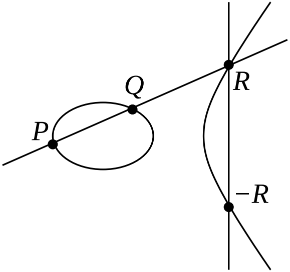
  Consider $(W - \alpha)(W - \beta) = W^2 - tW + q = W^2 + 3W + 5$, that is, $\alpha + \beta = -3$ and $\alpha\beta = 5$. But then $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 10 = -1$. Therefore, $|E_{25}| = 25 + 1 - (-1) = 27$.
- **Structure Theorem for E_q:**
  E_q is either cyclic or isomorphic to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ with $n_1, n_2 \geqslant 2$, $n_1 | n_2$, and $n_1 | (q - 1)$.

More on Divisors



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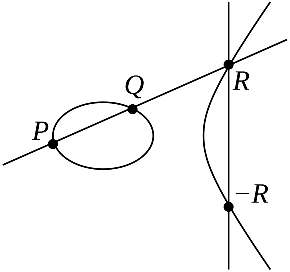

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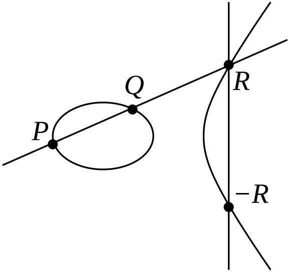

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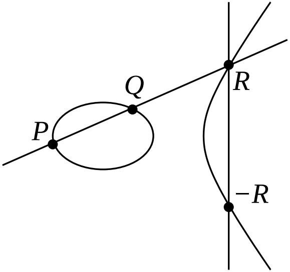

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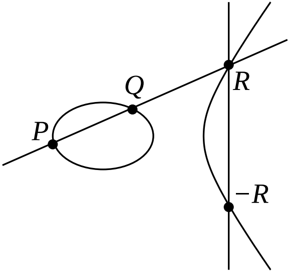

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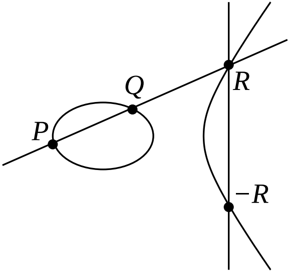

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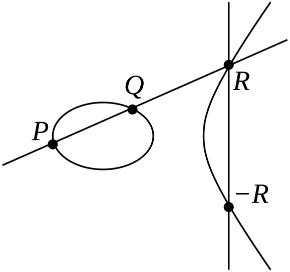

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- For both these cases of equivalence, the pertinent rational function is $L_{P,Q}/L_{P+Q,-(P+Q)}$ which can be easily computed. We can force this rational function to have leading coefficient 1.

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- **Weil reciprocity theorem:** If f and g are two non-zero rational functions on E such that $\mathrm{Div}(f)$ and $\mathrm{Div}(g)$ have disjoint supports, then

$$f(\mathrm{Div}(g)) = g(\mathrm{Div}(f)).$$

Weil Pairing: Definition

Let E be an elliptic curve defined over a finite field $K = \mathbb{F}_q$.
Take a positive integer m coprime to $p = \operatorname{char} K$.
Let μ_m denote the m-th roots of unity in $\bar{K}$.
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- $e_m(P_1, P_2)^m = f_1(mD_2)/f_2(mD_1) = f_1(\mathrm{Div}(f_2))/f_2(\mathrm{Div}(f_1)) = 1$ (by Weil reciprocity), that is, $e_m(P_1, P_2)$ is indeed an m-th root of unity.

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- The function $f_{n,P}$ is usually kept in the factored form.
- It is often not necessary to compute $f_{n,P}$ explicitly. The value of $f_{n,P}$ at some point Q is only needed.

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  - Return f.
- **Note:** One may supply a point $Q \in E$ and wish to compute the value $f_{n,P}(Q)$ (instead of the function $f_{n,P}$). In that case, the functions $L_{U,U}/L_{2U,-2U}$ and $L_{U,P}/L_{U+P,-(U+P)}$ should be evaluated at Q before multiplication with f.

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- For efficiency, one may avoid the division operations in Miller's loop by separately maintaining polynomial expressions for the numerator and the denominator of f. After the loop terminates, a single division is made.

Tate Pairing

Let E be an elliptic curve defined over $K = \mathbb{F}_q$ with $p = \operatorname{char} K$.
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Let $k = \operatorname{ord}_m(q)$ (the **embedding degree**), and $L = \mathbb{F}_{q^k}$.
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up to m-th powers.
- Let $k = \mathrm{ord}_m(q)$ be the embedding degree. The Tate pairing can be made unique by exponentiation to the power $(q^k - 1)/m$:

$$\hat{e}_m(P, Q) = (\langle P, Q\rangle_m)^{\frac{q^k-1}{m}}$$

$\hat{e}_m(P, Q)$ is called the **reduced Tate pairing**. The reduced pairing continues to exhibit bilinearity and non-degeneracy.

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- If the reduced pairing is desired, then a final exponentiation to the power $(q^k - 1)/m$ is made on the value returned by Miller's algorithm.

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- One takes $\mathbb{F}_q$ with $|q|$ about 160–300 bits and $k \leqslant 12$. Larger embedding degrees are impractical for implementation.

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- Distortion maps exist only for supersingular curves.

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- This is called the **twisted Tate pairing**.

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  - Curves of the form $Y^2 + aY = X^3 + bX + c$ are supersingular over fields of characteristic 2.
  - All supersingular curves over a finite field K of characteristic 2 have j-invariant equal to 0, and so are isomorphic over $\bar{K}$. The same result holds for $p = 3$.
  - Supersingular curves have small embedding degrees. The only possibilities are $1, 2, 3, 4, 6$.
  - If $\mathbb{F}_q$ is a prime field with $q \geqslant 5$, the only possibility is $k = 2$.
  - Non-supersingular curves are called **ordinary curves**.
  - It is difficult to locate ordinary curves with small embedding degrees.

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Part III

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- Representation of the Jacobian

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