Introduction to Cryptography

Dr. Abhijit Das

Indian Institute of Technology, Kharagpur

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Part I: Overview of cryptographic primitives

What is Cryptography?

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- Maintaining security and privacy is an ancient and primitive need.
- Particularly relevant for military and diplomatic applications.
- Wide deployment of the Internet makes everybody a user of cryptographic tools.

Message encryption

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- An eavesdropper (intruder, attacker, adversary, opponent, enemy) cannot decrypt C.

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- If there are many communicating pairs, the key storage requirement is high.

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- Algorithms are slow, in general.

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Asymmetric encryption

- Alice presses a self-locking padlock in order to lock the box. The locking process does not require a real key.
- Bob has the key to open the padlock.

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- Alice sends (C, L) to Bob.

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- Bob recovers K as $K = f_D(L, K_d)$.

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- Bob decrypts C as $M = f_d(C, K)$.

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- Alice puts K in a box, locks the box by L_A using K_A, and sends the box to Bob.
- Bob locks the box by L_B using K_B, and sends the doubly-locked box back to Alice.

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- A third party always finds the box locked either by L_A or L_B or both.

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- Alice computes $K_{AB} = f(A_e, A_d, B_e)$.

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- Bob computes $K_{BA} = f(B_e, B_d, A_e)$.
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- An intruder cannot compute this secret using A_e and B_e only.

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- Digital signatures are based on public-key techniques.
- Signature generation $\equiv$ Decryption (uses private key), and Signature verification $\equiv$ Encryption (uses public key).
- **Non-repudiation:** An entity should not be allowed to deny valid signatures made by him.

Signature with message recovery

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Drawback

- Algorithms are slow, not suitable for long messages.

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- Use Alice's public-key to generate $m' = f_v(s, K_e)$.

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Forging

- Verification is expected to fail if a key $K'_d \neq K_d$ is used to generate s.

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  - **One-time signatures:** Fast. Parameters are used only once.

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- Both symmetric and asymmetric techniques are used for entity authentication.

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- Bob computes $Q' = f(P')$.
- Bob compares Q' with the stored value Q.

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- Bob compares Q' with the stored value Q.
- $Q' = Q$ if and only if $P' = P$.

Weak authentication: Passwords

Set-up phase

- Alice supplies a secret password P to Bob.
- Bob transforms (typically encrypts) P to generate $Q = f(P)$.
- Bob stores Q for future use.

Authentication phase

- Alice supplies her password P' to Bob.
- Bob computes $Q' = f(P')$.
- Bob compares Q' with the stored value Q.
- $Q' = Q$ if and only if $P' = P$.
- If $Q' = Q$, Bob accepts Alice's identity.

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Drawbacks

- Alice reveals P itself to Bob. Bob may misuse this information.
- P resides in unencrypted form in the memory during the authentication phase. A third party having access to this memory obtains Alice's secret.

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- C and R may reveal to Bob or an eavesdropper some knowledge about Alice's secret.

Zero-knowledge protocol

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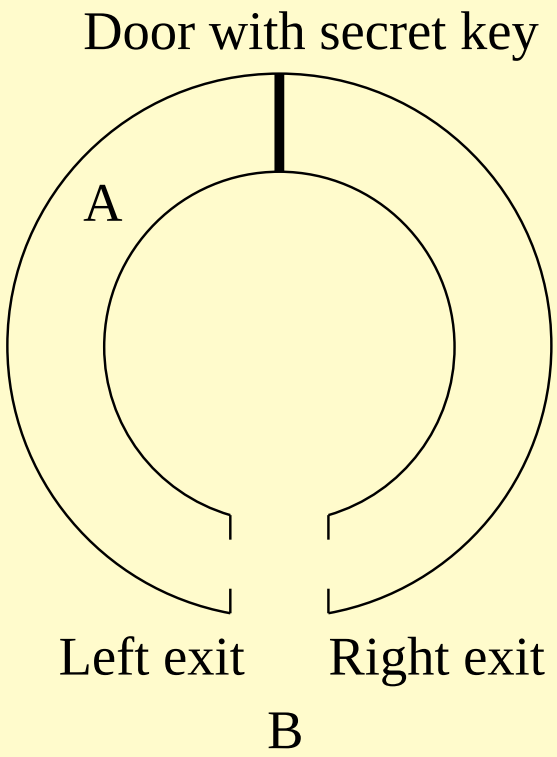
A real-life example



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- If a certificate is not in the CRL, and the signature of the CA on the certificate is verified, one gains the desired confidence of treating the public-key as authentic.

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- **Active attack**
  The attacker alters and/or deletes messages and even creates unauthorized messages.

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Part II: Symmetric cryptosystems

Block ciphers

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- A block cipher f of **block-size** n and **key-size** r is a function

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Block ciphers: Examples

Name	n, r
DES (Data Encryption Standard)	64, 56
FEAL (Fast Data Encipherment Algorithm)	64, 64
SAFER (Secure And Fast Encryption Routine)	64, 64
IDEA (International Data Encryption Algorithm)	64, 128
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- Number of **rounds** N_r for AES is 10/12/14 for key-sizes 128/192/256.
- AES **key schedule**: From K, generate round keys $K_0, K_1, \ldots, K_{4N_r+3}$.

A case study: AES (contd.)

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- **State:** AES represents a 128-bit message block as a 4×4 array of octets:

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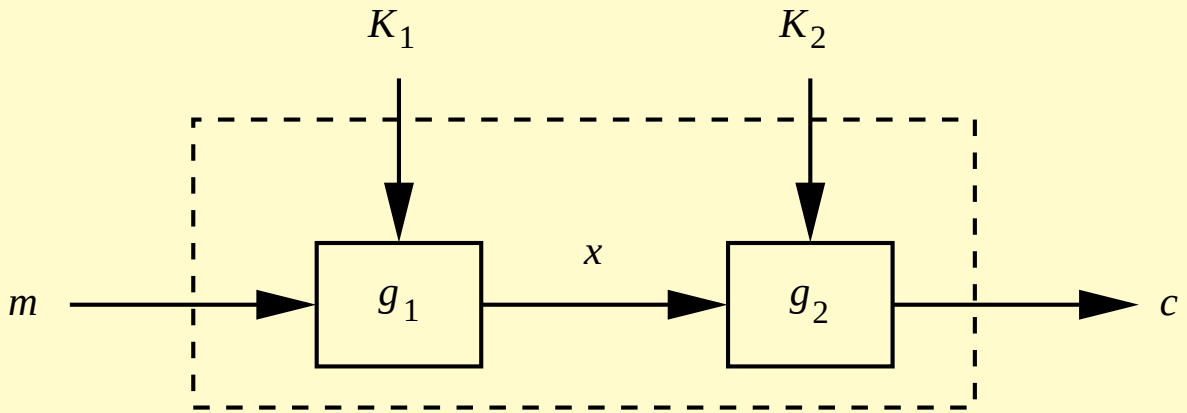
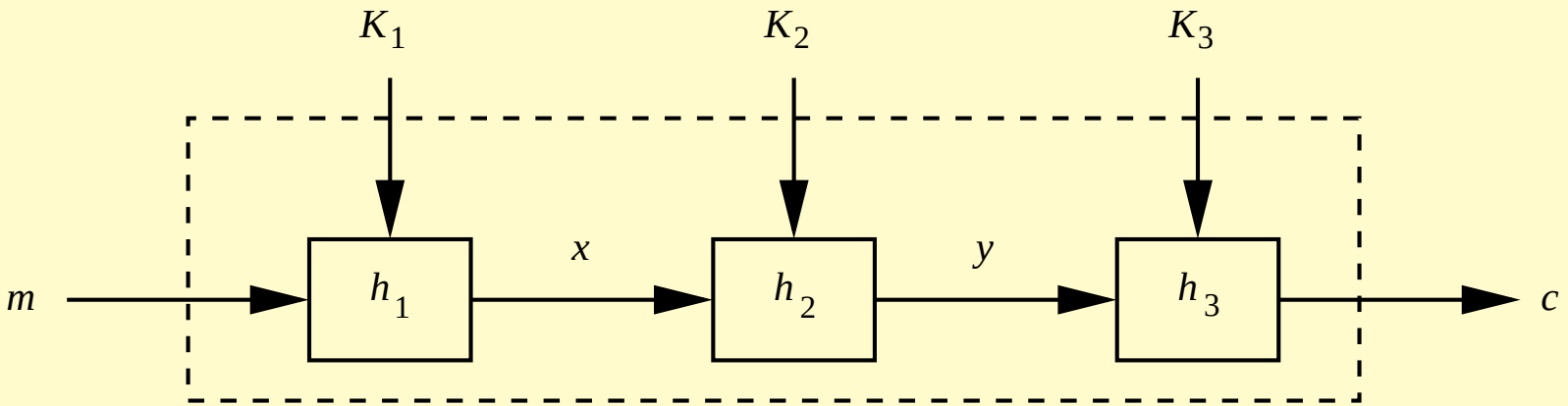
- Convert the state S to the plaintext block M.

Multiple encryption



(a) Double encryption



(b) Triple encryption

Modes of operation

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- **Linear and differential cryptanalysis:** By far the most sophisticated attacks on block ciphers. Impractical if sufficiently many rounds are used. AES is robust against these attacks.

Attacks on block ciphers (contd.)

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- **Meet-in-the-middle attack:** Applies to multiple encryption schemes. With m stages we get the equivalent security of $\lceil m/2 \rceil$ keys only.

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- **Vernam's one-time pad:** For a truly random key stream,

$$\Pr(c_i = 0) = \Pr(c_i = 1) = \frac{1}{2}$$

for each i, irrespective of the probabilities of the values assumed by m_i. This leads to **unconditional security**, that is, the knowledge of any number of plaintext-ciphertext bit pairs, does not help in decrypting a new ciphertext bit.

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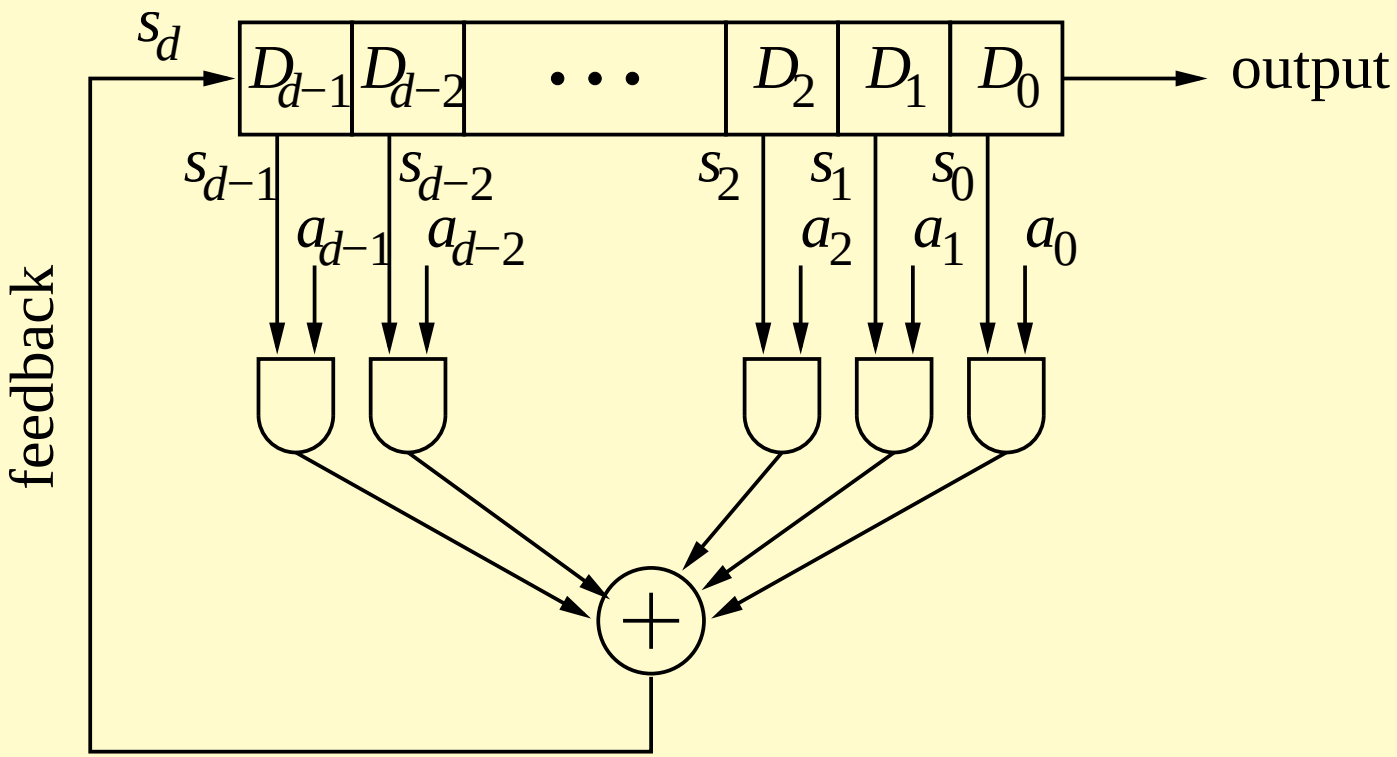
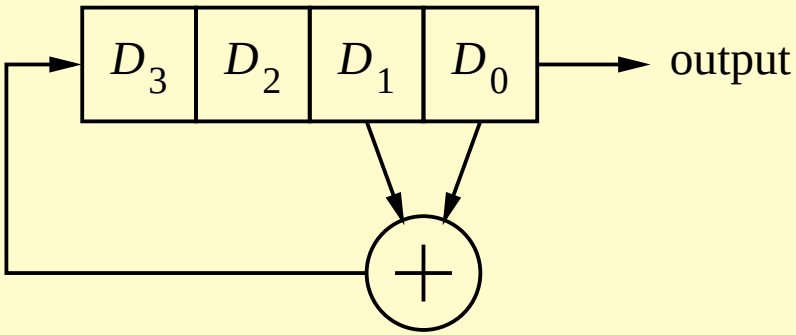
- Pseudorandom bit generators are vulnerable to compromise of seeds.
- Repeated use of the same key stream degrades security.

Linear Feedback Shift Registers (LFSR)

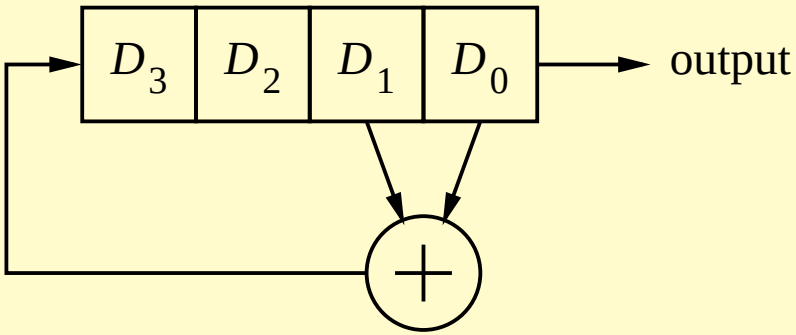


LFSR: Example



LFSR: Example

Time	D_3	D_2	D_1	D_0
0	1	1	0	1



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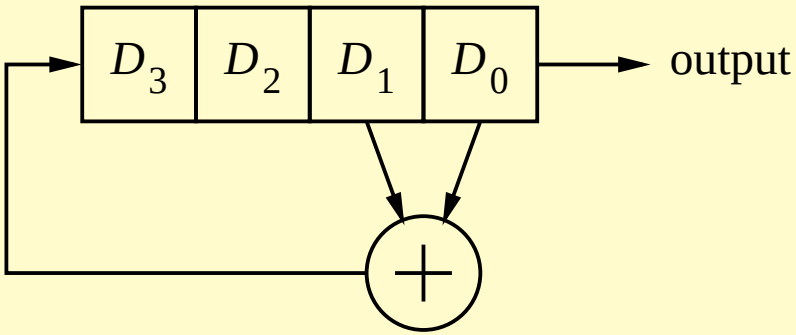

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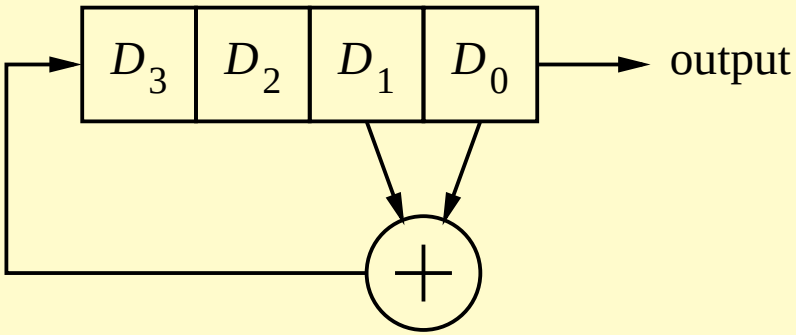

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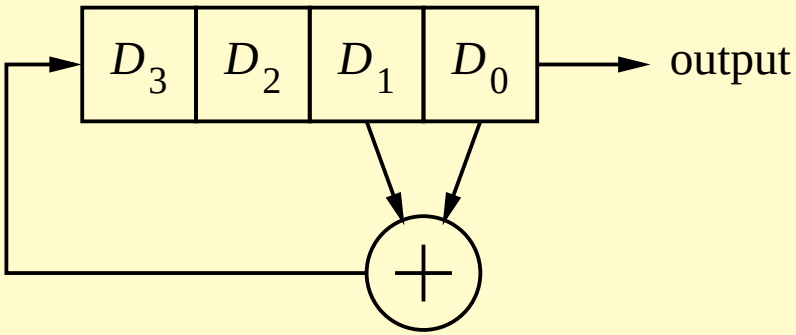

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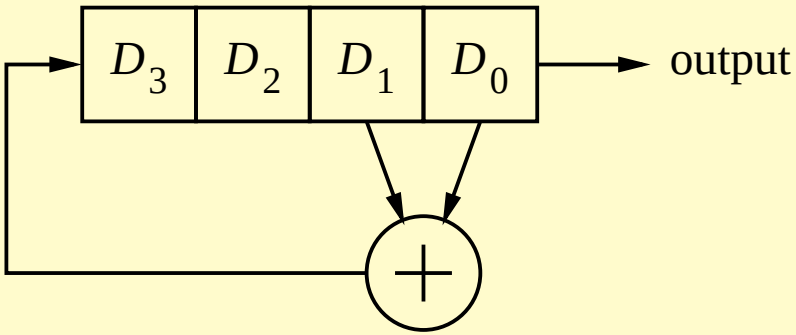

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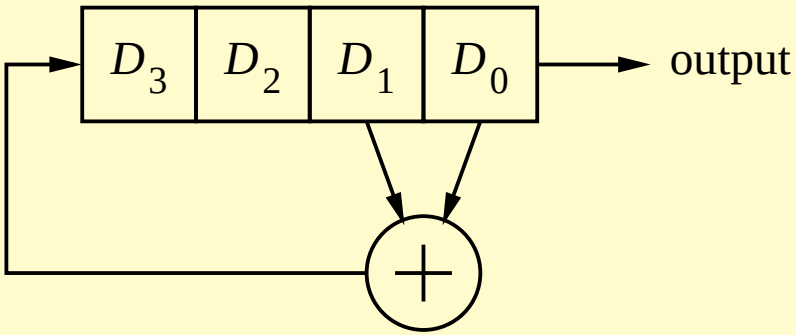

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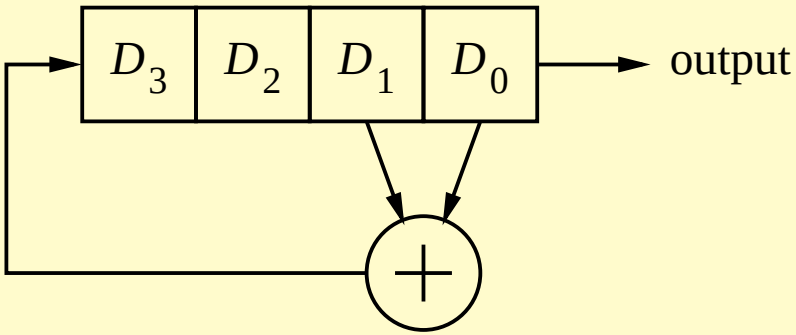

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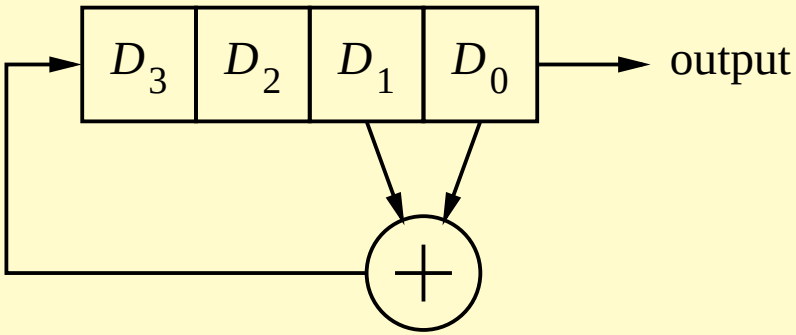

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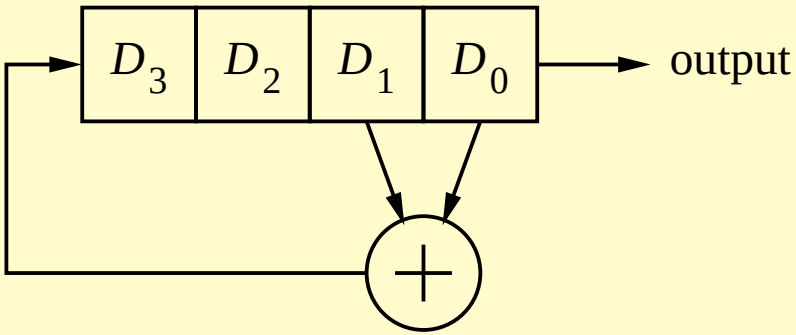

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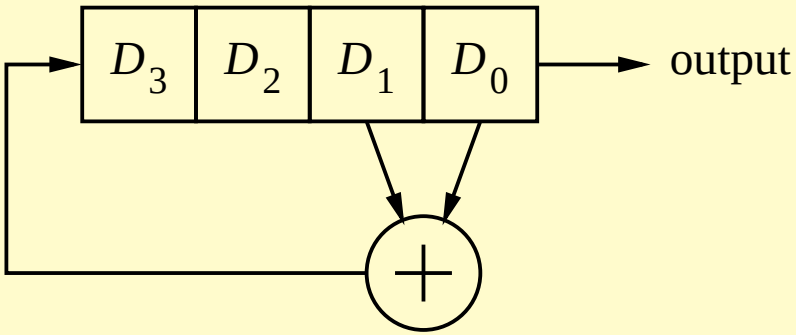

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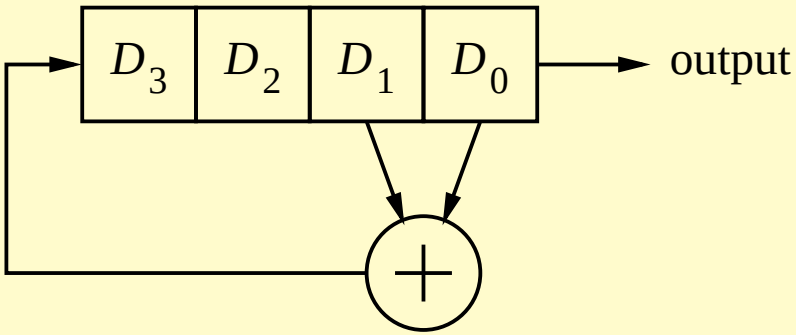

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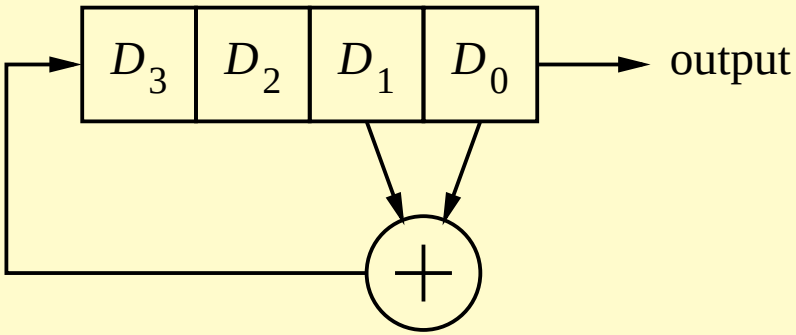

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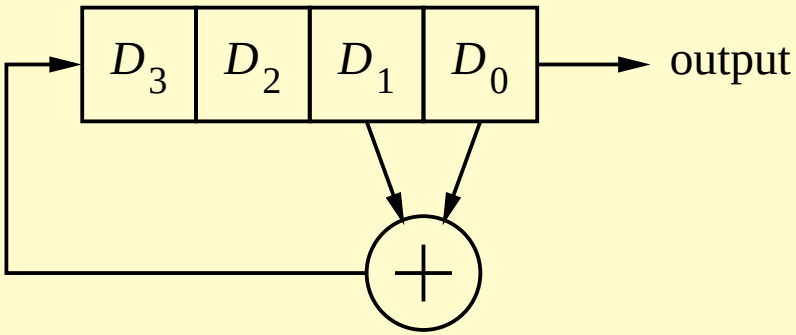

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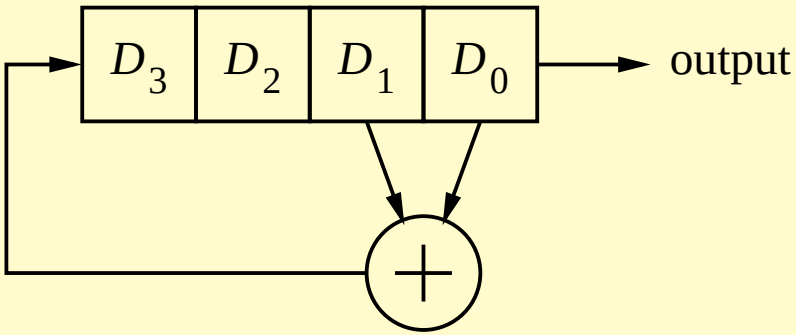

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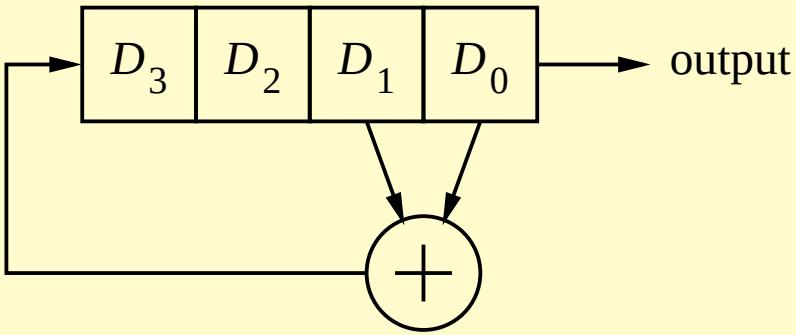

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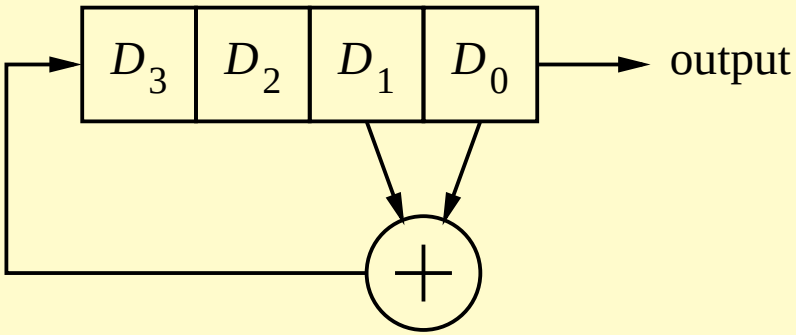

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- Each clock pulse changes the state as follows:

$$\begin{aligned}
t_0 &= s_1 \\
t_1 &= s_2 \\
&\vdots \\
t_{d-2} &= s_{d-1} \\
t_{d-1} &= a_0 s_0 + a_1 s_1 + a_2 s_2 + \cdots + a_{d-1} s_{d-1} \pmod 2.
\end{aligned}$$

LFSR: State transition (contd.)

- In the matrix notation $\mathbf{t} = \Delta_L \mathbf{s} \pmod 2$, where the **transition matrix** is

$$\Delta_L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{d-2} & a_{d-1} \end{pmatrix}.$$

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$$C_L(x) = 1 + a_{d-1}x + a_{d-2}x^2 + \cdots + a_1x^{d-1} + a_0x^d \in \mathbb{F}_2[X].$$

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- L is a maximum-length LFSR if and only if $C_L(x)$ is a primitive polynomial of $\mathbb{F}_2[x]$.

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- **Berlekamp-Massey attack**
  Suppose that the bits m_i and c_i for $2d$ consecutive values of i (say, $1, 2, \ldots, 2d$) are known to an attacker. Then $k_i = m_i \oplus c_i$ are also known for these values of i. Define the states $S_i = (k_i, k_{i+1}, \ldots, k_{i+d-1})$ of the LFSR. Then,

$$S_{i+1} = \Delta_L S_i \pmod 2$$

for $i = 1, 2, \ldots, d$. Treat each S_i as a column vector. Then,

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This reveals Δ_L, that is, the secret $a_0, a_1, \ldots, a_{d-1}$.

An attack on LFSR

- The linear relation of the feedback bit as a function of the current state in LFSRs invites attacks.
- **Berlekamp-Massey attack**
  Suppose that the bits m_i and c_i for $2d$ consecutive values of i (say, $1, 2, \ldots, 2d$) are known to an attacker. Then $k_i = m_i \oplus c_i$ are also known for these values of i. Define the states $S_i = (k_i, k_{i+1}, \ldots, k_{i+d-1})$ of the LFSR. Then,

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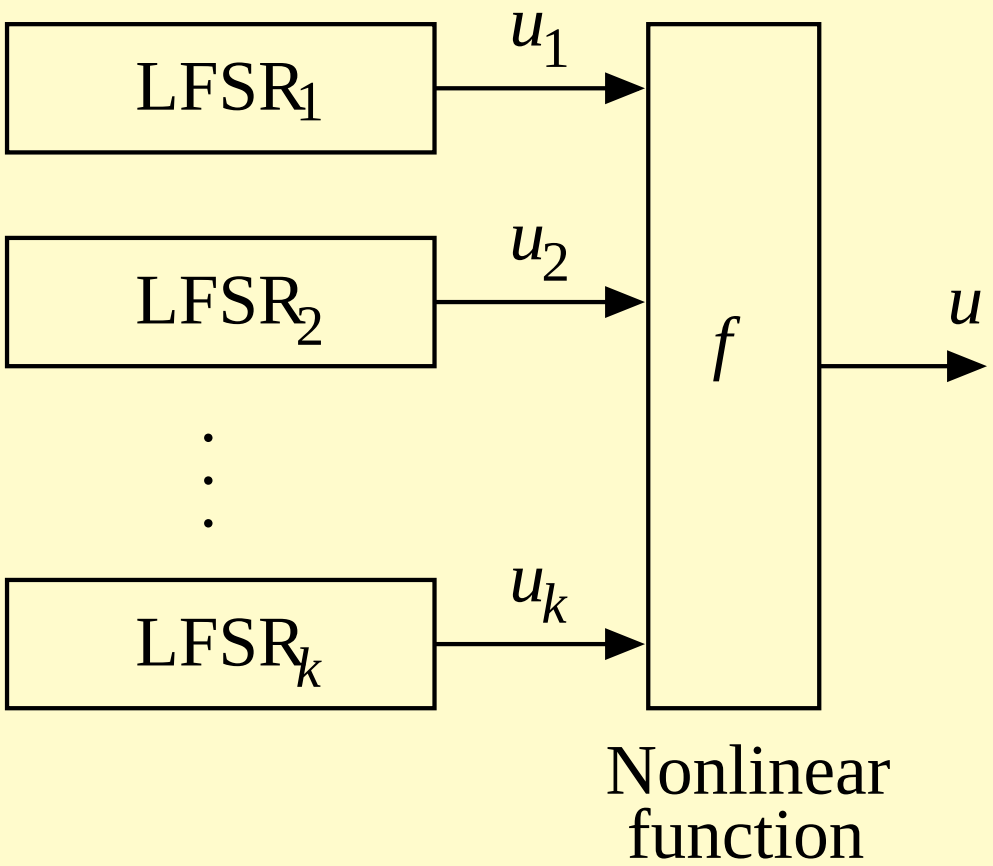
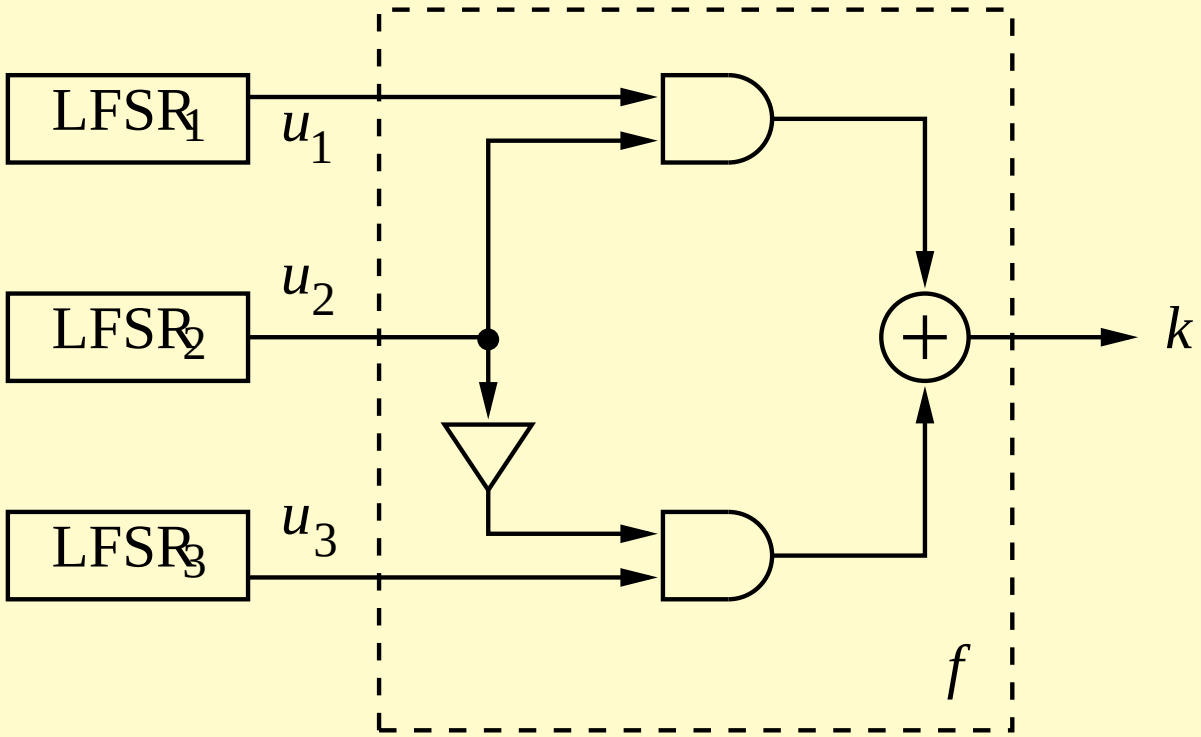
- **Remedy:** Introduce non-linearity to the LFSR output.

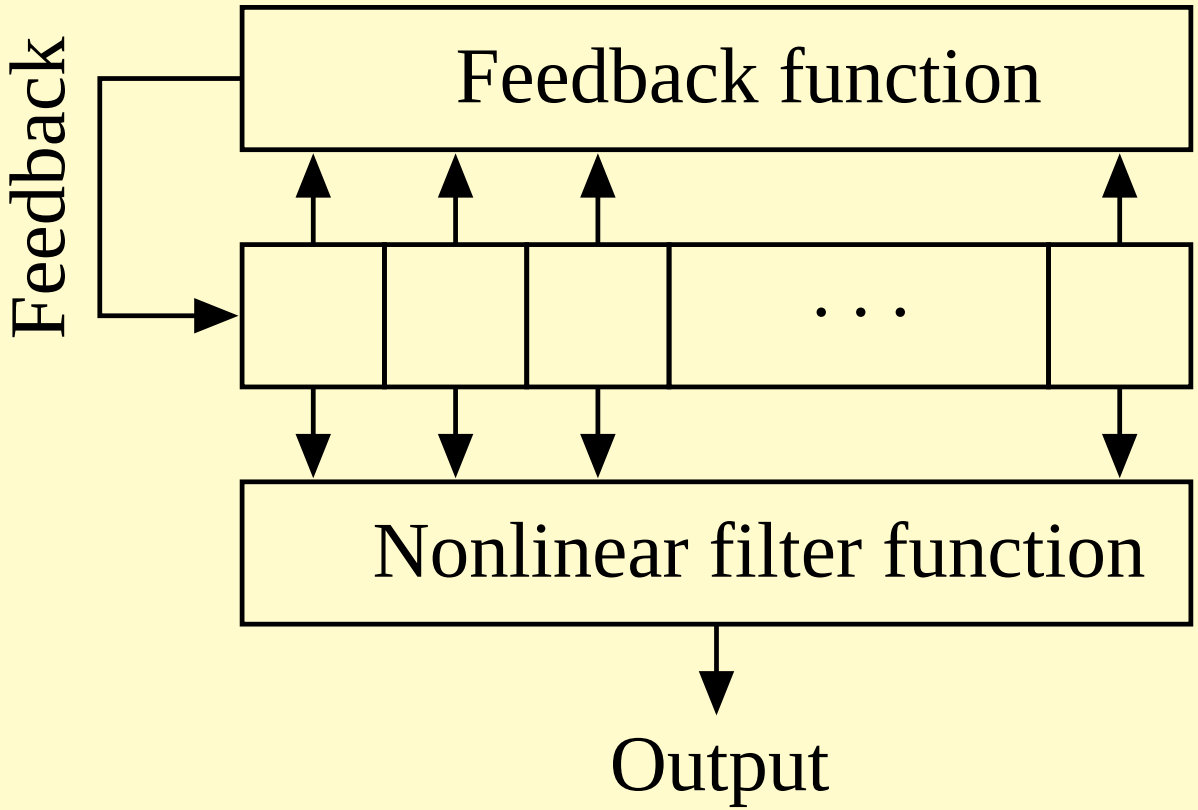
Nonlinear combination generator



The Geffe generator



Nonlinear filter generator



Hash functions

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- *Second pre-image resistance does not imply collision resistance*: Let S be a finite set of size $\geqslant 2$ and H a cryptographic hash function. Then

$$H'(x) = \begin{cases} 0^{n+1} & \text{if } x \in S, \\ 1 \,||\, H(x) & \text{otherwise,} \end{cases}$$

is second pre-image resistant but not collision resistant.

Hash functions (contd.)

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- *First pre-image resistance does not imply second pre-image resistance*: Let m be a product of two unknown big primes. Define $H'''(x) = (1 \,||\, x)^2 \pmod{m}$. H''' is first pre-image resistant, but not second pre-image resistant.

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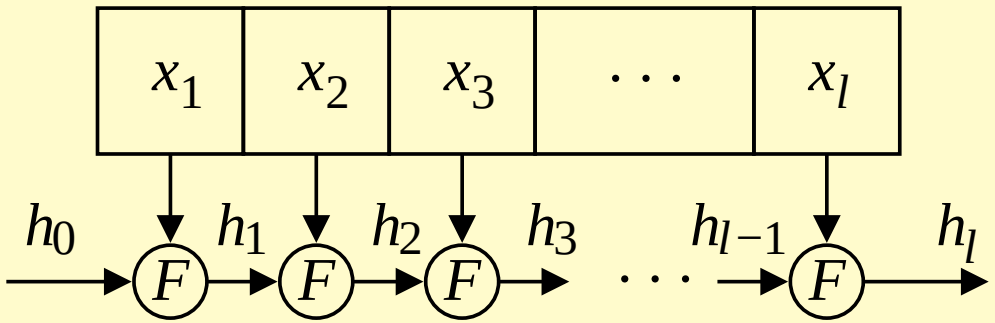

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Part III: Public-key cryptosystems

Intractable problems

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- Polynomial-time quantum algorithms are known for factoring integers and computing discrete logarithms in finite fields.

Introduction to number theory

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- **Common sets**

$$\begin{aligned}
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$$\gcd(a, b) = ua + vb.$$

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Therefore, $\gcd(899, 319) = 29$

Extended gcd computation

$$\begin{aligned} 29 &= 261 - 4 \times 58 \\ &= 261 - 4 \times (319 - 1 \times 261) = (-4) \times 319 + 5 \times 261 \\ &= (-4) \times 319 + 5 \times (899 - 2 \times 319) \\ &= 5 \times 899 + (-14) \times 319. \end{aligned}$$

Modular arithmetic

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- **Theorem:** $a \in \mathbb{Z}_n$ is invertible modulo n if and only if $\gcd(a, n) = 1$. In this case extended gcd gives $ua + vn = 1$. We may take $0 \leqslant u < n$. We have $u = a^{-1} \pmod{n}$.

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- **Inverse:** $\gcd(b, n) = 1 = (-45)b + 38n$, so
  $b^{-1} \pmod{n} = -45 + 257 = 212$.
- **Division:**
  $a/b \pmod{n} = ab^{-1} \pmod{n} = (127 \times 212) \text{ rem } 257 = 196$.

Modular exponentiation: Slow algorithm

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- **Example:** $n = 257$, $a = 127$, $e = 217$.

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$$\begin{aligned} a^2 &= a \times a = 195 \pmod{n}, \\ a^3 &= a^2 \times a = 195 \times 127 = 93 \pmod{n}, \\ a^4 &= a^3 \times a = 93 \times 127 = 246 \pmod{n}, \end{aligned}$$

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- **Example:** $n = 257$, $a = 127$, $e = 217$.

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a^2 &= a \times a = 195 \pmod{n}, \\
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a^4 &= a^3 \times a = 93 \times 127 = 246 \pmod{n}, \\
&\ldots \\
a^{216} &= a^{215} \times a = 131 \times 127 = 189 \pmod{n}, \\
a^{217} &= a^{216} \times a = 189 \times 127 = 102 \pmod{n}.
\end{aligned}$$

Modular exponentiation: Fast algorithm

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- Binary representation: $e = (e_{l-1}e_{l-2}\ldots e_1e_0)_2 = e_{l-1}2^{l-1} + e_{l-2}2^{l-2} + \cdots + e_1 2^1 + e_0 2^0$.

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- $a^e = \left(a^{2^{l-1}}\right)^{e_{l-1}} \left(a^{2^{l-2}}\right)^{e_{l-2}} \cdots \left(a^{2^1}\right)^{e_1} \left(a^{2^0}\right)^{e_0} \pmod{n}$.

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- Compute $a, a^2, a^{2^2}, a^{2^3}, \ldots, a^{2^{l-1}}$ and multiply those a^{2^i} modulo n for which $e_i = 1$. Also for $i \geqslant 1$, we have $a^{2^i} = \left(a^{2^{i-1}}\right)^2 \pmod{n}$.

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- $a^2 = 195 \pmod{n}$, $a^{2^2} = (195)^2 = 246 \pmod{n}$, $a^{2^3} = (246)^2 = 121 \pmod{n}$, $a^{2^4} = (121)^2 = 249 \pmod{n}$,

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- $a^2 = 195 \pmod{n}$, $a^{2^2} = (195)^2 = 246 \pmod{n}$, $a^{2^3} = (246)^2 = 121 \pmod{n}$, $a^{2^4} = (121)^2 = 249 \pmod{n}$, $a^{2^5} = (249)^2 = 64 \pmod{n}$,

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- $a^2 = 195 \pmod{n}$, $a^{2^2} = (195)^2 = 246 \pmod{n}$, $a^{2^3} = (246)^2 = 121 \pmod{n}$, $a^{2^4} = (121)^2 = 249 \pmod{n}$, $a^{2^5} = (249)^2 = 64 \pmod{n}$, $a^{2^6} = (64)^2 = 241 \pmod{n}$ and

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- $a^2 = 195 \pmod{n}$, $a^{2^2} = (195)^2 = 246 \pmod{n}$, $a^{2^3} = (246)^2 = 121 \pmod{n}$, $a^{2^4} = (121)^2 = 249 \pmod{n}$, $a^{2^5} = (249)^2 = 64 \pmod{n}$, $a^{2^6} = (64)^2 = 241 \pmod{n}$ and $a^{2^7} = (241)^2 = 256 \pmod{n}$.

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- $a^2 = 195 \pmod{n}$, $a^{2^2} = (195)^2 = 246 \pmod{n}$, $a^{2^3} = (246)^2 = 121 \pmod{n}$, $a^{2^4} = (121)^2 = 249 \pmod{n}$, $a^{2^5} = (249)^2 = 64 \pmod{n}$, $a^{2^6} = (64)^2 = 241 \pmod{n}$ and $a^{2^7} = (241)^2 = 256 \pmod{n}$.
- $a^e = 256 \times 241 \times 249 \times 121 \times 127 = 102 \pmod{n}$.

Euler totient function

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- Let $n \in \mathbb{N}$. Define

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}.$$

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- **Example:** If p is a prime, then $\phi(p) = p - 1$.
- **Example:** $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. We have $\gcd(0, 6) = 6$, $\gcd(1, 6) = 1$, $\gcd(2, 6) = 2$, $\gcd(3, 6) = 3$, $\gcd(4, 6) = 2$, and $\gcd(5, 6) = 1$. So $\mathbb{Z}_6^* = \{1, 5\}$, that is, $\phi(6) = 2$.

Euler totient function (contd.)

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- **Theorem:** Let $n = p_1^{e_1} \cdots p_r^{e_r}$ with distinct primes $p_i \in \mathbb{P}$ and with $e_i \in \mathbb{N}$. Then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_r}\right) = n\prod_{p \mid n}\left(1 - \frac{1}{p}\right).$$

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- **Fermat's little theorem:** Let $p \in \mathbb{P}$ and $a \in \mathbb{Z}$ with $p \nmid a$. Then $a^{p-1} = 1 \pmod{p}$.

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- **Euler's theorem:** Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$. Then $a^{\phi(n)} = 1 \pmod{n}$.

Multiplicative order

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- Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}_n^*$. Define $\mathrm{ord}_n\, a$ to be the smallest of the *positive* integers h for which $a^h = 1 \pmod{n}$.

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- **Example:** $n = 17$, $a = 2$. $a^1 = 2 \pmod{n}$, $a^2 = 4 \pmod{n}$, $a^3 = 8 \pmod{n}$, $a^4 = 16 \pmod{n}$, $a^5 = 15 \pmod{n}$, $a^6 = 13 \pmod{n}$, $a^7 = 9 \pmod{n}$, and $a^8 = 1 \pmod{n}$. So $\mathrm{ord}_{17}\, 2 = 8$.

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- **Theorem:** $a^k = 1 \pmod{n}$ if and only if $\mathrm{ord}_n\, a \mid k$.

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- **Example:** $n = 17$, $a = 2$. $a^1 = 2 \pmod{n}$, $a^2 = 4 \pmod{n}$, $a^3 = 8 \pmod{n}$, $a^4 = 16 \pmod{n}$, $a^5 = 15 \pmod{n}$, $a^6 = 13 \pmod{n}$, $a^7 = 9 \pmod{n}$, and $a^8 = 1 \pmod{n}$. So $\mathrm{ord}_{17}\, 2 = 8$.
- **Theorem:** $a^k = 1 \pmod{n}$ if and only if $\mathrm{ord}_n\, a \mid k$.
- **Theorem:** Let $h = \mathrm{ord}_n\, a$. Then, $\mathrm{ord}_n\, a^k = h/\gcd(h, k)$.

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- **Example:** $n = 17$, $a = 2$. $a^1 = 2 \pmod{n}$, $a^2 = 4 \pmod{n}$, $a^3 = 8 \pmod{n}$, $a^4 = 16 \pmod{n}$, $a^5 = 15 \pmod{n}$, $a^6 = 13 \pmod{n}$, $a^7 = 9 \pmod{n}$, and $a^8 = 1 \pmod{n}$. So $\mathrm{ord}_{17}\, 2 = 8$.
- **Theorem:** $a^k = 1 \pmod{n}$ if and only if $\mathrm{ord}_n\, a \mid k$.
- **Theorem:** Let $h = \mathrm{ord}_n\, a$. Then, $\mathrm{ord}_n\, a^k = h/\gcd(h, k)$.
- **Theorem:** $\mathrm{ord}_n\, a \mid \phi(n)$.

Primitive root

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- **Theorem (Gauss):** An integer $n > 1$ has a primitive root if and only if $n = 2, 4, p^e, 2p^e$, where p is an odd prime and $e \in \mathbb{N}$.

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- **Theorem (Gauss):** An integer $n > 1$ has a primitive root if and only if $n = 2, 4, p^e, 2p^e$, where p is an odd prime and $e \in \mathbb{N}$.
- **Example:** 3 is a primitive root modulo the prime $n = 17$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$3^k \pmod{17}$	1	3	9	10	13	5	15	11	16	14	8	7	4	12

14	15	16
2	6	1

Primitive root (contd.)

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- **Example:** $n = 2 \times 3^2 = 18$ has a primitive root 5 with order $\phi(18) = 6$:

k	0	1	2	3	4	5	6
$5^k \pmod{18}$	1	5	7	17	13	11	1

Primitive root (contd.)

- **Example:** $n = 2 \times 3^2 = 18$ has a primitive root 5 with order $\phi(18) = 6$:

k	0	1	2	3	4	5	6
$5^k \pmod{18}$	1	5	7	17	13	11	1

- **Example:** $n = 20 = 2^2 \times 5$ does not have a primitive root. We have $\phi(20) = 8$, and the orders of the elements of $\mathbb{Z}_{20}^*$ are $\text{ord}_{20}\, 1 = 1$, $\text{ord}_{20}\, 3 = \text{ord}_{20}\, 7 = \text{ord}_{20}\, 13 = \text{ord}_{20}\, 17 = 4$, and $\text{ord}_{20}\, 9 = \text{ord}_{20}\, 19 = 2$.

Discrete logarithm

Discrete logarithm

- Let $p \in \mathbb{P}$, g a primitive root modulo p, and $a \in \{1, 2, \ldots, p-1\}$. Then there exists a unique integer $x \in \{0, 1, 2, \ldots, p-2\}$ such that $g^x = a \pmod{p}$. We call x the *index* or *discrete logarithm* of a to the base g. We denote this by $x = \mathrm{ind}_g\, a$.

Discrete logarithm

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- Indices follow arithmetic modulo $p-1$.

$$\begin{aligned} \mathrm{ind}_g(ab) &= \mathrm{ind}_g\, a + \mathrm{ind}_g\, b \pmod{p-1}, \\ \mathrm{ind}_g(a^e) &= e\, \mathrm{ind}_g\, a \pmod{p-1}. \end{aligned}$$

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- Take $p = 17$ and $g = 3$.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
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- $\text{ind}_3\, 6 = 15$ and $\text{ind}_3\, 11 = 7$. Since $6 \times 11 = 15 \pmod{17}$, we have $\text{ind}_3\, 15 = \text{ind}_3\, 6 + \text{ind}_3\, 11 = 15 + 7 = 6 \pmod{16}$.

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- The converse is only believed to be true.

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- **Key generation**
  The recipient generates two random large primes p, q, computes $n = pq$ and $\phi(n) = (p-1)(q-1)$, finds a random integer e with $\gcd(e, \phi(n)) = 1$, and determines an integer d with $ed = 1 \pmod{\phi(n)}$.

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  Input: Ciphertext c and the recipient's private key (n, d).
  Output: Plaintext $m = c^d \pmod{n}$.

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- Let $p = 257$, $q = 331$, so that $n = pq = 85067$ and $\phi(n) = (p-1)(q-1) = 84480$. Take $e = 7$, so that $d = e^{-1} = 60343 \pmod{\phi(n)}$.
  Public key: $(85067, 7)$.
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- Recover $m = c^d = (53384)^{60343} = 34152 \pmod{n}$.
- Decryption by an exponent d' other than d does not give back m. For example, take $d' = 38367$. We have $m' = c^{d'} = (53384)^{38367} = 71303 \pmod{n}$.

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- **Note:** The message can be recovered uniquely even when $m \notin \mathbb{Z}_n^*$.

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  Input: Signed message (m, s) and signer's public key (n, e).
  Output: "Signature verified" if $s^e = m \pmod{n}$,
  "Signature not verified" if $s^e \neq m \pmod{n}$.

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- Verification of a forged signature $(m, s) = (3759, 42954)$ gives $s^e = (42954)^{19823} = 22968 \pmod{n}$. Since $s^e \neq m \pmod{n}$, the forged signature is not verified.

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- In practice, we require the size of n to be $\geqslant 1024$ bits with each of p, q having nearly half the size of n.

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- Alice computes $g^{ab} = (g^b)^a \pmod p$.
- Bob computes $g^{ab} = (g^a)^b \pmod p$.
- The quantity $g^{ab} \pmod p$ is the secret shared by Alice and Bob.

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- Practically, we require p to be of size $\geqslant 1024$ bits. The security does not depend on the choice of g. However, a and b must be sufficiently randomly chosen.

ElGamal encryption

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- **Key generation**
  The recipient selects a random big prime p and a primitive root g modulo p, chooses a random $d \in \{2, 3, \ldots, p-2\}$, and computes $y = g^d \pmod{p}$.
  Public key: (p, g, y).
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- **Encryption**
  Input: Plaintext $m \in \mathbb{Z}_p$ and recipient's public key (p, g, y).
  Output: Ciphertext (s, t).
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  Compute $s = g^{d'} \pmod{p}$ and $t = my^{d'} \pmod{p}$.
- **Decryption**
  Input: Ciphertext (s, t) and recipient's private key (p, g, d).
  Output: Recovered plaintext $m = ts^{-d} \pmod{p}$.

ElGamal encryption (contd.)

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- **Correctness:** We have $s = g^{d'} \pmod p$ and $t = my^{d'} = m(g^d)^{d'} = mg^{dd'} \pmod p$. Therefore, $m = tg^{-dd'} = t(g^{d'})^{-d} = ts^{-d} \pmod p$.

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  - The recipient retrieves $m = ts^{-d} = 1597 \times (52958)^{-23632} = 29485 \pmod p$.

Security of ElGamal encryption

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- An eavesdropper knows g, p, y, s, t, where $y = g^d \pmod p$ and $s = g^{d'} \pmod p$. Determining m from (s, t) is equivalent to computing $g^{dd'} \pmod p$, since $t = mg^{dd'} \pmod p$. (Here, m is masked by the quantity $g^{dd'} \pmod p$.) But d, d' are unknown to the attacker. So the ability to solve the DHP lets the eavesdropper break ElGamal encryption.

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- Practically, we require p to be of size $\geqslant 1024$ bits for achieving a good level of security.

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- **Key generation**
  Like ElGamal encryption, one chooses p, g and computes a key-pair (y, d) where $y = g^d \pmod p$. The public key is (p, g, y), and the private key is (p, g, d).

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- **Signature generation**
  Input: Message $m \in \mathbb{Z}_p$ and signer's private key (p, g, d).
  Output: Signed message (m, s, t).
  Generate a random session key $d' \in \{2, 3, \ldots, p-2\}$.
  Compute $s = g^{d'} \pmod p$ and
  $t = d'^{-1}(H(m) - dH(s)) \pmod{p-1}$.

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- **Signature verification**
  Input: Signed message (m, s, t) and signer's public key (p, g, y).
  Set $a_1 = g^{H(m)} \pmod p$ and $a_2 = y^{H(s)} s^t \pmod p$.
  Output "signature verified" if and only if $a_1 = a_2$.

ElGamal signature (contd.)

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- **Correctness:** $H(m) = dH(s) + td' \pmod{p-1}$. So $a_1 = g^{H(m)} = (g^d)^{H(s)}(g^{d'})^t = y^{H(s)}s^t = a_2 \pmod{p}$.

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  - Take $p = 104729$ and $g = 89$. The signer chooses the private exponent $d = 72135$ and so $y = g^d = 98771 \pmod{p}$.

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- **Example:**
  - Take $p = 104729$ and $g = 89$. The signer chooses the private exponent $d = 72135$ and so $y = g^d = 98771 \pmod{p}$.
  - Let $m = 23456$ be the message to be signed. The signer chooses the session exponent $d' = 3951$ and computes $s = g^{d'} = 14413 \pmod{p}$ and $t = d'^{-1}(m - ds) = (3951)^{-1}(23456 - 72135 \times 14413) = 17515 \pmod{p-1}$.

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  - Verification involves computation of $a_1 = g^m = 29201 \pmod{p}$ and $a_2 = y^s s^t = (98771)^{14413} \times (14413)^{17515} = 29201 \pmod{p}$. Since $a_1 = a_2$, the signature is verified.

ElGamal signature (contd.)

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- **Forging:** A forger chooses $d' = 3951$ and computes $s = g^{d'} = 14413 \pmod{p}$. But computation of t involves d which is unknown to the forger. So the forger randomly selects $t = 81529$. Verification of this forged signature gives $a_1 = g^m = 29201 \pmod{p}$ as above. But $a_2 = y^s s^t = (98771)^{14413} \times (14413)^{81529} = 85885 \pmod{p}$, that is, $a_1 \neq a_2$ and the forged signature is not verified.

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  - Computation of s can be done by anybody. However, computation of t involves the signer's private exponent d. If the forger can solve the DLP modulo p, then d can be computed from the public-key y, and the correct signature can be generated.

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  - The prime p should be large (of bit-size $\geqslant 1024$) in order to preclude this attack.

Some other encryption algorithms

Encryption algorithm	Security depends on
Rabin encryption	Square-root problem
Goldwasser-Micali encryption	Quadratic residuosity problem
Blum-Goldwasser encryption	Square-root problem
Chor-Rivest encryption	Subset sum problem
XTR	DLP
NTRU	Closest vector problem in lattices

Some other digital signature algorithms

Signature algorithm	Security depends on
Rabin signature	Square-root problem
Schnorr signature	DLP
Nyberg-Rueppel signature	DLP
Digital signature algorithm (DSA)	DLP
Elliptic curve version of DSA (ECDSA)	DLP in elliptic curves
XTR signature	DLP
NTRUSign	Closest vector problem

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- Alice computes Bob's signature $s = \rho^{-1}\sigma \pmod{n}$ on M.

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- Therefore, $s = \rho^{-1}\sigma = m^d = H(M)^d \pmod{n}$.

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Examples

Chaum-van Antwerpen undeniable signature scheme
RSA-based undeniable signature scheme

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- Bob accepts Alice's identity if and only if $r' = r$.

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- Before sending the decrypted plaintext r', Alice confirms that Bob actually knows the plaintext r.

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  Bob computes $w' = m^\epsilon r^e \pmod{n}$.
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- **Correctness**

  $w' = m^{\epsilon} r^e = m^{\epsilon}(cs^{\epsilon})^e = m^{\epsilon}(cm^{-d\epsilon})^e = (m^{1-ed})^{\epsilon} c^e = c^e = w \pmod{n}$.

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  - The quantity s^{ϵ} is blinded by the random commitment c.
  - As a witness for c, Alice presents its encrypted version w.

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  - The check $w' \neq 0$ precludes the case $c = 0$ which lets a claimant succeed always.

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- A revoked certificate cannot be used to establish the authenticity of a public key.

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  - Name, address and other personal details of the entity.
  - The public key of the entity. The key pair may be generated by either the entity or the CA. If the CA generates the key pair, then the private key is handed over to the entity by trusted couriers.

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  The certificate is digitally signed by the private key of the CA.
- If signatures cannot be forged, nobody other than the CA can generate a valid certificate for an entity.

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- **Certificate Revocation List** (**CRL**): The CA maintains a list of revoked certificates.
- If Alice wants to use Bob's public key, she obtains the certificate for Bob's public key. If the CA's signature is verified on this certificate and if the certificate is not found in the CRL, then Alice gains the desired confidence to use Bob's public key.

Part IV: Public-key cryptanalysis

Integer factoring algorithms

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Let n be the integer to be factored.

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Older algorithms

- Trial division (efficient if all prime divisors of n are small)
- Pollard's rho method
- Pollard's $p-1$ method (efficient if $p-1$ has only small prime factors for some prime divisor p of n)
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In the worst case these algorithms take exponential (in $\log n$) running time.

Modern algorithms

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Subexponential running time:

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Algorithm	Inventor(s)	Running time
Continued fraction method (CFRAC)	Morrison & Brillhart (1975)	$L(n, 1/2, c)$
Quadratic sieve method (QSM)	Pomerance (1984)	$L(n, 1/2, 1)$
Cubic sieve method (CSM)	Reyneri	$L(n, 1/2, 0.816)$
Elliptic curve method (ECM)	H. W. Lenstra (1987)	$L(n, 1/2, c)$
Number field sieve method (NFSM)	A. K. Lenstra, H. W. Lenstra, Manasse & Pollard (1990)	$L(n, 1/3, 1.923)$

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  $3 \times 833 = 2500 - 1 = 50^2 - 1^2 = (50 - 1) \times (50 + 1) = 49 \times 51$.
  $\gcd(50 - 1, 833) = 49$ is a non-trivial factor of 833.

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To find integers $x, y \in \mathbb{Z}_n$ such that $x^2 = y^2 \pmod{n}$. Unless $x = \pm y \pmod{n}$, $\gcd(x - y, n)$ is a non-trivial divisor of n.

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If n is composite (but not a prime power), then for a randomly chosen pair (x, y) with $x^2 = y^2 \pmod{n}$, the probability that $x \neq \pm y \pmod{n}$ is at least $1/2$.

The Quadratic Sieve Method (QSM)

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Suppose $T(c)$ factors over small primes $p_1, p_2, \ldots, p_t$:

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This is called a **relation**.

The left side is already a square.
The right side is also a square if each α_i is even.
But this is very rare.

QSM (contd.)

Collect many relations:

$$\left.\begin{array}{lrcl} \text{Relation 1:} & (H+c_1)^2 & = & p_1^{\alpha_{11}} p_2^{\alpha_{12}} \cdots p_t^{\alpha_{1t}} \\ \text{Relation 2:} & (H+c_2)^2 & = & p_1^{\alpha_{21}} p_2^{\alpha_{22}} \cdots p_t^{\alpha_{2t}} \\ \cdots & & & \\ \text{Relation } r\text{:} & (H+c_r)^2 & = & p_1^{\alpha_{r1}} p_2^{\alpha_{r2}} \cdots p_t^{\alpha_{rt}} \end{array}\right\} \pmod{n}.$$

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Let $\beta_1, \beta_2, \ldots, \beta_r \in \{0, 1\}$.

$$\left[(H+c_1)^{\beta_1}(H+c_2)^{\beta_2}\cdots(H+c_r)^{\beta_r}\right]^2 = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_t^{\gamma_t} \pmod{n}.$$

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Tune $\beta_1, \beta_2, \ldots, \beta_r$ to make each γ_i even.

QSM (contd.)

$$\begin{aligned}
\alpha_{11}\beta_1 + \alpha_{21}\beta_2 + \cdots + \alpha_{r1}\beta_r &= \gamma_1, \\
\alpha_{12}\beta_1 + \alpha_{22}\beta_2 + \cdots + \alpha_{r2}\beta_r &= \gamma_2, \\
&\cdots \\
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\end{aligned}$$

Linear system with t equations and r variables $\beta_1, \beta_2, \ldots, \beta_r$:

$$\begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{r1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{r2} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{1t} & \alpha_{2t} & \cdots & \alpha_{rt} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod 2.$$

QSM (contd.)

For $r \geqslant t$, there are non-zero solutions for $\beta_1, \beta_2, \ldots, \beta_r$. Take

$$\begin{aligned} x &= (H + c_1)^{\beta_1}(H + c_2)^{\beta_2} \cdots (H + c_r)^{\beta_r} \pmod{n}, \\ y &= p_1^{\gamma_1/2} p_2^{\gamma_2/2} \cdots p_t^{\gamma_t/2} \pmod{n}. \end{aligned}$$

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If $x \neq \pm y \pmod{n}$, then $\gcd(x - y, n)$ is a non-trivial factor of n.

Let $p = p_i$ be a small prime.

$p \mid T(c)$ implies $(H + c)^2 = n \pmod{p}$.

If n is not a quadratic residue modulo p, then $p \nmid T(c)$ for any c.

Consider only the small primes p modulo which n is a quadratic residue.

Example of QSM: Parameters

$n = 7116491$.

$H = \lceil \sqrt{n} \rceil = 2668$.

Take all primes < 100 modulo which n is a square:

$$B = \{2, 5, 7, 17, 29, 31, 41, 59, 61, 67, 71, 79, 97\}.$$

$t = 13$.

Take $r = 13$. (In practice, one takes $r \approx 2t$.)

Example of QSM: Relations

$$
\left.
\begin{aligned}
\text{Relation 1:} \quad (H+3)^2 &= 2 \times 5^3 \times 71 \\
\text{Relation 2:} \quad (H+8)^2 &= 5 \times 7 \times 31 \times 41 \\
\text{Relation 3:} \quad (H+49)^2 &= 2 \times 41^2 \times 79 \\
\text{Relation 4:} \quad (H+64)^2 &= 7 \times 29^2 \times 59 \\
\text{Relation 5:} \quad (H+81)^2 &= 2 \times 5 \times 7^2 \times 29 \times 31 \\
\text{Relation 6:} \quad (H+109)^2 &= 2 \times 7 \times 17 \times 41 \times 61 \\
\text{Relation 7:} \quad (H+128)^2 &= 5^3 \times 71 \times 79 \\
\text{Relation 8:} \quad (H+145)^2 &= 2 \times 71^2 \times 79 \\
\text{Relation 9:} \quad (H+182)^2 &= 17^2 \times 59^2 \\
\text{Relation 10:} \quad (H+228)^2 &= 5^2 \times 7^2 \times 17 \times 61 \\
\text{Relation 11:} \quad (H+267)^2 &= 2 \times 7^2 \times 17 \times 29 \times 31 \\
\text{Relation 12:} \quad (H+382)^2 &= 7 \times 59 \times 67 \times 79 \\
\text{Relation 13:} \quad (H+411)^2 &= 2 \times 5^4 \times 31 \times 61
\end{aligned}
\right\} \pmod{n}.
$$

Example of QSM: Linear System

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \pmod 2.$$

Example of QSM: Solution of Relations

$(\beta_1, \beta_2, \beta_3, \ldots, \beta_{13})$	x	y	$\gcd(x-y, n)$
$(0,0,0,0,0,0,0,0,0,0,0,0,0)$	1	1	7116491
$(1,0,1,0,0,0,1,0,0,0,0,0,0)$	1755331	560322	1847
$(0,0,1,0,0,0,0,1,0,0,0,0,0)$	526430	459938	1847
$(1,0,0,0,0,0,1,1,0,0,0,0,0)$	7045367	7045367	7116491
$(0,0,0,0,0,0,0,0,1,0,0,0,0)$	2850	1003	1847
$(1,0,1,0,0,0,1,0,1,0,0,0,0)$	6916668	6916668	7116491
$(0,0,1,0,0,0,0,1,1,0,0,0,0)$	5862390	5862390	7116491
$(1,0,0,0,0,0,1,1,1,0,0,0,0)$	3674839	6944029	1847
$(0,1,0,0,1,1,0,0,0,0,1,0,1)$	1079130	3965027	3853
$(1,1,1,0,1,1,1,0,0,0,1,0,1)$	5466596	1649895	1
$(0,1,1,0,1,1,0,1,0,0,1,0,1)$	5395334	1721157	1
$(1,1,0,0,1,1,1,1,0,0,1,0,1)$	6429806	3725000	3853
$(0,1,0,0,1,1,0,0,1,0,1,0,1)$	1196388	5920103	1
$(1,1,1,0,1,1,1,0,1,0,1,0,1)$	1799801	3818773	3853
$(0,1,1,0,1,1,0,1,1,0,1,0,1)$	5081340	4129649	3853
$(1,1,0,0,1,1,1,1,1,0,1,0,1)$	7099266	17225	1

Algorithms for computing discrete logarithms

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Worst-case complexity: Exponential in $\log p$

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Based on the index calculus method (ICM)

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Algorithm	Inventor(s)	Running time
Basic ICM	Western & Miller (1968)	$L(p, 1/2, c)$
Linear sieve method (LSM) Residue list sieve method Gaussian integer method	Coppersmith, Odlyzko & Schroeppel (1986)	$L(p, 1/2, 1)$
Cubic sieve method (CSM)	Reyneri	$L(p, 1/2, 0.816)$
Number field sieve method (NFSM)	Gordon (1993)	$L(p, 1/3, 1.923)$

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For random $j \in \{1, 2, \ldots, p-2\}$, try to factor $g^j \pmod p$ over B.

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For random $j \in \{1, 2, \ldots, p-2\}$, try to factor $g^j \pmod p$ over B.

Relation: $g^j = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} \pmod p$

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Factor base: First t primes $B = \{p_1, p_2, \ldots, p_t\}$

To compute $d_i = \text{ind}_g\, p_i$ for $i = 1, 2, \ldots, t$

For random $j \in \{1, 2, \ldots, p-2\}$, try to factor $g^j \pmod p$ over B.

Relation: $g^j = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} \pmod p$

Linear equation in t variables $d_1, d_2, \ldots, d_t$:

$$j = \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_t d_t \pmod{p-1}$$

The basic ICM: Precomputation (contd.)

Generate $r \geqslant t$ relations for different values of j:

$$\left.\begin{array}{lrcl}
\text{Relation 1:} & j_1 & = & \alpha_{11}d_1 + \alpha_{12}d_2 + \cdots + \alpha_{1t}d_t \\
\text{Relation 2:} & j_2 & = & \alpha_{21}d_1 + \alpha_{22}d_2 + \cdots + \alpha_{2t}d_t \\
\cdots & & & \\
\text{Relation } r\text{:} & j_r & = & \alpha_{r1}d_1 + \alpha_{r2}d_2 + \cdots + \alpha_{rt}d_t
\end{array}\right\} \pmod{p-1}.$$

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Solve the system modulo $p-1$ to determine $d_1, d_2, \ldots, d_t$.

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Substitute the values of $d_1, d_2, \ldots, d_t$ to get $\text{ind}_g\, a$.

The basic ICM: Example (Precomputation)

Paramaters: $p = 839$, $g = 31$, $B = \{2, 3, 5, 7, 11\}$, $t = 5$, $r = 10$.

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Relations

$$\left.\begin{array}{lrcl}
\text{Relation 1:} & g^{118} & = & 2^3 \times 5^2 \\
\text{Relation 2:} & g^{574} & = & 2^7 \times 5 \\
\text{Relation 3:} & g^{318} & = & 2^2 \times 3^3 \\
\text{Relation 4:} & g^{46} & = & 2^7 \\
\text{Relation 5:} & g^{786} & = & 2^2 \times 3^3 \times 7 \\
\text{Relation 6:} & g^{323} & = & 2 \times 3 \times 11 \\
\text{Relation 7:} & g^{606} & = & 3^4 \\
\text{Relation 8:} & g^{252} & = & 2^3 \times 3^2 \times 7 \\
\text{Relation 9:} & g^{160} & = & 3 \times 5^2 \\
\text{Relation 10:} & g^{600} & = & 2 \times 3^3 \times 5
\end{array}\right\} \pmod{p}.$$

The basic ICM: Example (Precomputation)

$$\begin{pmatrix} 3 & 0 & 2 & 0 & 0 \\ 7 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{pmatrix} = \begin{pmatrix} 118 \\ 574 \\ 318 \\ 46 \\ 786 \\ 323 \\ 606 \\ 252 \\ 160 \\ 600 \end{pmatrix} \pmod{p-1}.$$

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$$\left.\begin{array}{lcccl} d_1 & = & \mathrm{ind}_{31}\, 2 & = & 246 \\ d_2 & = & \mathrm{ind}_{31}\, 3 & = & 780 \\ d_3 & = & \mathrm{ind}_{31}\, 5 & = & 528 \\ d_4 & = & \mathrm{ind}_{31}\, 7 & = & 468 \\ d_5 & = & \mathrm{ind}_{31}\, 11 & = & 135 \end{array}\right\} \pmod{p-1}.$$

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Take $a = 561$.

$$ag^{312} = 600 = 2^3 \times 3 \times 5^2 \pmod{p}, \quad \text{that is,}$$

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Take $a = 89$.

$$ag^{342} = 99 = 3^2 \times 11 \pmod{p}, \quad \text{that is,}$$
$$\text{ind}_{31}\, 89 = -342 + 2 \times 780 + 135 = 515 \pmod{p-1}.$$

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Take $a = 625$.

$$ag^{806} = 70 = 2 \times 5 \times 7 \pmod{p}, \quad \text{that is,}$$
$$\text{ind}_{31}\, 625 = -806 + 246 + 528 + 468 = 436 \pmod{p-1}.$$

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Remedies: Shield the decrypting device from external measurements, recheck computations, add random delays.

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Remedy: Use trustworthy (like open-source) software.

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Selected references

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