Turing Machines

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- Was there a model that could capture the computability of many such machines?
- Turing machine is one such model.
- We will see examples of equivalent models for computability.

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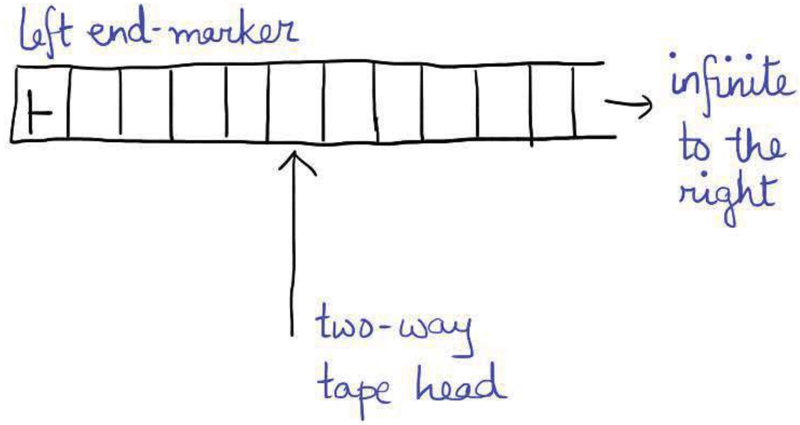
- This is like taking in a piece of program as input and simulating it.
- Flipside: We will also see that many problems that try to simulate an input machine are unsolvable by Turing machines.

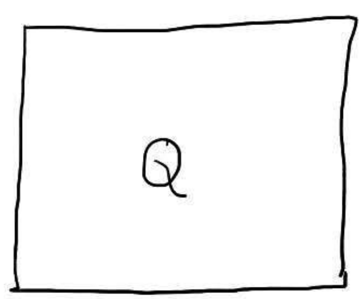
Description of the Turing Machine Model

- Single read-write tape, infinite on the right



- Tape head pointing at current cell, that can move in both directions, but not beyond left end-marker

- Finite set of states
  - Finite control



Finite representation of a TM

- $M = (Q, \Sigma, \Gamma, \vdash, \Box, \delta, s, t, r)$
- Q - finite set of states
- Σ - finite input alphabet, Γ - finite tape alphabet that contains Σ
- $\vdash \in \Gamma - \Sigma$ - left endmarker
- $\Box \in \Gamma - \Sigma$ - blank symbol
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ - transition function
- $s \in Q$ - start state, $t \in Q$ - accept state, $r \neq t \in Q$ - reject state

Transitions

- $\delta(p, a) = (q, b, d)$: If the current state is p and the tape head is scanning a, then write down b in place of a, move tape head towards direction $d \in \{L, R\}$, and transition to state q.

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- Once t or r states are entered, then they cannot be left: for all $b \in \Gamma$ there exist $c, c' \in \Gamma, d, d' \in \{L, R\}$ such that
  (i) $\delta(t, b) = (t, c, d)$
  (ii) $\delta(r, b) = (r, c', d')$.

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- **Configuration**: $\alpha \in Q \times \{y\Box^{\omega} | y \in \Gamma^*\} \times \mathbb{N}$ gives a snapshot of the TM currently.
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  Configuration $\alpha = (p, z, n)$ denotes that the TM is in state p, has z on its read/write tape and the tape head is at the n^{th} position
- **Start configuration** on input $x \in \Sigma^*$ is unique: $(s, \vdash x\Box^\omega, 0)$

Configurations

- **Next configuration relation** $\rightarrow^1_M$ for a string $z \in \Gamma^*$: let z_n be the n^{th} symbol, and $s^n_b(z)$ be the string obtained from z by replacing z_n by the alphabet $b \in \Gamma$.

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- If $\delta(p, z_n) = (q, b, L)$, then $(p, z, n) \rightarrow_M^1 (q, s_b^n(z), n-1)$.

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- If $\delta(p, z_n) = (q, b, L)$, then $(p, z, n) \rightarrow_M^1 (q, s_b^n(z), n - 1)$.
- If $\delta(p, z_n) = (q, b, R)$, then $(p, z, n) \rightarrow_M^1 (q, s_b^n(z), n + 1)$.

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- $\alpha \rightarrow^0_M \alpha$,
- $\alpha \rightarrow^{n+1}_M \beta$ if $\alpha \rightarrow^n_M \gamma \rightarrow^1_M \beta$ for some γ,
- $\alpha \rightarrow^*_M \beta$ if $\alpha \rightarrow^n_M \beta$ for some $n \geq 0$

Acceptance and Rejection

- TM *accepts* $x \in \Sigma^*$: $(s, \vdash x\Box^\omega, 0) \rightarrow^*_M (t, y, n)$ for some y, n. This set is denoted as $L(M)$ for TM M.

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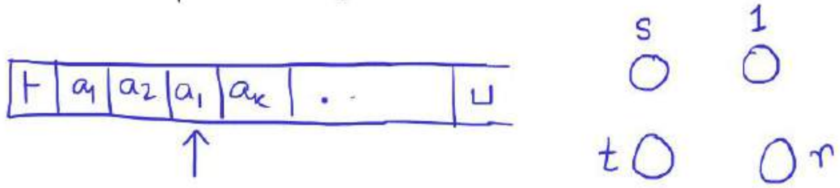
- TM *rejects* $x \in \Sigma^*$: $(s, \vdash x\Box^\omega, 0) \rightarrow^*_M (r, y, n)$ for some y, n.
- TM *halts* on $x \in \Sigma^*$ if it either accepts or rejects x. It may *loop*.

Design a TM that accepts even length strings.

Language is a <u>regular set</u>

Algorithm: As the tape head moves from $\vdash$ to the rest of the input, the finite control remembers the parity of the input so far.



Transition function:

	$\vdash$	a_1	a_2	$\dots$	a_k	$\sqcup$
s	$(1,\vdash,R)$	$(1,a_1,R)$	$(1,a_2,R)$		$(1,a_k,R)$	$(r,\sqcup,R)$
1	$(r,\vdash,R)$	(s,a_1,R)	(s,a_2,R)		(s,a_k,R)	$(t,\sqcup,R)$
t	$(t,\vdash,R)$	(t,a_1,R)	(t,a_2,R)		(t,a_k,R)	$(t,\sqcup,R)$
r	$(r,\vdash,R)$	(r,a_1,R)	(r,a_2,R)		(r,a_k,R)	$(r,\sqcup,R)$

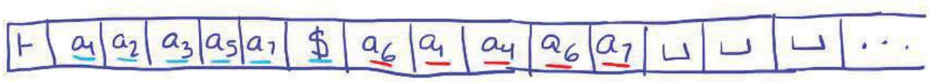
Design a TM that accepts a string if it is of odd length and the middle element is a $ symbol

Language is (a) Not a regular set
(b) CFL

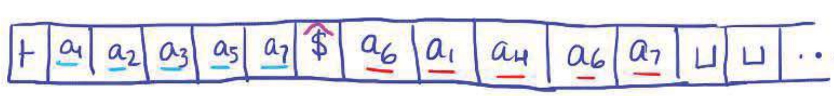
Algorithm:

Step 1: Check if input string is of odd length.

Step 2: Finding the middle element:



Finite control remembers that last blue mark could not be matched with a red mark.



Middle element detected and marked.

Step 3: Check if middle element is \$.

This can be done using finite control: transition to accept state 't' if $\hat{\$}$ is read on the current cell.

Algorithm in words:

1. Check that the string is of odd length: Using the finite control determine the parity of the length of the string in one pass.
2. Determine the middle element: Starting from $\vdash$ mark the first unmarked alphabet in Σ by 'underlining'. Remember this _ marking was done and move right till the rightmost unmarked alphabet in Σ. Mark it by 'overlining'. Remember this ‾ marking and move left till you hit $\vdash$. Repeat above step as long as possible. If at some point the very next cell after a _ marking contains an alphabet marked with ‾, the the previous cell contains the middle element: Remember this using a special state in the finite control and move left.

4. Checking that the middle element is \$: If you are in the special state and the current alphabet is \$, then go to accept state.
5. Anything not working out in steps 1-4, go to reject state 'r'.

Design a TM that writes a copy of the input string after the input string. Eg. If x is the string, in the end the tape contains xx

This is NOT a decision problem - TM can still do this operation.

Algorithm:

- Starting from $\vdash$, when you see first unmarked alphabet $a \in \Sigma$, mark it by rewriting with $\underline{a}$. In finite control, remember a was seen (Σ is finite).
- Move right till you see $\sqcup$ for the first time. Rewrite with $\bar{a}$.
- Continue till there are no unmarked alphabets in the tape. (All are marked with $\_$ or $\bar{\ }$.
- Move left till $\vdash$.
- In one pass to the right, rewrite each $\underline{a}$ by a and each $\bar{a}$ by a.

There is a TM that accepts the language $\{a^n b^n c^n | n \geq 0\}$

The language is not a CFL.

Algorithm:

Step 1: Check if input string is of the form $a^* b^* c^*$.

Step 2: When $\sqcup$ is seen for the first time, it is rewritten with $\dashv$ (right end-marker, good practice, defines right endpoint of string written on the tape)

Step 3: Move left, mark first c, with $\hat{c}$
then first b, with $\hat{b}$
then first a, with $\hat{a}$

Step 4: When $\vdash$ is hit, move right
mark first a, with $\hat{a}$
first b, with $\hat{b}$
first c, with $\hat{c}$

Step 5: Continue making left and right passes.
In each pass, 1 a, b, c gets marked.
If in any pass, this does not happen - reject
If all alphabets are seen to be marked
in a pass - accept.

There is a TM that accepts the language $\{ww | w \in \Sigma^*\}$

Language is not a CFL.

Algorithm:-

Step 1: Check the parity of the input string

Step 2: Mark the first half of the string with $\_$
second half of the string with $\bar{\ }$

Step 3: Move left till $\vdash$ is reached.
Move right to the first alphabet marked with $\_$.
Unmark it (rewrite $\underline{a}$ to a)
In finite control remember a was seen.

Step 4: Move right till first alphabet marked $\bar{\ }$ is hit.
Check if the unmarked version is same as the alphabet remembered in the finite control.
Unmark it (rewrite $\bar{a}$ to a)

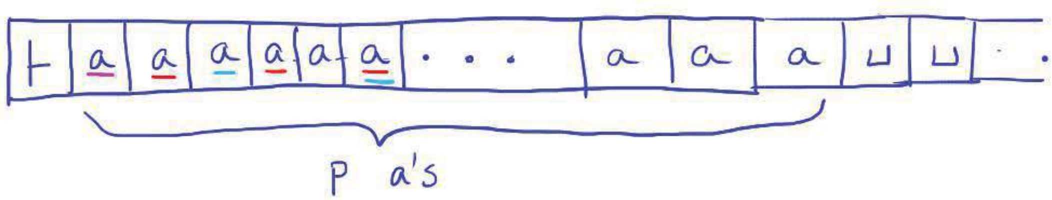
Step 5: Repeat steps 3 and 4 till every alphabet gets unmarked or some mismatch has been detected for rejection.

If no rejection condition has arisen, accept the string.

There is a TM that accepts the language $\{a^p | p \text{ is prime }\}$

The language is not a CFL

Algorithm: Implementing Eratosthenes' sieve:



Step 1: - Mark the first a in the input.

Step 2: - Choose the first unmarked a. Suppose it is at the i^{th} position. Mark all a's in positions that are multiples of i (If a ⊔ is encountered, all multiples are marked)

Step 3: - Continue Step 2 till all a's are marked.

STEP 2: For an i, marking all positions that are multiples of i.

- Mark the i^{th} a as $\hat{a}$. Move to $\vdash$.
- Move right, mark a as a' till $\hat{a}$ is hit (remember in finite control)
- Move left to $\vdash$. For a' to the left match an a after $\hat{a}$ (using markings) and mark this a as a'.
- Upon reaching $\hat{a}$ match to an a, and mark it with $\underset{\vee}{\underline{a}}$.
- Move left to previous $\hat{a}$ or $\underset{\vee}{\underline{a}}$ and match a's in between to a's after $\underset{\vee}{\underline{a}}$ - mark these with a'. At $\underset{\vee}{\underline{a}}$, matched a will be marked with $\underset{\vee}{\underline{a}}$. Continue till $\sqcup$ reached.
- In one pass, unmark all a' to a, and mark all $\hat{a}$ and $\underset{\vee}{\underline{a}}$ to $\underline{a}$.

R.E and Recursive Sets

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- A TM is *total* if it halts on all inputs: for any input it either accepts or rejects the input.
- A set of strings L is:
  (i) *recursively enumerable* (r.e) if $L = L(M)$ where M is a TM.
  (ii) *co-r.e* if $\overline{L}$ is r.e.
  (iii) *recursive* if $L = L(M)$ where M is a total TM (always halts on all inputs).

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- A property P of strings in Σ^* is a subset of Σ^*.
- A property P is *decidable* if the set of strings with P is a recursive set. Eg. The TM for the language $\{ww | w \in \Sigma^*\}$
- P is *semidecidable* if the set of strings with P is r.e.
- Can you think of an example of a semidecidable set? Will see examples of such properties which are semidecidable but not decidable, etc.