More on Regular Sets

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- Expansions of numbers divisible by 3:
  $0 = 0$
  $3 = 11$
  $6 = 110$
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- Apriori no structural pattern can be deciphered as before.

- How does the string change when we read one more bit to the right: [$\#y$ is decimal number represented in binary by y]
  $\#(x0) = 2\#x + 0$
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- $\#\epsilon = 0$ for convenience, same as $\#0$.

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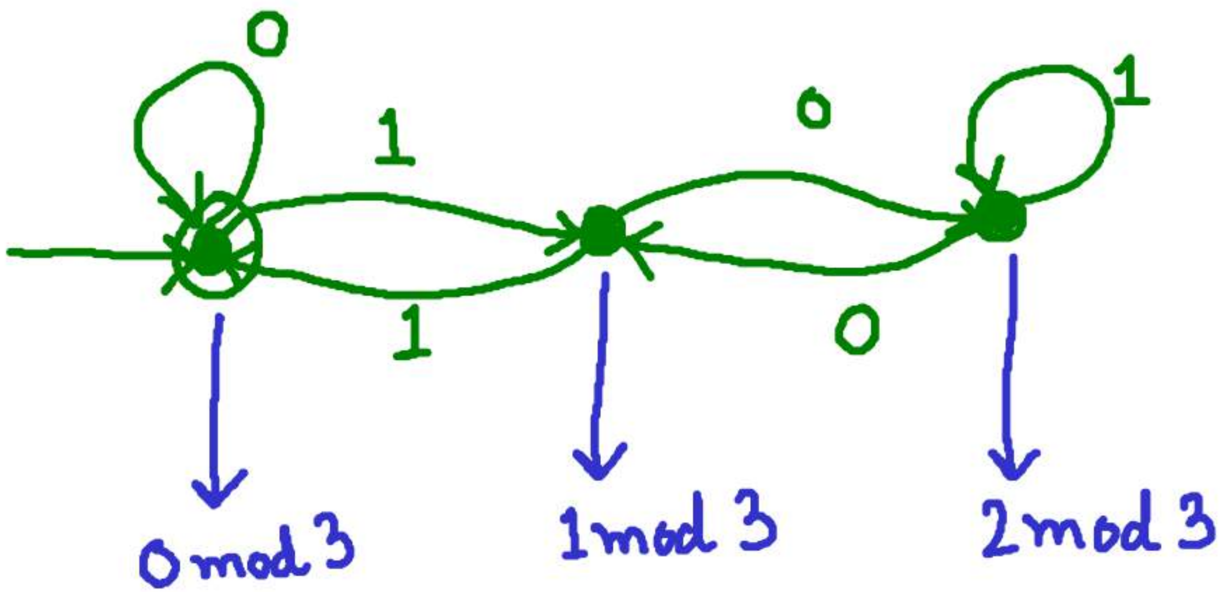
- Assume that we are starting with the ϵ string and so we start with the 0 state.
- Intuitively we want to end at the 0 state because it is supposed to represent $0 \mod 3$ (Final state).
- Transition function: $\delta(q, c) = (2q + c) \mod 3$

Transition Diagram

$$\delta(q, c) = (2q + c) \bmod 3$$



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- Base: $x = \epsilon$
  $\hat{\delta}(0, \epsilon) = 0$ by definition of $\hat{\delta}$.
  $= \#\epsilon$
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- $\hat{\delta}(0, xc) = \delta(\hat{\delta}(0, x), c)$
  $= \delta(\#x \bmod 3, c)$ [IH]
  $= (2(\#x \bmod 3) + c) \bmod 3$ [Definition of δ]
  $= (2(\#x) + c) \bmod 3$ [Property of mod function]
  $= \#xc \bmod 3$ [Property of strings x and xc]

Closure properties of regular sets A and B

- Union $A \cup B$, Intersection $A \cap B$, Complement $\neg A$, Concatenation AB, A^*.

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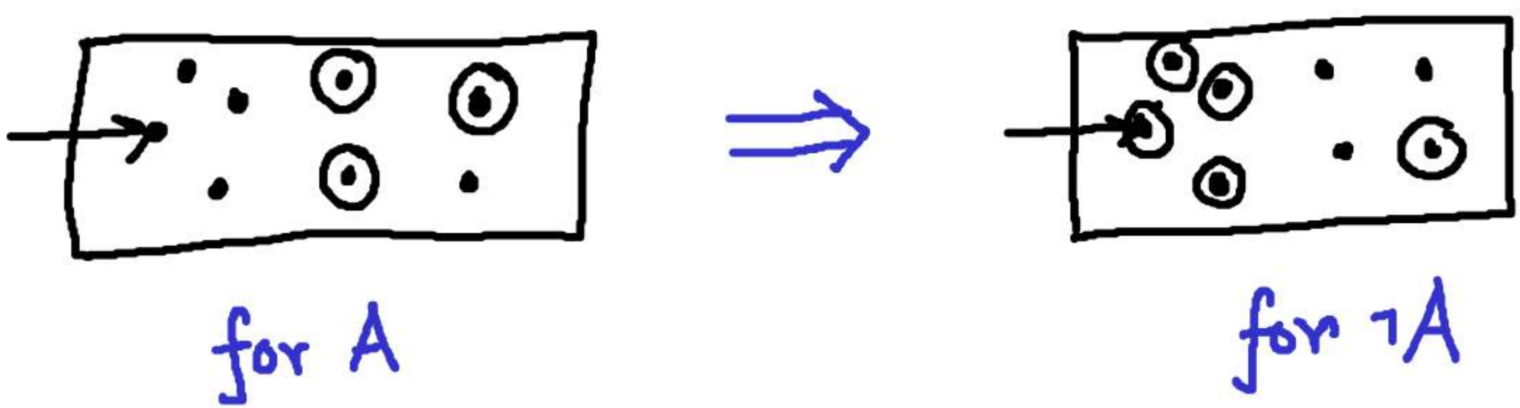
- Union $A \cup B$, Intersection $A \cap B$, Complement $\neg A$, Concatenation AB, A^*.
- Show that if A is a regular set then so is $\neg A$: This means $A = L(M)$ for a DFA M. Make all previous non-final states as current final states and all previous final states as current non-final states – this accepts $\neg A$.



Product construction: Closure under intersection

- A and B are 2 regular sets. M_1 and M_2 are respective DFAs. Design a DFA for the set $A \cap B$.

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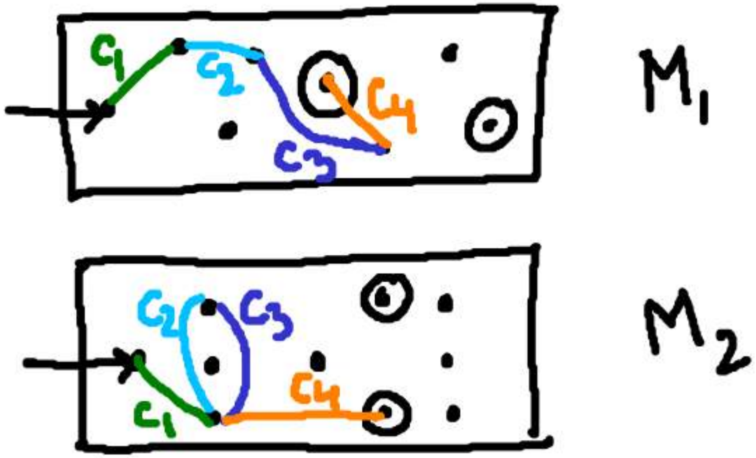
- *A* and *B* are 2 regular sets. M_1 and M_2 are respective DFAs. Design a DFA for the set $A \cap B$.
- Intuitively, I am simultaneously following a path in each DFA M_1 and M_2. If both paths end in final states of respective DFAs then I accept.



Product construction: Closure under intersection

- $M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$
  $Q_3 = Q_1 \times Q_2$, $s_3 = (s_1 \times s_2)$
  $F_3 = (F_1 \times F_2)$
  $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$

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  $\delta_3((p, q), a) = (\delta_1(p, a), \delta_2(q, a))$
- M_3 a **product automaton** of M_1 and M_2.

Multistep transition function $\hat{\delta}_3$

- Inductive definition revisited:
  $\hat{\delta}_3((p, q), \epsilon) = (p, q)$
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- Base: $x = \epsilon$
  $\hat{\delta}_3((p, q), \epsilon) = (p, q) = (\hat{\delta}_1(p, \epsilon), \hat{\delta}_2(q, \epsilon))$

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- $\hat{\delta}_3((p,q),xc) = \delta_3(\hat{\delta}_3((p,q),x),c)$
  $= \delta_3((\hat{\delta}_1(p,x),\hat{\delta}_2(q,x)),c)$ [I.H]
  $= (\delta_1(\hat{\delta}_1(p,x),c),\delta_2(\hat{\delta}_2(q,x),c))$ [Definition of δ_3]
  $= (\hat{\delta}_1(p,xc),\hat{\delta}_2(q,xc))$

- **Theorem**: $L(M_3) = L(M_1) \cap L(M_2)$

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- Proof: $x \in L(M_3)$ means $\hat{\delta}_3(s_3, x) \in F_3$
  means $\hat{\delta}_3((s_1, s_2), x) \in F_3$
  means $(\hat{\delta}_1(s_1, x), \hat{\delta}_2(s_2, x)) \in F_3$ [from Lemma 1]
  means $(\hat{\delta}_1(s_1, x), \hat{\delta}_2(s_2, x)) \in F_1 \times F_2$ [defn. of F_3]
  means $\hat{\delta}_1(s_1, x) \in F_1$ and $\hat{\delta}_2(s_2, x) \in F_2$
  means $x \in L(M_1) \cap L(M_2)$.

Closure under Union

De Morgan's Law – $A \cup B = \neg(\neg A \cap \neg B)$.
Construct (1) DFA for $\neg A$ and $\neg B$,
(2) then product DFA for $\neg A$ and $\neg B$ accepting $C = \neg A \cap \neg B$,
(3) then DFA for $\neg C$
to get DFA for $A \cup B$.