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  $k = 0 \implies A \to a$ kind of productions.
- Note: No $A \to \epsilon$ kind of productions. So ϵ as a string cannot be generated!

Normal Forms

For any CFC G, there is a CFG G' in Chomsky Normal Form (and a CFG G'' in Greibach Normal Form) such that
$L(G) - \{\epsilon\} = L(G')$ $(= L(G''))$

Conversion of CFGs to Normal Forms

- ϵ-production of the form $A \to \epsilon$, $A \in N$.

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- ϵ-production of the form $A \to \epsilon$, $A \in N$.
- Unit production of the form $A \to B$, $A, B \in N$.
- Normal forms: These productions have to be removed without changing $L(G) \setminus \{\epsilon\}$.

Conversion of CFGs to Normal Forms

- Let $\hat{P}$ be the smallest set of productions containing P and closed under two rules:
  (a) if $A \to \alpha B \beta, B \to \epsilon \in \hat{P}$, then $A \to \alpha\beta \in \hat{P}$,
  (b) if $A \to B, B \to \gamma \in \hat{P}$, then $A \to \gamma \in \hat{P}$.

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- $\hat{P}$: Keep adding new productions using the above rules on top of P.
- Each new RHS is a substring of an old RHS. P is finite: So $\hat{P}$ is finite.

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- $\hat{G} = (N, \Sigma, \hat{P}, S)$.
- As $P \subseteq \hat{P}$, $L(G) \subseteq L(\hat{G})$.
- $L(\hat{G}) = L(G)$: simulate each new production in two steps by the two old productions that created the new production.

Property of $\hat{G}$

- *Claim*: Minimum length derivation of $x \in \Sigma^* - \{\epsilon\}$ such that $S \to^*_{\hat{G}} x$ does not use any ϵ- or unit productions.
  Implication: They can be deleted from $\hat{G}$ to get $\tilde{G}$!

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- (i) Suppose for contradiction that an ϵ-production $B \rightarrow \epsilon$ is used in the minimum length derivation:
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- This implies, there is $A \rightarrow \alpha B \beta$ such that
  $S \rightarrow^m_{\hat{G}} \eta A \theta \rightarrow^1_{\hat{G}} \eta\alpha B \beta\theta \rightarrow^n_{\hat{G}} \gamma B \delta \rightarrow^1_{\hat{G}} \gamma\delta \rightarrow^k_{\hat{G}} x$.

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- But by rule (a), $A \rightarrow \alpha\beta \in \hat{P}$:
  $S \rightarrow^m_{\hat{G}} \eta A \theta \rightarrow^1_{\hat{G}} \eta\alpha\beta\theta \rightarrow^n_{\hat{G}} \gamma\delta \rightarrow^k_{\hat{G}} x$ is a shorter derivation ($\rightarrow\leftarrow$).

Property of $\hat{G}$

- (ii) Suppose for contradiction that a unit-production $A \to B$ is used in the minimum length derivation of x: $S \to^*_{\hat{G}} \gamma A \delta \to^1_{\hat{G}} \gamma B \delta \to^*_{\hat{G}} x$ (γ or δ must be non-null if $x \neq \epsilon$).

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- (ii) Suppose for contradiction that a unit-production $A \to B$ is used in the minimum length derivation of x:
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- This implies, there is a $B \to \alpha$ such that
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- But by rule (b), $A \rightarrow \alpha \in \hat{P}$:
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$S \to^m_{\hat{G}} \gamma A\delta \to^1_{\hat{G}} \gamma\alpha\delta \to^n_{\hat{G}} \eta\alpha\theta \to^k_{\hat{G}} x$ is a shorter derivation ($\to\leftarrow$).
- $\tilde{G}$: after throwing away ϵ- and unit productions from $\hat{G}$.

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- After this, all productions are of the form $A \to a$ or $A \to B_1 B_2 \ldots B_k$, $k \geq 2$ and $B_i \in N$.
- Take an $A \to B_1 B_2 \ldots B_k$ with $k \geq 3$ and introduce a new nonterminal C;
  Replace with $A \to B_1 C$ and $C \to B_2 B_3 \ldots B_k$.
  Keep doing this till all RHS's are of length at most 2.

Example 1

Balanced Parentheses

Start with $S \rightarrow [S] | SS | \epsilon$.

CNF for non-empty Balanced Parentheses:

$S \rightarrow AB | AC | SS, C \rightarrow SB, A \rightarrow [, B \rightarrow]$.

Example 2

$\{a^n b^n | n \geq 0\}$.
Start with $S \to aSb|\epsilon$.
CNF for $\{a^n b^n | n \geq 0\} - \{\epsilon\} = \{a^n b^n | n \geq 1\}$:
$S \to AB|AC, C \to SB, A \to a, B \to b$.