Equivalence of CFG and PDA

CFG $\rightarrow$ PDA $\rightarrow$ accepts by empty stack

$G = (N, \Sigma, P, S) \longmapsto (\{*\}, \Sigma, \Gamma, S, \delta, *, \emptyset) = M$

$\Gamma = N \cup \Sigma$

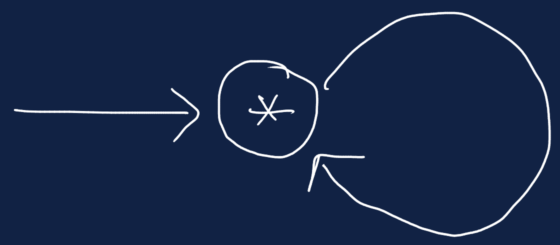
$A \rightarrow \gamma \qquad ((*, \epsilon, A), (*, \gamma)) \in \delta$

$\forall a \in \Sigma \qquad ((*, a, a), (*, \epsilon)) \in \delta$

$\mathcal{L}(G) = \mathcal{L}(M)$ [Exercise: Use induction]

M simulates a leftmost derivation of x in the language.

$E \to \epsilon \mid aEb \mid bEa \mid EE \qquad L = \mathcal{L}(E) = \{ w \in \{a,b\}^* \mid \#a(w) = \#b(w) \}$



$\epsilon, E/\epsilon$

$\epsilon, E/aEb$

$\epsilon, E/bEa$

$\epsilon, E/EE$

$a, a/\epsilon$

$b, b/\epsilon$

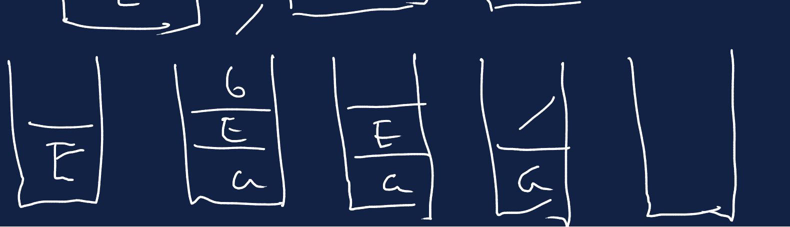
$baabba \in L$

$E \to EE \to EEE \to bEaEE$

$\to baEE \to baaEbE$

$\to baabE \to baabbEa \to baabba$

baabba



PDA $\to$ CFG

$M \longmapsto G$

Two-step process

(1) $M \longmapsto M'$ ← has only one state, accepts by empty stack

(2) $M' \longmapsto G$ ← reverse of the CFG → PDA conversion process

Step (2)

$$((*, a, A), (*, \gamma)) \in \delta$$

introduce $A \to a\gamma$

(straightforward to prove correctness)

Step (1)

Is M' general enough?

Yes.

$$M \longmapsto M' = (\{*\}, \Sigma, \Gamma', \perp', \delta', *, \emptyset)$$

accepts by empty stack

$$\Big\downarrow$$

$$(Q, \Sigma, \Gamma, \perp, \delta, s, \{t\})$$

only final state

M accepts by <u>both</u> empty stack and final state

M' "<u>stores</u>" the state information of M in its stack.

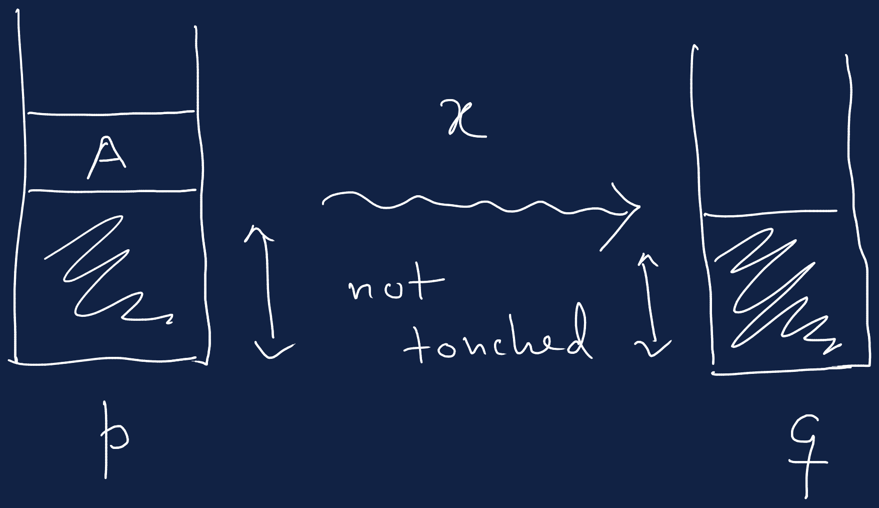
$$\Gamma' = Q \times \Gamma \times Q$$

$$\perp' = \langle s \perp t \rangle$$

$$\langle p\ A\ q \rangle$$

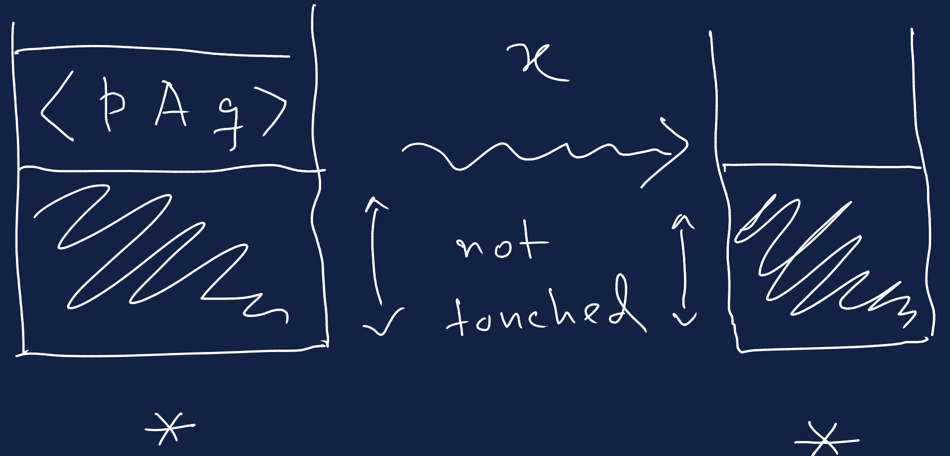
$$(p, A, q\rangle$$

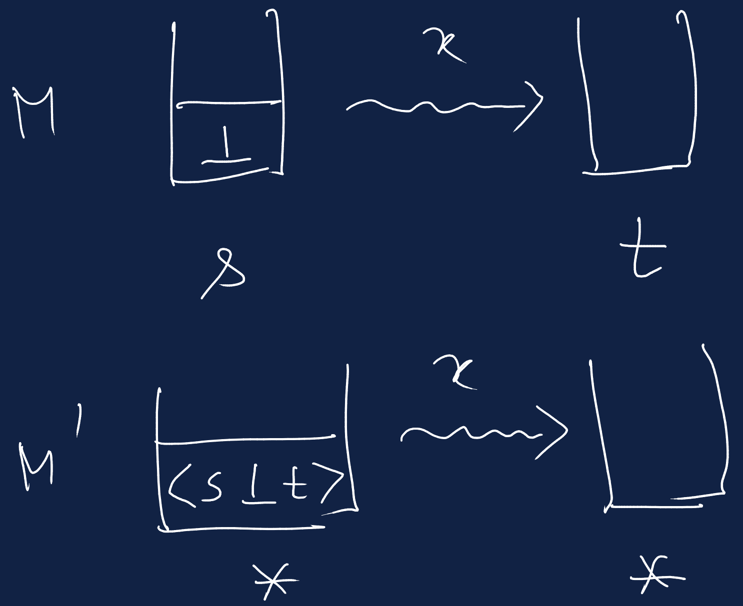
M



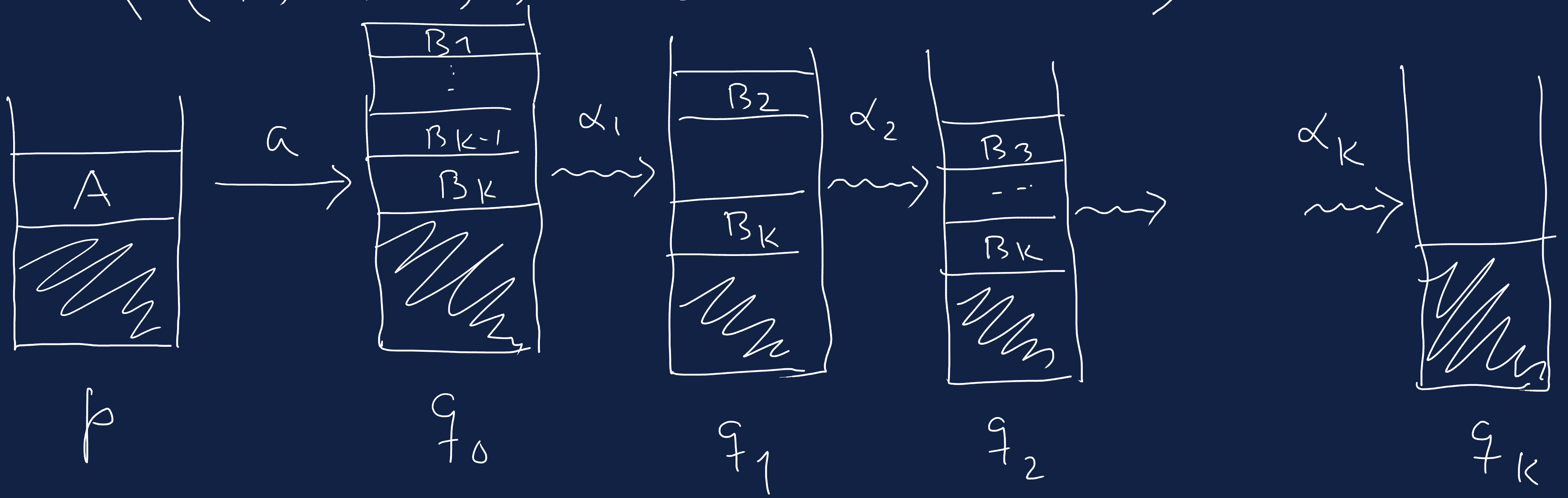
$$\perp' = \langle s \perp t \rangle$$

M'





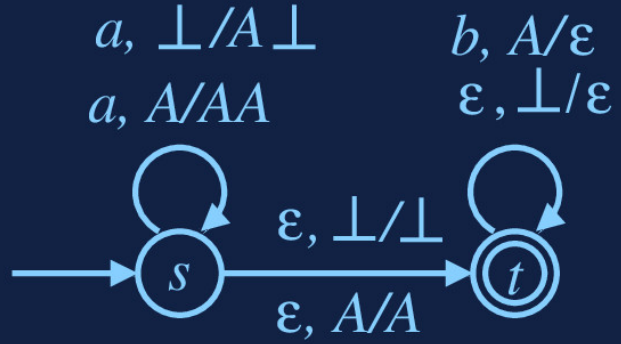
$M \qquad ((p, a, A), (q_0, B_1 B_2 \cdots B_k)) \in \delta$



$a \in \Sigma \cup \{\epsilon\}$ $\qquad$ $x = a\alpha_1\alpha_2 \cdots \alpha_k$, M' guesses $\underline{q_1, q_2, \ldots, q_k}$

$$((*, a, \langle p\, A\, q_k \rangle), (*, \langle q_0 B_1 q_1 \rangle \langle q_1 B_2 q_2 \rangle \cdots \langle q_{k-1} B_k q_k \rangle))$$

$\{ a^nb^n \mid n \geq 0 \}$



$((s,a,\perp),(s,A\perp))$

$((s,a,A),(s,AA))$

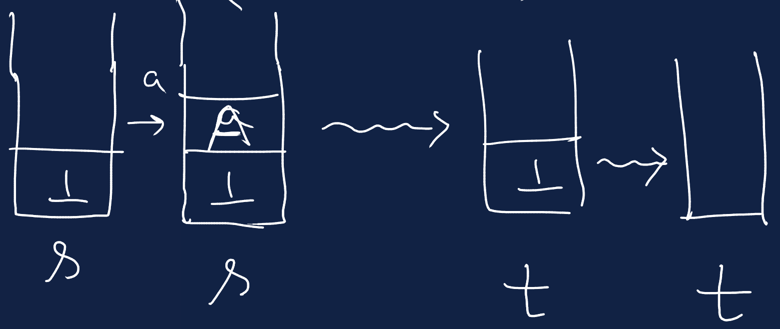
$((s,\varepsilon,\perp),(t,\perp))$

$((s,\varepsilon,A),(t,A))$

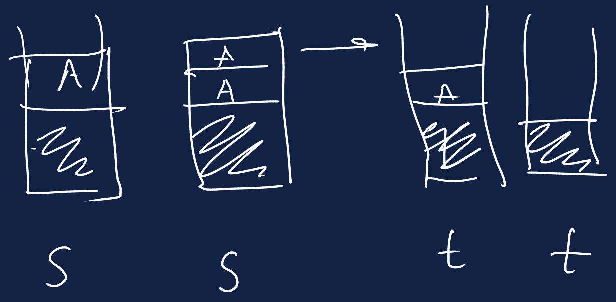
$((t,b,A),(t,\varepsilon))$

$((t,\varepsilon,\perp),(t,\varepsilon))$

useless $\left\{ \begin{matrix} \langle s\ A\ s\rangle & \langle s\perp s\rangle \\ \langle t\ A\ s\rangle & \langle t\perp s\rangle \end{matrix} \right.$



$\langle s\perp t\rangle$



$\langle s\perp t\rangle$	$\perp_s$
$\langle s\,A\,t\rangle$	A_s
$\langle t\perp t\rangle$	$\perp_t$
$\langle t\,A\,t\rangle$	A_t

$((*, a, \langle s\perp t\rangle), (*, \langle s\,A\,t\rangle\langle t\perp t\rangle))$

$((*, a, \langle s\,A\,t\rangle), (*, \langle s\,A\,t\rangle\langle t\,A\,t\rangle))$

$((*, \varepsilon, \langle s\perp t\rangle), (*, \langle t\perp t\rangle))$

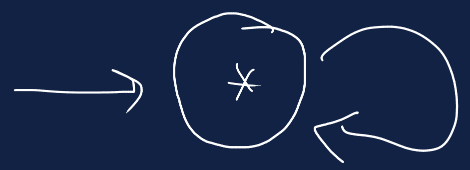
$((*, \varepsilon, \langle s\,A\,t\rangle), (*, \langle t\,A\,t\rangle))$

$((*, b, \langle t\,A\,t\rangle), (*, \varepsilon))$

$((*, \varepsilon, \langle t\perp t\rangle), (*, \varepsilon))$

$\Gamma = \{\bot_s, \bot_t, \hat{A}_s, \hat{A}_t\}$

aabb



$a, \bot_s / A_s \bot_t$

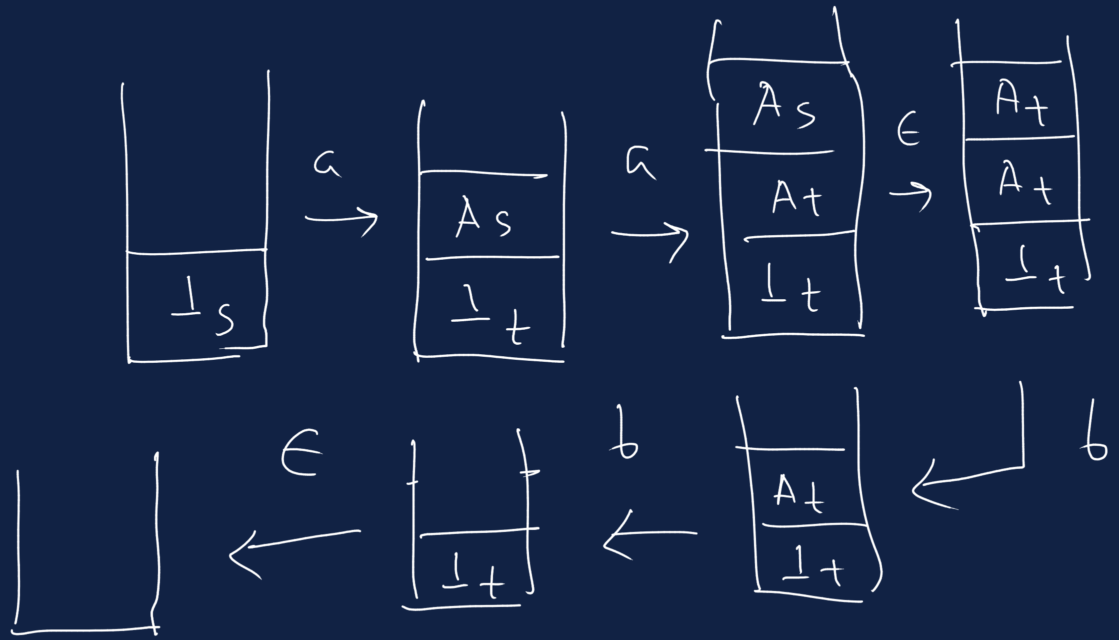
$a, \hat{A}_s / A_s \hat{A}_t$

$\epsilon, \bot_s / \bot_t$

$\epsilon, A_s / \hat{A}_t$

$b, \hat{A}_t / \epsilon$

$\epsilon, \bot_t / \epsilon$



Formal proof

$\mathcal{L}(M') = \mathcal{L}(M)$

Induction