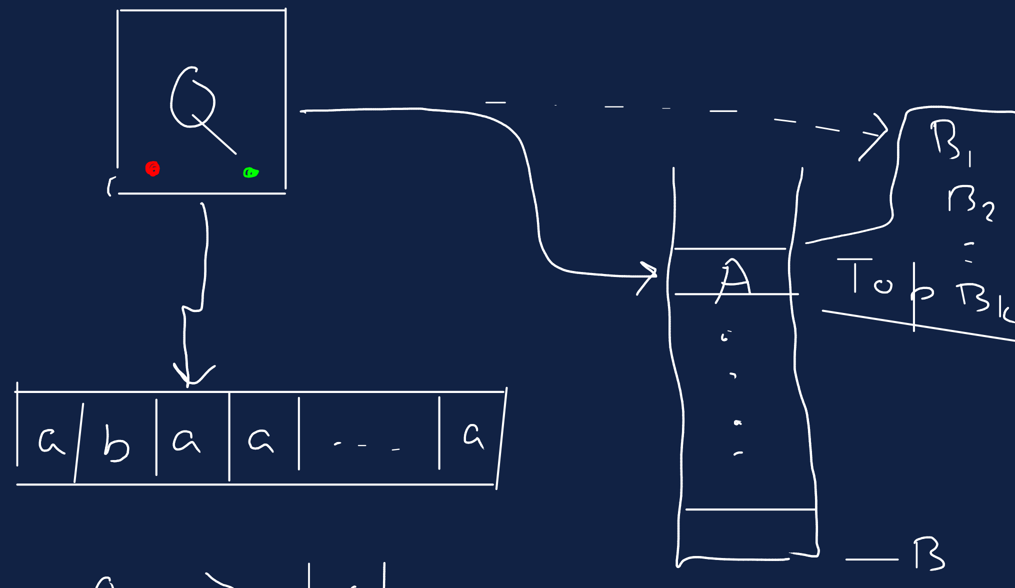
Pushdown Automata (PDA)

- $\Sigma \to$ input alphabet
- $\Gamma \to$ stack alphabet
- $\bot \in \Gamma$ (initial) → bottom-of-stack
- $Q \to$ a finite set of states
- $s \to$ a start state
- $F \to$ a set of final states
- $\delta \to$ a transition function



$k = 0 \Rightarrow$ pop

$k = 1, B_1 = A$

$\Rightarrow$ no change

stack

pushdown store

Input: p (state $\in Q$), $a \in \Sigma \cup \{\epsilon\}$, $A \in \Gamma$

Output: q (a state $\in Q$, may be the same as p)

A is replaced by $B_1 B_2 \ldots B_k \in \Gamma^*$ $(k \geq 0)$

Init:

state s

$(Q, \Sigma, \Gamma, \perp, \delta, s, F)$

Input x yet to be read

Stack

Finally — Going to a final state ($f \in F$)

[stack may contain anything]

— By emptying the stack

[$F = \emptyset$, no role of F]

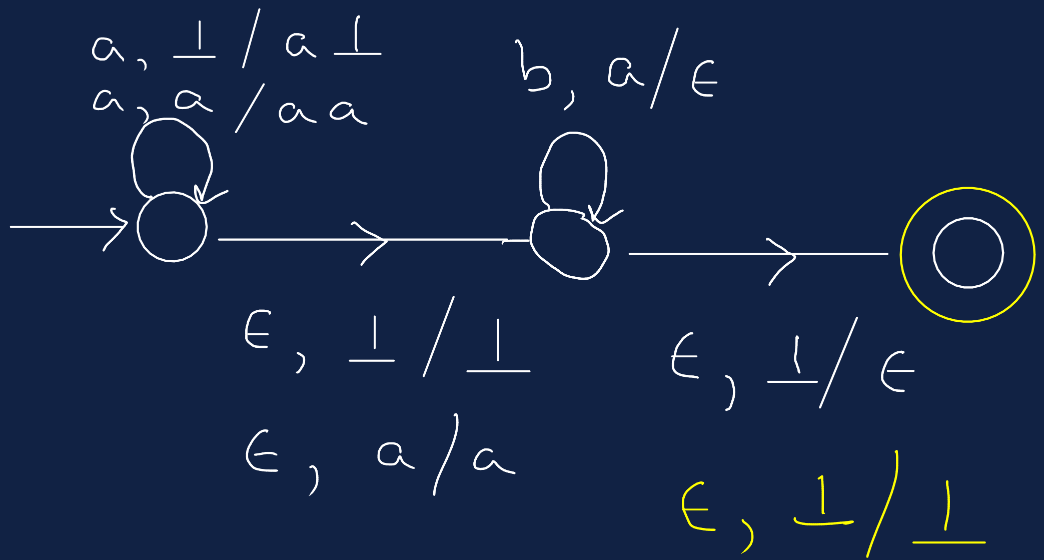
- Each transition requires some symbol at the top of the stack
- ϵ-transitions are allowed
- A PDA is non-deterministic by definition

Examples

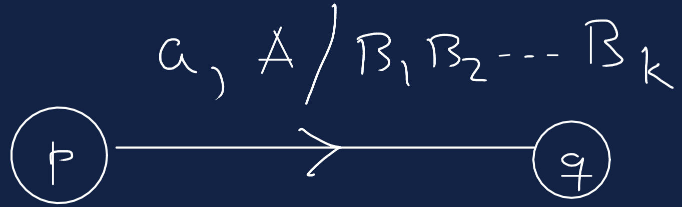
1. $\{a^n b^n \mid n \geq 0\}$

$\Sigma = \{a, b\}$

$\Gamma = \{\perp, a\}$



$\delta(p, a, A) \longrightarrow (q, B_1 B_2 \cdots B_k)$



Exercise:

$\{a^m b^n \mid m \geq n \geq 0\}$

2. $E = \{ w \in \{a, b\}^* \mid \#a(w) = \#b(w) \}$

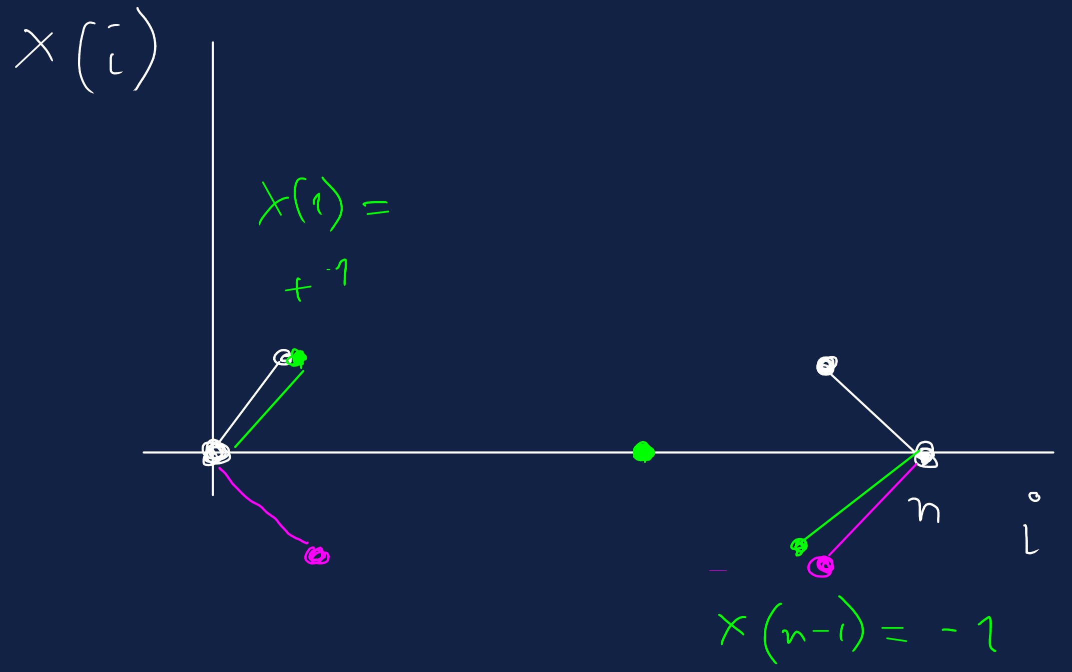
$$\begin{array}{ccccc} a_1 & a_2 & \cdots & & a_n \\ \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ 0 & 1 & 2 & & n \end{array}$$

$x(i) = \#a$ read so far $- \#b$ read so far

$x(0) = 0$

$x(n) = 0$

$x(i) - x(i-1) \in \{+1, -1\}$



Case 1

$a_1 = a$

$a_n = b$

Case 2

$a_1 = b$

$a_n = a$

Case 3

$a_1 = a$

$a_n = a$

$E \to \epsilon \mid aEb \mid bEa \mid EE$

A PDA for E

Acceptance by empty stack

$a, +/++$

$a, -/\epsilon$

$a, \perp/+\perp$

$b, +/\epsilon$

$b, -/--$

$b, \perp/-\perp$

$\epsilon, \perp/\epsilon$

Remember $X(i)$ in the stack

$X(i) = j > 0$ $\quad j$ $+$'s

$X(i) = -k < 0$ $\quad k$ $-$'s

ToS

$+ \xrightarrow{a} \begin{matrix} + \\ + \end{matrix}$

$\xrightarrow{b}$ pop

$- \xrightarrow{a}$ pop

$\xrightarrow{b} \begin{matrix} - \\ - \end{matrix}$

abaab

$\perp$ is exposed

3. $G = \{ w \in \{a, b\}^* \mid \#a(w) \geq \#b(w) \}$

Add this to the previous PDA

$\epsilon, + / \epsilon$ $\leftarrow$ throw away some excess +'s

$\perp$ is exposed at the end

$\not\Leftrightarrow$ $(\#a(w) - \#b(w))$ excess +'s have to be flushed

4 $L = \{ ww \mid w \in \{a, b\}^* \}$ is not CF

$\sim L$ is context-free

→ ① strings of odd length

② strings of the form (even length) → $2n$

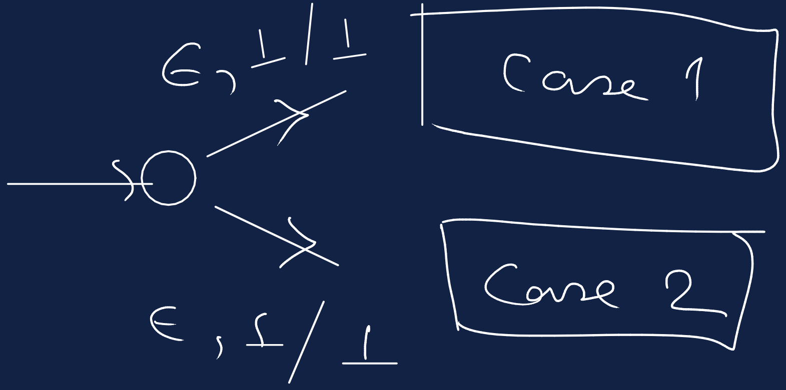
$c = a, \; d = b$

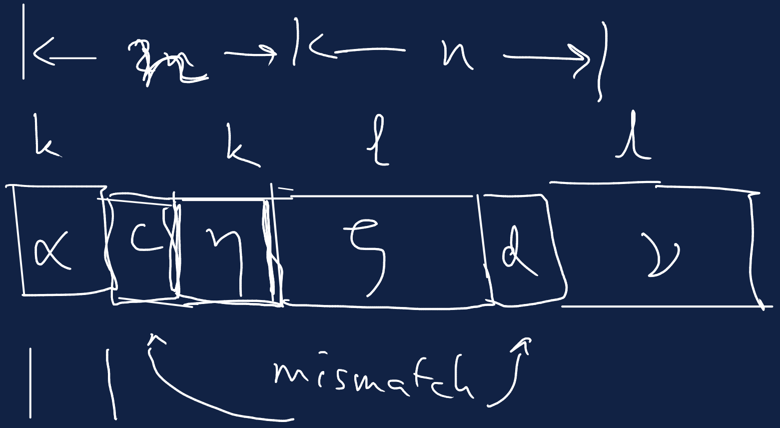
$c = b, \; d = a$

A PDA

— Nondeterministic

guess bet^n ① and ②





Read α : For each symbol, push ✓ to stack

$\downarrow$ nondeterministically decide that α is over

Read and remember c in the finite control

Read η : For each symbol, pop ✓

Read ζ : For each symbol, push ✓

$\downarrow$ nondet decide that ζ is over

Read d and compare with c — $c = d \longrightarrow$ reject

$c \neq d$ proceed

Read ν : For each symbol, pop ✓

If $\perp$ is exposed, go to a final state and accept.