Myhill–Nerode Relations

R – regular language

$M, N \longrightarrow$ two DFA with $\mathcal{L}(M) = \mathcal{L}(N) = R$.

$\hookrightarrow$ no inaccessible states

$M/\approx$

$N/\approx$

essentially the same machine

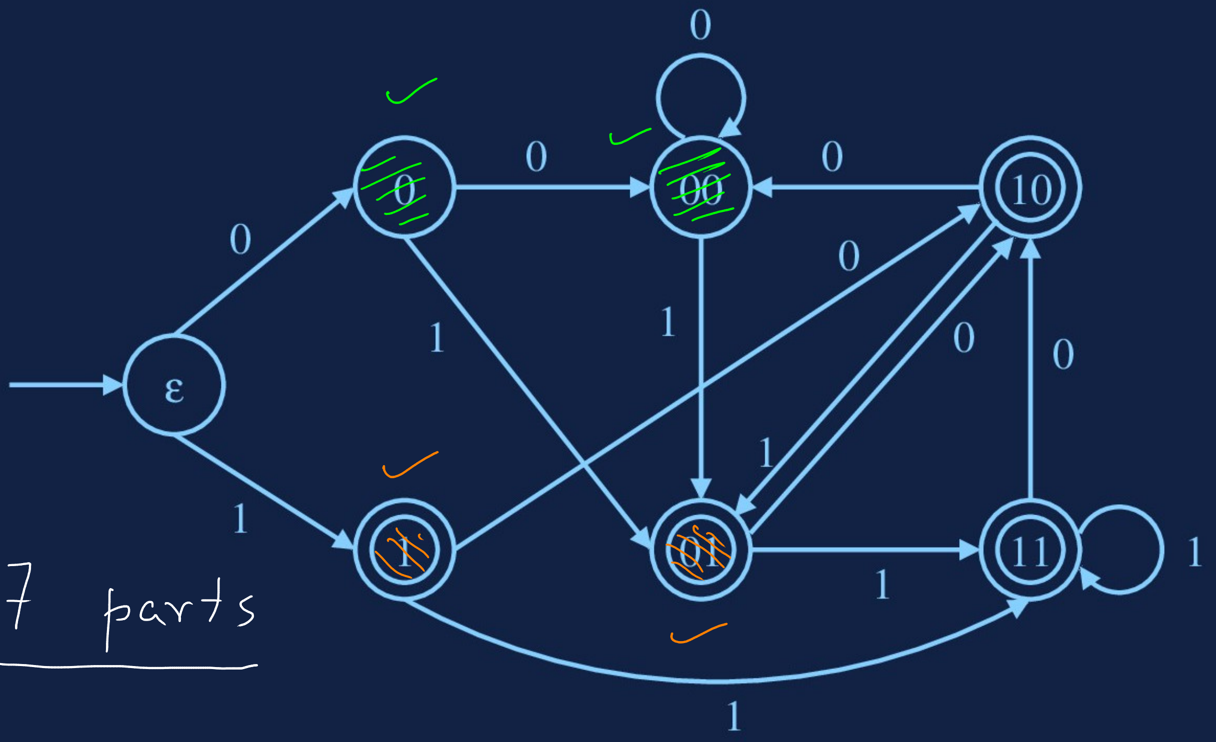
isomorphic

property of R

jump from machines to languages

and from languages to machines

R = {x ∈ {0,1}* | x contains at least one 1 in its last two positions}



7 parts

L_q, $q \in Q$, partition Σ^*

$\varepsilon \notin R$, $0 \notin R$, $1 \in R$

$x \in \Sigma^*$

$\hat{\delta}(s, x)$ – a unique state

$L_\varepsilon = \{\varepsilon\}$

$L_0 = \{0\}$, $L_1 = \{1\}$

$L_{00} = \mathcal{L}((0+1)^* 00\}$

$L_{01} = \mathcal{L}((0+1)^* 01\}$

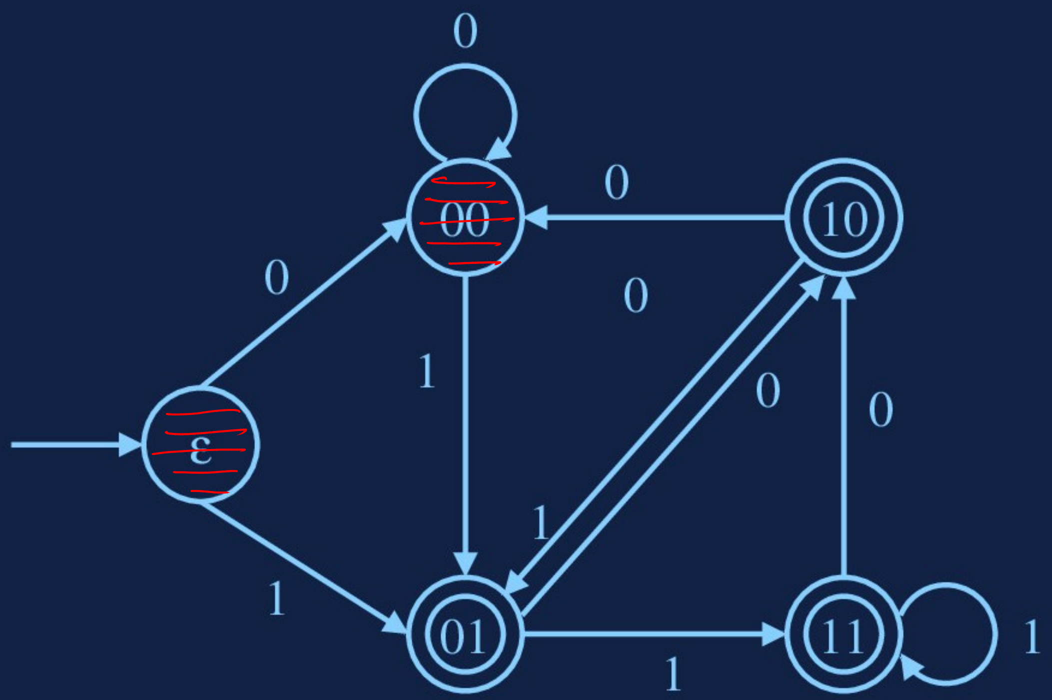
$L_{10} = \mathcal{L}((0+1)^* 10\}$

$L_{11} = \mathcal{L}((0+1)^* 11\}$

Each $q \in Q$ is associated with $\emptyset \neq L_q \subseteq \Sigma^*$

- $\bigcup_{q \in Q} L_q = \Sigma^*$
- $L_p \cap L_q = \emptyset$ if $p \neq q$

$R = \{x \in \{0,1\}^* \mid x$ contains at least one 1 in its last two positions$\}$



$L_\epsilon = \{\epsilon\}$

$L_{00} = \{0\} \cup \mathcal{L}((0+1)^* 00)$

$L_{01} = \{1\} \cup \mathcal{L}((0+1)^* 01)$

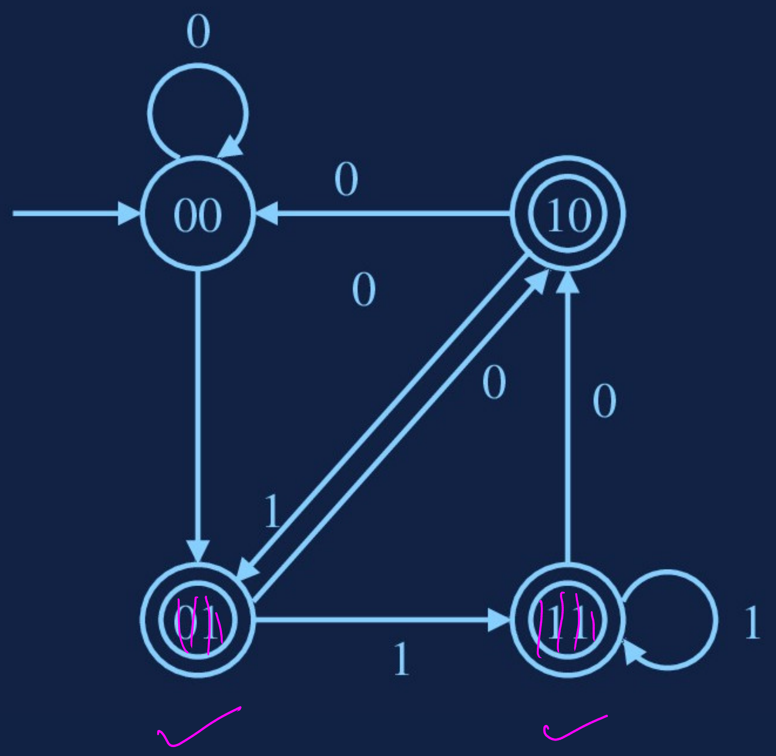
$L_{10} = \mathcal{L}((0+1)^* 10)$

$L_{11} = \mathcal{L}((0+1)^* 11)$

A new partition of Σ^*

5 parts (coarser than the previous partition)

$R = \{x \in \{0,1\}^* \mid x$ contains at least one 1 in its last two positions$\}$



$L_{00} = \{\epsilon, 0\} \cup \mathcal{L}((0+1)^* 00\}$

$L_{01} = \{1\} \cup \mathcal{L}((0+1)^* 01\}$

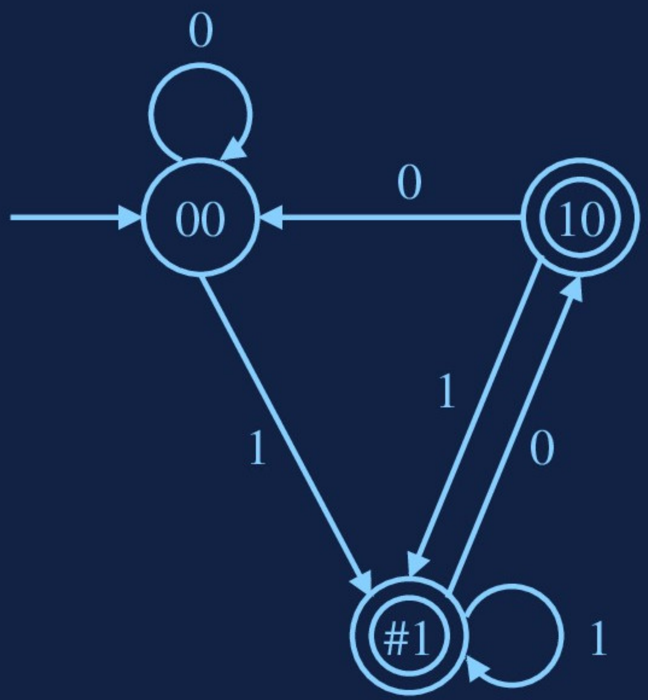
L_{10} — same as before

L_{11} —

Partition of Σ^* in 4 parts.

— even coarser

R = {x ∈ {0,1}* | x contains at least one 1 in its last two positions}



$L_{00} = \{\epsilon, 0\} \cup \mathcal{L}((0+1)^* 00)$

$L_{10} = \mathcal{L}((0+1)^* 10)$

$L_{\#1} = \mathcal{L}((0+1)^* 1)$

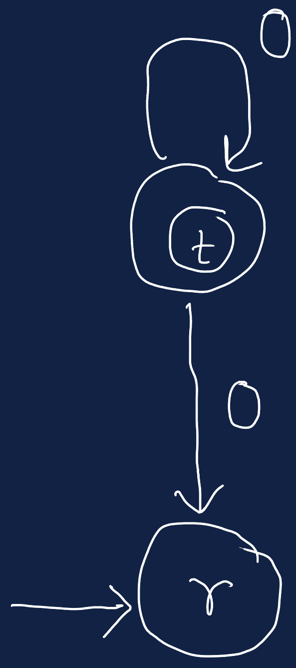
Partition of Σ^* in $\underline{\underline{3}}$ parts

Coarser

- No further collapsing possible
- The partition cannot be mady any coarser.

Not all partitions are realizable by DFA

R = {x ∈ {0,1}* | x contains at least one 1 in its last two positions}



not allowed for a DFA.

01
10 $\in R$

0 comes

$0\underline{10} \in R$

$1\underline{00} \notin R$

a final state (t) → R, $\Sigma^* - R$ ← a non-final state (r)

coarsest possible partition for R.

fine → more parts

coarse → fewer parts

Partitions and Equivalence Relations

$S \to$ a set

Equivalence relations on S have a one-to-one correspondence with all partitions of S

$\equiv$ partition — set of all equivalence classes

partition — Two elements are equivalent $\Leftrightarrow$ They belong to the same part.

Equivalence relations on Σ^*.

From DFA to Relations

$M = (Q, \Sigma, \delta, s, F)$ $\mathcal{L}(M) = R$

Define $\equiv_M$ on Σ^* s.t. $x \equiv_M y \iff \hat{\delta}(s, x) = \hat{\delta}(s, y)$

0 Equivalence relation

1 - Right congruence $x \equiv_M y \Rightarrow xa \equiv_M ya \quad \forall a \in \Sigma$

2 $\equiv_M$ refines R $x \equiv_M y \Rightarrow (x \in R \iff y \in R)$

3 $\equiv_M$ has finite index → the no of equivalence classes

An equivalence relation on Σ^* satisfying 1, 2 and 3 is called a Myhill-Nerode relation

From Relations to DFA

Input : A Myhill-Nerode relation $\equiv$ on Σ^*

Goal : To construct a DFA $M_\equiv = (Q, \Sigma, \delta, s, F)$

$$Q = \{ [x] \mid x \in \Sigma^* \} \longrightarrow \text{finite}$$

$$s = [\epsilon]$$

$$F = \{ [x] \mid x \in R \} \longrightarrow x \in R \iff [x] \in F$$

$$\delta([x], a) = [xa] \longrightarrow$$

well-defined by the right congruence property

$$\delta([x], a) = [xa]$$

$$\delta([x], y) = [xy]$$ Prove by induction on $|y|$.

Theorem: $\mathcal{L}(M_{\equiv}) = R$.

Proof: $x \in \mathcal{L}(M_{\equiv}) \iff \hat{\delta}([\epsilon], x) \in F$

$$\iff [x] \in F$$

$$\iff x \in R$$

$$M \longmapsto \equiv_M$$

$$\equiv \longmapsto M_{\equiv}$$

$$\equiv \longmapsto M_{\equiv} \longmapsto \equiv_{M_{\equiv}}$$

$$M \longmapsto \equiv_M \longmapsto M_{\equiv_M}$$

<u>Theorem</u> : $\equiv$ and $\equiv_{M_{\equiv}}$ are the same.

$$M_{\equiv} = (Q, \Sigma, \delta, s, F)$$

$$Q = \{ [x] \mid x \in \Sigma^* \}$$

$$s = [\epsilon], \quad F = \{ [x] \mid x \in R \}$$

$$\delta([x], a) = [xa]$$

$$\begin{aligned} x \equiv_{M_{\equiv}} y &\iff \hat{\delta}([\epsilon], x) = \hat{\delta}([\epsilon], y) \\ &\iff [x] = [y] \\ &\iff x \equiv y. \end{aligned}$$

$$M \longmapsto \equiv_M \longmapsto M_{\equiv_M}$$

$= (Q, \Sigma, \delta, s, F)$

individual states

$= (Q', \Sigma, \delta', s', F')$

equivalence classes

The same machine with renaming of states

$M = (Q_M, \Sigma, \delta_M, s_M, F_M)$

$N = (Q_N, \Sigma, \delta_N, s_N, F_N)$

M and N are isomorphic if $\exists$ a bijection $f : Q_M \to Q_N$ s.t.

(i) $\delta(s_M) = s_N$ (ii) $q \in F_M \iff f(q) \in F_N$

(iii) $f(\delta_M(q, a)) = \delta_N(f(q), a)$

M and $M_{\equiv_M}$ are isomorphic.

$$\begin{tikzcd}
M \arrow[d, equal] & M_{\equiv_M} \arrow[r] & (Q', \Sigma, \delta', s', F') \arrow[r] & Q' = \{ [x] \mid x \in \Sigma^* \} \\
(Q, \Sigma, \delta, s, F) \arrow[r] & \equiv_M & &
\end{tikzcd}$$

$x \equiv_M y \iff \hat{\delta}(s, x) = \hat{\delta}(s, y)$

$f : Q' \longrightarrow Q$

$[x] \longmapsto \hat{\delta}(s, x)$

– f is a bijection

injective $f([x]) = f([y])$

$\Rightarrow \hat{\delta}(s, x) = \hat{\delta}(s; y)$

$\Rightarrow x \equiv_M y$

$\Rightarrow [x] = [y]$

(i) $f(s') = f([\epsilon]) = \hat{\delta}(s, \epsilon) = s$

(iii) $f(\delta'([x], a)) = f([xa])$

$= \hat{\delta}(s, [xa]) = \delta(\hat{\delta}(s, x), a)$

$= \delta(f(x), a)$.

Surjective follows from that M does not contain inaccessible states.

Let R be a regular language.

<u>Theorem</u>: There is a one-to-one correspondence between

(1) the set of all DFA whose language is R

(2) the set of all Myhill-Nerode relations (that refine R).

<u>Theorem</u> one-to-one correspondence

(1) the set of all DFA

(2) the set of all MN relations.

Talk about R in algebraic terms.