Pumping Lemma (continued)

Example: $L = \{a^m b^n c^2 \mid m, n \geq 0,\ m \geq n\} \subseteq \{a, b, c\}^*$ is not regular

Proof: Suppose that L is regular.

Let k be a pumping-lemma constant for L.

Take $\alpha = xyz$ with $x = a^k$, $y = b^k$, $z = c^2$. $\qquad \alpha \in L$

$|y| = k \geq k$

$y = uvw$ such that (1) $|v| > 0$

(3) $\alpha_i = xuv^i wz \in L$ for all $i \geq 0$.

Take $i = 2$ $\qquad v = b^l$ for some $l > 0$.

$\alpha_2 = a^k b^{k+l} c^2 \in L$

Since $k < k+l$, we have $\alpha_2 \notin L$

Example: $L = \{ a^{n!} \mid n \geq 0 \} \subseteq \{a\}^*$.

Suppose that L is regular. Let k be a PLC for L.

Take $x = z = \epsilon$ and $y = a^{k!}$ $[\alpha = xyz = y \in L]$

$|y| = k! \geq k$

$y = uvw$ s.t. $|v| > 0$, $|uv| \leq k$,

$\alpha_i = xuv^iwz = uv^iw \in L \ \forall i \geq 0$.

Take $i = (k+1)! + 1$

$v = a^l$ for $1 \leq l \leq k$

$\alpha_i \in L$ $|\alpha_i| = k! + l(k+1)! = p!$ for some $p > k+1$

$$\underbrace{1 + l(k+1)}_{\text{not a multiple of } k+1} = \overbrace{(k+1)(k+2)(k+3)\cdots p}^{\text{a multiple of } k+1}$$

$$\boxed{k+1 \geq 1+1 = 2}$$

Using the closure properties of regular languages

Example: $L = \{ w \in \{a,b\}^* \mid \#a(w) = \#b(w) \}$

ϵ, abbaab, aabb, baab $\in L$

Suppose that L is regular.

$\mathcal{L}(a^* b^*)$ is regular.

$L \cap \mathcal{L}(a^* b^*)$ is also regular [closure under intersection]

$= \{ a^n b^n \mid n \geq 0 \}$

$\longrightarrow$ contradiction

Exercise: Prove using the PL.

Example: $L = \{ a^m b^n \mid m \geq n \}$

Suppose L is regular $\Rightarrow$ rev $L = \{ b^n a^m \mid m \geq n \}$ is also regular [closure under reversal]

$L' = \{ a^n b^m \mid m \geq n \}$ is also regular

$= \{ a^m b^n \mid m \leq n \}$ is also regular

$L \cap L'$ is regular

$= \{ a^m b^n \mid m = n \}$

Ultimate Periodicity

Def: A subset $S \subseteq \mathbb{N}_0$ is called ultimately periodic if there exist constants $n_0 \in \mathbb{N}_0$ and $p \in \mathbb{N}$ such that for all $n \geq n_0$, $n \in S \Leftrightarrow n+p \in S$.

p is called a period of S.

Example (1) $\{\underline{0, 1, 4, 8}, 10, 15, 20, 25, 30, 35, \ldots\}$ is u.p. with period 5 (or 10, 20, 15, 1000).

$(n_0 = 10)$

(2) $\{1, 3, 8\} \cup \{10, 13, 16, 19, 22, 25, \ldots\}$ $\rightarrow 3$ is u.p.

$(n_0 = 10)$

$\cup \{11, 13, 15, 17, 19, 21, 23, \ldots\}$ $\rightarrow 2$ with 6 as a period

$= \{1, 3, 8, 1\underline{0}, 11, 13, 15, \underline{16}, 17, 19, 21, 22, 23, 25, \ldots\}$ $\rightarrow$ $\mathrm{lcm}(2,3) = 6$

Theorem: Let $L \subseteq \{a\}^*$. Then $\{m \mid a^m \in L\}$ is ultimately periodic if and only if L is regular.

Proof: $\{m \mid a^m \in L\}$ is infinite

[Exercise: Handle the case when this is a finite set.]

Exercise: Any finite subset of $\mathbb{N}_0$ is u.p.

[if] Let D be a DFA with $\mathcal{L}(D) = L$.

deterministic

for any state p of D, $\delta(p, a)$ is unique.

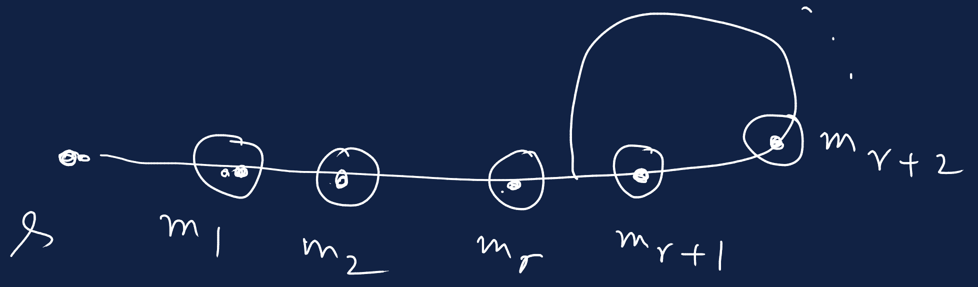
Some state must eventually be repeated.

$\{m \mid a^m \in L\}$ is u.p. where a period p is the length of the cycle.

[Only if] $\{m \mid a^m \in L\}$ is u-p.

$$\{m_1, m_2, \ldots, m_r \mid m_{r+1}, m_{r+2}, \ldots, m_{r+1}+p, m_{r+2}+p, \ldots, m_{r+1}+2p, m_{r+2}+2p, \ldots\}$$

n_0



D

$\mathcal{L}(D) = L$

Applications

- $\{a^{n^2} \mid n \geq 0\}$ — not regular

  $\{n^2 \mid n \geq 0\} = \{0, 1, 4, 9, 16, 25, 36, 49, \dots\}$ is not ultimately periodic

- $\{a^{2^n} \mid n \geq 0\}$ — not regular

  $\{2^n \mid n \geq 0\} = \{1, 2, 4, 8, 16, 32, 64, \dots\}$ is not u.p.

- $\{a^{n!} \mid n \geq 0\}$ — not regular

  $\{n! \mid n \geq 0\} = \{1, 2, 6, 24, 120, 720, 4320, \dots\}$ is not u.p.

What about a general alphabet Σ

$L \subseteq \Sigma^*$ $\quad$ $\text{lengths}(L) = \{ m \in \mathbb{N}_0 \mid m = |w| \text{ for some } w \in L \}$

L is regular $\overset{?}{\Leftarrow}\!\!\!\!\!\overset{?}{\Rightarrow}$ $\text{lengths}(A)$ is ultimately periodic.

$\Leftarrow$ Not true

$L = \{ a^n b^n \mid n \geq 0 \}$

$\text{lengths}(L) = \{0, 2, 4, 6, 8, 10, 12, \ldots\}$ is u.p.

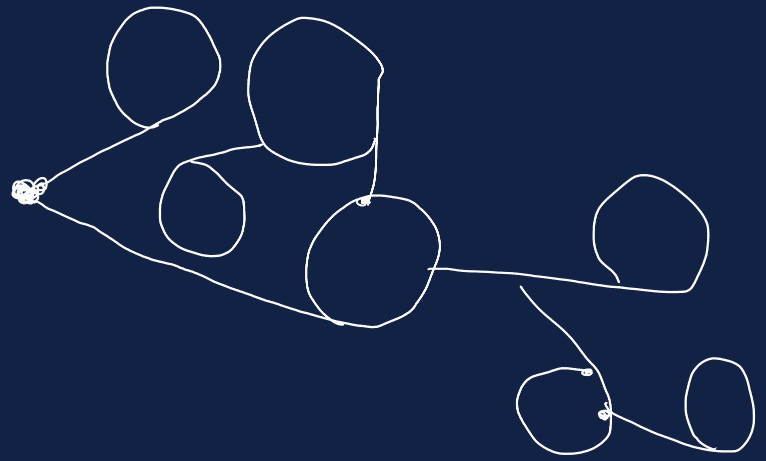
But L is <u>not</u> regular.

$\Rightarrow$ <u>True</u>

L is regular $\Rightarrow$ lengths(L) is ultimately periodic.

$|\Sigma|$ is finite.

$L = \mathcal{L}(D)$ for a DFA D.



D may have multiple cycles (if $|\Sigma| \geq 2$)

$\boxed{\text{finitely many}}$

lengths(L) is ultimately periodic with a period equal to the lcm of all the cycle lengths.

finite no. finite

Can also be proved using homomorphism

Example : $L = \{ w \in \{a, b\}^* \mid \#a(w) = (\#b(w))^2 \}$

$$\text{lengths}(A) = \{ n + n^2 \mid n \geq 0 \}$$

is <u>not</u> ultimately periodic

$\Rightarrow$ L is <u>not</u> regular.

- Using a DFA
- Pumping lemma
- Closure properties
- Ultimate periodicity

To prove certain languages are <u>not</u> regular.