CS21201 Discrete Structures

Tutorial 9

Recurrences

1. The triangular numbers are defined as $t_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for $n \geqslant 0$. Define $a_n = \sum_{i=0}^{n} t_i$ for $n \geqslant 0$. Find a recurrence relation for a_n, and solve it. Use the standard procedure for solving linear recurrences. Do not use any summation formula.

Solution The recurrence is $a_n - a_{n-1} = t_n = \frac{n(n+1)}{2}$ for $n \geqslant 1$ with the initial condition $a_0 = t_0 = 0$. The characteristic equation is $r-1=0$, that is, $r=1$. The non-homogeneous part is $\frac{n(n+1)}{2} \times 1^n$. Therefore the particular solution is $a_n^{(p)} = n(Un^2+Vn+W)$. Putting this in the recurrence gives

$$n(Un^2+Vn+W) = (n-1)(U(n-1)^2+V(n-1)+W) + \frac{n(n+1)}{2}.$$

Put $n=0$ (or collect the constant terms from both sides) to get $-U+V-W=0$. Put $n=1$ to get $U+V+W=1$. Finally, put $n=-1$ to get $-(U-V+W) = -2(4U-2V+W)$, that is, $4U-2V+W=0$. The first two equations give $V = U+W = \frac{1}{2}$. The third equation gives $4U+W=1$, that is, $3U=\frac{1}{2}$, that is, $U=\frac{1}{6}$, so $W=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$. This we have obtained

$$a_n^{(p)} = \frac{1}{6}n(n^2+3n+2) = \frac{1}{6}n(n+1)(n+2).$$

It then follows that $a_n = A + \frac{1}{6}n(n+1)(n+2)$. Put $n=0$ to get $A=0$, so

$$a_n = \frac{1}{6}n(n^2+3n+2) = \frac{1}{6}n(n+1)(n+2) \text{ for all } n \geqslant 0.$$

2. Let a_n, $n \geqslant 0$, be the count of strings over $\{0,1,2\}$ containing no consecutive 1's and no consecutive 2's. Find a recurrence relation for a_n, and solve it.

Solution Let b_n, c_n, d_n, $n \geqslant 1$, denote the counts of the strings of the desired form that start with $0,1,2$, respectively. Let us also take $b_0 = 1$, $c_0 = 0$ and $d_0 = 0$. We have the following equations involving these.

$$\begin{aligned}
a_n &= b_n + c_n + d_n \text{ for all } n \geqslant 0,\\
b_n &= a_{n-1} \text{ for all } n \geqslant 1,\\
c_n &= b_{n-1} + d_{n-1} = a_{n-1} - c_{n-1} \text{ for all } n \geqslant 1,\\
d_n &= b_{n-1} + c_{n-1} = a_{n-1} - d_{n-1} \text{ for all } n \geqslant 1.
\end{aligned}$$

Adding the last three equations gives

$$a_n = 3a_{n-1} - (c_{n-1}+d_{n-1}) = 3a_{n-1} - (a_{n-1}-b_{n-1}) = 2a_{n-1} + b_{n-1} = 2a_{n-1} + a_{n-2}$$

for all $n \geqslant 2$. The initial conditions are $a_0 = 1$, $a_1 = 3$. The characteristic equation of the sequence is $r^2 - 2r - 1 = 0$. The roots of the characteristic equation are $1+\sqrt{2}, 1-\sqrt{2}$.

So the general solution of this recurrence is of the form

$$a_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n.$$

The initial conditions give

$$\begin{aligned}
a_0 &= 1 = A+B,\\
a_1 &= 3 = A(1+\sqrt{2}) + B(1-\sqrt{2}).
\end{aligned}$$

Solving gives $A = \frac{1+\sqrt{2}}{2}$ and $B = \frac{1-\sqrt{2}}{2}$.

Therefore, the final solution is:

$$a_n = \frac{1}{2}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right] \text{ for all } n \geqslant 0.$$

3. Let D_n, $n \geqslant 1$, denote that number of derangements (permutations without fixed points) of $1, 2, 3, \ldots, n$.

(a) Prove that $D_n = (n-1)(D_{n-1} + D_{n-2})$ for all $n \geqslant 3$.

(b) Deduce that $D_n = nD_{n-1} + (-1)^n$ for all $n \geqslant 3$.

(c) Solve for D_n.

Solution (a) Let $x_1, x_2, \ldots, x_n$ be a derangement of $1, 2, \ldots, n$. Then $x_n = i$ for some $i \in \{1, 2, \ldots, n-1\}$. Fix an i, and consider the following two cases:

Case 1: $x_i = n$. Then, $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}$ is a derangement of $1, 2, \ldots, i-1, i+1, \ldots, n-1$, and the number of such derangements is D_{n-2}.

Case 2: $x_i \neq n$. Rename n as i occurring somewhere in the first $n-1$ positions (except the i-th one). But then, $x_1, x_2, \ldots, x_{n-1}$ is a derangement of $1, 2, \ldots, n-1$. The number of such derangements is D_{n-1}.

Now, varying i over all of the $n-1$ allowed values, we get:

$$D_n = (n-1)\big(D_{n-1} + D_{n-2}\big) \text{ for all } n \geqslant 3 \qquad (\text{Note that } D_1 = 0,\ D_2 = 1).$$

(b) $D_n = (n-1)(D_{n-1} + D_{n-2}) \quad \Rightarrow \quad D_n - nD_{n-1} = (-1)\big[D_{n-1} - (n-1)D_{n-2}\big]$

We recursively simplify the right-hand side of the equation above to get:

$$\begin{aligned} D_n - nD_{n-1} &= (-1)\big[D_{n-1} - (n-1)D_{n-2}\big] \\ &= (-1)^2\big[D_{n-2} - (n-2)D_{n-3}\big] \\ &= (-1)^3\big[D_{n-3} - (n-3)D_{n-4}\big] \\ &= \cdots = (-1)^{n-2}\big[D_2 - 2D_1\big] \end{aligned}$$

Since $D_1 = 0$ and $D_2 = 1$, we get: $D_n - nD_{n-1} = (-1)^{n-2} \quad \Rightarrow \quad D_n = nD_{n-1} + (-1)^n$.

(c) Diving both side of the given recurrence by $n!$, we get

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$$

Unwinding gives

$$\begin{aligned} \frac{D_n}{n!} &= \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \frac{D_{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \frac{D_{n-3}}{(n-3)!} + \frac{(-1)^{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} \\ &= \cdots = \frac{D_0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \end{aligned}$$

$$\therefore D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

4. Let a_n, $n \geqslant 1$, satisfy $a_1 = 1$, and $a_n = \begin{cases} 2a_{n-1} & \text{if } n \text{ is odd} \\ 2a_{n-1} + 1 & \text{if } n \text{ is even} \end{cases}$ for $n \geqslant 2$. Develop a recurrence relation for a_n, that holds for both odd and even n, and solve it.

Solution For both cases, $a_n - a_{n-2} = 2(a_{n-1} - a_{n-3}) \Rightarrow a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$. Moreover, $a_2 = 3$ and $a_3 = 6$.

The characteristics equation is $r^3 - 2r^2 - r + 2 = 0$. Solving this, we get the three roots as $-1, 1, 2$.

So the general solution is $\quad a_n = A(-1)^n + B1^n + C2^n$.

The initial conditions give $a_1 = -A + B + 2C = 1$, $a_2 = A + B + 4C = 3$ and $a_3 = -A + B + 8C = 6$.

Solving these, we get $A = \frac{1}{6}$, $B = -\frac{1}{2}$, and $C = \frac{5}{6}$.

Therefore $a_n = \frac{1}{6}(-1)^n - \frac{1}{2}1^n + \frac{5}{6}2^n = \frac{1}{6}\big[5 \times 2^n + (-1)^n - 3\big]$.

5. Solve the following recurrence relation, and deduce the closed-form expression for a_n.

$$a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3} + 2^n \text{ (for } n \geqslant 3) \quad \text{with } a_0 = 1,\ a_1 = 1,\ a_2 = \frac{83}{5}.$$

Solution The characteristic equation is $r^3 - r^2 - 8r + 12 = 0$, that is, $(r-2)^2(r+3) = 0$, that is, $r = 2$ (double root) and $r = -3$. Therefore, the homogeneous solution is $a_n^{(h)} = (A + Bn)2^n + C(-3)^n$.

Since 2 is a root of the characteristic equation with multiplicity 2, the particular solution is $a_n^{(p)} = Dn^2 2^n$. The recurrence gives

$$\begin{aligned} Dn^2 2^n &= D(n-1)^2 2^{n-1} + 8D(n-2)^2 2^{n-2} - 12D(n-3)^2 2^{n-3} + 2^n \\ \Rightarrow \quad Dn^2 &= \frac{D}{2}(n-1)^2 + 2D(n-2)^2 - \frac{3}{2}D(n-3)^2 + 1. \end{aligned}$$

Comparing the constant terms (coefficients of n^0) in above equation, we find

$$0 = \frac{D}{2} + 8D - \frac{27}{2} + 1 \quad \Rightarrow \quad D = \frac{1}{5}.$$

So the general form of the solution is

$$a_n = a_n^{(h)} + a_n^{(p)} = (A + Bn)2^n + C(-3)^n + \frac{1}{5}n^2 2^n.$$

The initial conditions give the equations

$$\begin{aligned} a_0 = 1 = A + C \quad &\Rightarrow \quad C = 1 - A, \\ a_1 = 1 = 2(A+B) - 3C + \frac{2}{5} \quad &\Rightarrow \quad 5A + 2B = \frac{18}{5}, \\ a_2 = \frac{83}{5} = 4(A + 2B) + 9C + \frac{16}{5} \quad &\Rightarrow \quad -5A + 8B = \frac{22}{5}. \end{aligned}$$

Solving the system gives $A = \frac{2}{5}$, $B = \frac{4}{5}$, $C = \frac{3}{5}$. Hence, the final solution is

$$a_n = \left(\frac{2}{5} + \frac{4}{5}n\right)2^n + \frac{3}{5}(-3)^n + \frac{1}{5}n^2 2^n = \frac{1}{5}\left[(1+2n)2^{n+1} + (-1)^n 3^{n+1} + n^2 2^n\right].$$

6. Solve the recurrence relation $a_n = na_{n-1} + n(n-1)a_{n-2} + n!$ for $n \geqslant 2$, with $a_0 = 0$, $a_1 = 1$.

Solution Diving both side of the given recurrence by $n!$, we get,

$$\frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!} + \frac{a_{n-2}}{(n-2)!} + 1$$

Let, $b_n = \frac{a_n}{n!}$. So, we have, $b_n = b_{n-1} + b_{n-2} + 1$ with $b_0 = 0,\ b_1 = 1$.

Homogeneous Solution: $b_n^{(h)} = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ (similar as Fibonacci sequence).

Particular Solution: $b_n^{(p)} = U \times 1^n = U$.

From the recurrence, we get $U = U + U + 1 \Rightarrow U = -1$.

Final Solution (general form): $b_n = b_n^{(h)} + b_n^{(p)} = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n - 1$.

Using the initial conditions $b_0 = 0,\ b_1 = 1$, we get $A_1 = \frac{3+\sqrt{5}}{2\sqrt{5}}$ and $A_2 = -\frac{3-\sqrt{5}}{2\sqrt{5}}$.

Therefore $b_n = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - 1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right] - 1$, so

$$a_n = n!b_n = \frac{n!}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2} - \sqrt{5}\right] \quad \text{for } n \geqslant 0$$

7. Solve the recurrence relation: $a_0 = 1$, $a_1 = 2$, and $a_n = \dfrac{2a_{n-1}^3}{a_{n-2}^2}$ for $n \geqslant 2$.

Solution Taking log (base 2) of the given recurrence, we get

$$\log a_n = \log 2 + 3\log a_{n-1} - 2\log a_{n-2}$$

Let $b_n = \log a_n$. So we have $b_n = 3b_{n-1} - 2b_{n-2} + 1$ with $b_0 = 0,\ b_1 = 1$.

Homogeneous Solution: The characteristics equation is $r^2 - 3r + 2 = 0$. Since the roots of this equation are 1 and 2, we can write $b_n^{(h)} = A_1 \times 1^n + A_2 \times 2^n = A_1 + A_2 2^n$.

Particular Solution: $b_n^{(p)} = nU1^n = Un$.

From the recurrence, we get $Un = 3U(n-1) - 2U(n-2) + 1 \Rightarrow U = -1$.

Final Solution (general form): $b_n = b_n^{(h)} + b_n^{(p)} = A_1 + A_2 2^n - n$. Using the initial conditions, we get $b_0 = 0,\ b_1 = 1$. These give $A_1 = -2$ and $A_2 = 2$. Therefore $b_n = -2 + 2 \times 2^n - n = 2^{n+1} - n - 2$, that is,

$$a_n = 2^{b_n} = 2^{2^{n+1}-n-2} \quad \text{for } n \geqslant 0.$$

8. How many lines are printed by the call $f(n)$ for an integer $n \geqslant 0$?

```
void f ( int n )
{
   int m;
   printf("Hi\n");
   m = n - 1;
   while (m >= 0) { f(m); m -= 2; }
}
```

Solution Suppose that L_n be the number of lines printed by the call $f(n)$. We have $L_0 = 1, L_1 = 2$, and

$$L_n = \begin{cases} 1 + \sum_{k=0}^{(n-1)/2} L_{2k}, & \text{if } n \text{ is odd} \\ 1 + \sum_{k=0}^{(n-2)/2} L_{2k+1}, & \text{if } n \text{ is even} \end{cases} \quad \text{for } n \geqslant 2$$

In both the cases, we derive $L_n - L_{n-2} = L_{n-1}$. This is the same recurrence as satisfied by the Fibonacci numbers. Moreover, $L_0 = 1 = F_2$ and $L_2 = 2 = F_3$. It therefore follows that $L_n = F_{n+2}$ for all $n \geqslant 0$, that is,

$$L_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right].$$

9. Consider a non-homogeneous recurrence of the form

$$a_n = c_{k-1}a_{n-1} + c_{k-2}a_{n-2} + \cdots + c_0 a_{n-k} + p_1(n)s_1^n + p_2(n)s_2^n.$$

Here, $c_{k-1}, c_{k-2}, \ldots, c_0$ are constants (with $c_0 \neq 0$), $p_1(n)$ and $p_2(n)$ are non-zero polynomials in n, and s_1, s_2 are distinct non-zero constants. Propose a method to solve this recurrence.

Solution Consider two new sequences u_n and v_n satisfying the recurrences

$$u_n = c_{k-1}u_{n-1} + c_{k-2}u_{n-2} + \cdots + c_0 u_{n-k} + p_1(n)s_1^n,$$

and

$$v_n = c_{k-1}v_{n-1} + c_{k-2}v_{n-2} + \cdots + c_0 v_{n-k} + p_2(n)s_2^n.$$

These recurrences can be solved because the non-homogeneous part in each is in the standard form. Adding the two recurrences gives

$$u_n + v_n = c_{k-1}(u_{n-1} + v_{n-1}) + c_{k-2}(u_{n-2} + v_{n-2}) + \cdots + c_0(u_{n-k} + v_{n-k}) + p_1(n)s_1^n + p_2(n)s_2^n.$$

Therefore $a_n = u_n + v_n$ is the solution of the given recurrence.

What about the initial conditions? Suppose that the values of $a_0, a_1, a_2, \ldots, a_{k-1}$ are supplied. We need to choose $u_0, u_1, u_2, \ldots, u_{k-1}$ and $v_0, v_1, v_2, \ldots, v_{k-1}$ so that $a_n = u_n + v_n$ for $n = 0, 1, 2, \ldots, k-1$ as well. To ensure that, we can choose the initial conditions for the u and v sequences in any manner we like. For example, we can take $u_n = a_n$ and $v_n = 0$ for $n = 0, 1, 2, \ldots, k-1$.

Additional Exercises

10. Consider a linear recurrence relation with constant coefficients having characteristic equation $(x-r)^{\mu}$ for some $\mu \in \mathbb{N}$, and with a non-homogeneous part $f(n) = n^t r^n$. Using the theory of generating functions, prove that the particular solution for this recurrence relation is of the form

$$n^{\mu}(u_t n^t + u_{t-1} n^{t-1} + \cdot + u_2 t^2 + u_1 t + u_0) r^n.$$

11. Foosia and Barland play a long series of ODI matches. In the first game, Foosia bats first. After that, the team that wins a match must bat first in the next match. For each team, the probability of win is p if it bats first. Assume that $0 < p < 1$. Find the probability p_n that Foosia wins the n-th match. What is $\lim_{n\to\infty} p_n$?

12. Pell numbers are defined as $P_0 = 0$, $P_1 = 2$, $P_n = 2P_{n-1} + P_{n-2}$ for $n \geqslant 2$.

(a) Deduce a closed-form formula for P_n.
(b) Prove that $\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ for all $n \geqslant 1$.
(c) Prove that $\lim_{n\to\infty} \dfrac{P_{n-1} + P_n}{P_n} = \sqrt{2}$.
(d) Prove that if P_n is prime, then n is also prime.

13. The Pell–Lucas numbers are defined as $Q_0 = Q_1 = 2$, $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geqslant 2$.

(a) Deduce a closed-form formula for Q_n.
(b) Prove that $Q_n = P_{2n}/P_n$ for all $n \geqslant 1$.

14. Let $a_0 = 1$, and $a_n = \frac{5}{2} a_{n-1} - a_{n-2}$ for all $n \geqslant 2$. Find a_1 such that the sequence a_n converges.

15. A set of natural numbers is called *selfish* if it contains its size as a member. Let s_n denote the number of selfish subsets of $\{1,2,3,\ldots,n\}$ for $n \geqslant 1$. Develop a recurrence relation for s_n, and solve it.

16. Let us call a selfish set A minimal if no proper subset of A is selfish. Let S_n denote the number of minimal selfish subsets of $\{1,2,3,\ldots,n\}$. Develop a recurrence relation for S_n, and solve it.

17. Let a_n, $n \geqslant 0$, denote the number of binary strings of length n, not containing the pattern 101. Develop a recurrence relation for a_n, and solve it.

18. Solve the recurrence relation: $a_0 = 1$, $a_1 = 2$, $a_n = a_{n-1} + 2a_{n-2} + n^2 + 2^n$ for $n \geqslant 2$.

19. Solve the recurrence relation: $a_0 = 1$, $a_1 = 2$, $a_n = 4a_{n-2} + 2^n + n3^n$ for $n \geqslant 2$.

20. Solve the recurrence relation: $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-1} + a_{n-2} - a_{n-3} + n^2 + n + (-1)^n$ for $n \geqslant 3$.

21. Solve the recurrence relation $na_n = (n+1)a_{n-1} + 2n$ for $n \geqslant 1$, with the initial condition $a_0 = 0$.

22. Solve the recurrence relation $a_0 = \frac{2}{3}$, and $a_n = 2a_{n-1}^2 - 1$ for $n \geqslant 1$.

23. A sequence a_n is defined recursively as $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and $a_n = \dfrac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$ for $n \geqslant 4$. Prove that a_n is an integer for all $n \in \mathbb{N}$.

24. Consider the recurrence relation $a_n = a_{n-1} + 3a_{n-2} - a_{n-3}$ for $n \geqslant 3$. Find a matrix A such that $\begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = A \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$. Express $\begin{pmatrix} a_{3n} \\ a_{3n+1} \\ a_{3n+2} \end{pmatrix}$ in terms of A and a_0, a_1, a_2, for all $n \in \mathbb{N}_0$.

25. So far, we have solved recurrence relations. In this exercise, we reverse this process, that is, from a sequence, we generate a recurrence relation, of which the given sequence is a solution. We concentrate only on linear recurrence relations with constant coefficients (homogeneous/non-homogeneous). Solve the following parts for the given sequences with the orders of the recurrence relations as specified.

(a) $(2+\sqrt{3})^n + (2-\sqrt{3})^n$, order two.
(b) $2^n + 3^n$, order two.

(c) $2^n + 3^n$, order one.
(d) $2^n + n3^n$, order two.
(e) $2^n + n3^n$, order one.
(f) $2^n + n3^n + n^2 4^n$, order three.
(g) $2^n + n3^n + n^2 4^n$, order two.
(h) $2^n + n3^n + n^2 4^n$, order one.

26. Let $A(x) = 1 + \frac{1}{\sqrt{1-2x}}$ be the generating function of a sequence a_n, $n \geqslant 0$. Develop a recurrence relation for the sequence.

27. Let a_n denote the number of strings w of length n over the alphabet $\{A, C, G, T\}$ such that the number of T in w is a multiple of 3. Find a closed-form expression for a_n.

28. How many lines are printed by the call $g(n, 0, 0)$ for an integer $n \geqslant 0$?

```
void g ( int n, int i, int flag )
{
   if (i == n) { printf("Hola\n"); return; }
   g(n, i+1, flag); g(n, i+1, flag); g(n, i+1, flag);
   if (flag == 0) g(n, i+1, 1);
}
```

29. **(a)** How many strings of length n over the alphabet $\{A, C, G, T\}$ are there, in which T never appears after A? Notice that there is no restriction on the appearances of T before the first occurrence of A.

(b) Modify the function `g()` of the last exercise so as to print precisely the strings of Part (a).

30. How many strings of length n over the alphabet $\{A, C, G, T\}$ are there, in which the pattern TT (two consecutive T's) never appears after A? Note that TT may appear before the first occurrence of A, and that single isolated T's may appear after A.