CS21201 Discrete Structures

Tutorial 8

Generating Functions

1. Find the generating function of the sequence $1, 2, 0, 3, 4, 0, 5, 6, 0, 7, 8, 0, \ldots$.

Solution Let us decompose the sequence as follows.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
		−3			−6			−9			−12	0	0	−15	...
			−1	−1		−2	−2		−3	−3		−4	−4		...
1	2	0	3	4	0	5	6	0	7	8	0	9	10	0	...

Therefore the generating function of the given sequence is

$$
\begin{aligned}
& \Big(1+2x+3x^2+4x^3+\cdots\Big) - 3x^2\Big(1+2x^3+3x^6+4x^9+\cdots\Big) - (x^3+x^4)(1+2x^3+3x^6+4x^9+\cdots) \\
= \; & \frac{1}{(1-x)^2} - \frac{3x^2}{(1-x^3)^2} - \frac{x^3(1+x)}{(1-x^3)^2} \\
= \; & \frac{1+2x+x^3}{(1-x^3)^2}.
\end{aligned}
$$

2. Let $A(x)$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. Express the generating function of the sequence $a_0 + a_1, a_2 + a_3, a_4 + a_5, a_6 + a_7, \ldots$ in terms of $A(\,)$.

Solution The generating function of the given sequence is

$$
\begin{aligned}
& (a_0+a_1)+(a_2+a_3)x+(a_4+a_5)x^2+(a_6+a_7)x^3+\cdots \\
= \; & \Big(a_0+a_2x+a_4x^2+a_6x^3+\cdots\Big)+\Big(a_1+a_3x+a_5x^2+a_7x^3+\cdots\Big) \\
= \; & \left(\frac{A(\sqrt{x})+A(-\sqrt{x})}{2}\right)+\left(\frac{A(\sqrt{x})-A(-\sqrt{x})}{2\sqrt{x}}\right).
\end{aligned}
$$

3. Let F_n, $n \geqslant 0$, denote the Fibonacci numbers. Prove that $\displaystyle\sum_{n\in\mathbb{N}_0} \frac{F_n}{2^n} = 2$.

Solution The generating function of the Fibonacci sequence is

$$
\sum_{n\in\mathbb{N}_0} F_n x^n = \frac{x}{1-x-x^2} = \frac{x}{(1-\rho x)(1-\bar{\rho}x)} = \frac{A}{1-\rho x} + \frac{B}{1-\bar{\rho}x},
$$

where $\rho = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$ is the golden ratio, $\bar{\rho} = \frac{1-\sqrt{5}}{2} - 0.6180339887\cdots$ is the conjugate of the golden ratio, and A, B are constant real numbers. Since $|\rho/2|$ and $|\bar{\rho}/2|$ are less than 1, the power series converge for $x = \frac{1}{2}$, and we get

$$
\sum_{n\in\mathbb{N}_0} \frac{F_n}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}-(\frac{1}{2})^2} = 2.
$$

4. Let a_n, $n \geqslant 0$, be the sequence satisfying

$$
\begin{aligned}
a_0 &= 1, \\
a_n &= 2+2a_0+2a_1+2a_2+\cdots+2a_{n-2}+a_{n-1} \quad \text{for } n \geqslant 1.
\end{aligned}
$$

Deduce that the generating function of this sequence is $\frac{1+x}{1-2x-x^2}$. Solve for a_n.

Solution The generating function of the sequence is

$$\begin{aligned}
A(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\
&= 1 + (2 + a_0)x + (2 + 2a_0 + a_1)x^2 + (2 + 2a_0 + 2a_1 + a_2)x^3 + \cdots + \\
&\quad (2 + 2a_0 + 2a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1})x^n + \cdots \\
&= 1 + 2x\left(1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots\right) + 2x^2\left(a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots\right) + \\
&\quad x\left(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + \cdots\right) \\
&= 1 + \frac{2x}{1-x} + 2x^2\left(\frac{A(x)}{1-x}\right) + xA(x).
\end{aligned}$$

It therefore follows that

$$\left(1 - \frac{2x^2}{1-x} - x\right)A(x) = 1 + \frac{2x}{1-x},$$

that is,

$$(1 - 2x - x^2)A(x) = 1 + x,$$

that is,

$$A(x) = \frac{1+x}{1-2x-x^2}.$$

Now, use the fact that $1 - 2x - x^2 = \left(1 - \left(1+\sqrt{2}\right)x\right)\left(1 - \left(1-\sqrt{2}\right)x\right)$.

5. The generating function $A(x)$ of a sequence $a_0, a_1, a_2, a_3, \ldots$ satisfies $A'(x) = 1 + A(x)$. Prove that $A(x) = (a_0 + 1)e^x - 1$.

Solution We have $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$. Equating the coefficients of $x^0, x^1, x^2, x^3, \ldots$ from the two sides gives $a_1 = a_0 + 1$, $2a_2 = a_1$, that is, $a_2 = (a_0+1)/2!$, $3a_3 = a_2$, that is, $a_3 = (a_0+1)/3!$, and so on.

6. A test has four sections. Section A contains many questions of 2 marks each. Section B has many questions of 5 marks each. Section C has a single question of 4 marks. Section D has a single question of 1 mark. Assume that the questions in Sections A, B and D are of objective numerical type. You either get full marks or zero. The Section C question is essay-type, and you can get an integer mark in the range $[0,4]$. In how many ways can you get a total of n marks?

Solution The relevant generating function is

$$\begin{aligned}
&(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)(1 + x + x^2 + x^3 + x^4)(1 + x) \\
= &\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)(1+x) \\
= &\frac{1}{(1-x)^2} \\
= &1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots.
\end{aligned}$$

The desired answer is therefore $n+1$.

7. Consider the sequence $a_0, a_1, a_2, \ldots$ defined recursively as follows.

$$\begin{aligned}
a_0 &= 0, \\
a_1 &= 1, \\
a_2 &= 2, \\
a_n &= 2a_{n-2} + a_{n-3} + 2 \text{ for all } n \geqslant 3.
\end{aligned}$$

(a) Derive a closed-form expression for the generating function $A(x)$ of the sequence.

Solution We have

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \sum_{n\geqslant 3} a_n x^n \\ &= x + 2x^2 + \sum_{n\geqslant 3} (2a_{n-2} + a_{n-3} + 2)x^n \\ &= x + 2x^2 + 2x^2 \sum_{n\geqslant 3} a_{n-2}x^{n-2} + x^3 \sum_{n\geqslant 3} a_{n-3}x^{n-3} + 2x^3 \sum_{n\geqslant 3} x^{n-3} \\ &= x + 2x^2 + 2x^2\big(A(x) - 0\big) + x^3 A(x) + \frac{2x^3}{1-x} \\ &= (2x^2 + x^3)A(x) + \frac{x + x^2}{1-x}. \end{aligned}$$

$$A(x) = \frac{x + x^2}{(1-x)(1-2x^2-x^3)} = \frac{x(1+x)}{(1-x)(1+x)(1-x-x^2)} = \frac{x}{(1-x)(1-x-x^2)}.$$

(b) From the closed-form expression of $A(x)$ derived in Part (a), establish that $a_n = F_{n+2} - 1$ for all $n \geqslant 0$, where $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.

Solution We can write

$$\frac{x}{(1-x)(1-x-x^2)} = \frac{A}{1-x} + \frac{Bx + c}{1-x-x^2}.$$

Solving gives $A = -1$ and $B = C = 1$, that is,

$$A(x) = \frac{1+x}{1-x-x^2} - \frac{1}{1-x}.$$

The OGF of the Fibonacci sequence is $\frac{x}{1-x-x^2}$, that is, $\frac{x}{1-x-x^2}$ generates $F_0, F_1, F_2, \ldots, F_n, \ldots$. This implies that $\frac{1}{1-x-x^2}$ generates $F_1, F_2, F_3, \ldots, F_{n+1}, \ldots$. Finally, $\frac{1}{1-x}$ generates $1, 1, 1, \ldots$. Therefore, we have $a_n = F_n + F_{n+1} - 1 = F_{n+2} - 1$ for all $n \geqslant 0$.

8. Let u, v, s, t be positive constant values. Consider the sequence $a_0, a_1, a_2, \ldots$ defined recursively as follows.

$$\begin{aligned} a_0 &= u, \\ a_1 &= v, \\ a_n &= sa_{n-1} + ta_{n-2} \text{ for all } n \geqslant 2. \end{aligned}$$

For $n \geqslant 3$, we have $a_n = sa_{n-1} + ta_{n-2} = s\big(sa_{n-2} + ta_{n-3}\big) + ta_{n-2} = (s^2 + t)a_{n-2} + sta_{n-3}$. In view of this, consider the sequence $b_0, b_1, b_2, \ldots$ defined recursively as follows.

$$\begin{aligned} b_0 &= u, \\ b_1 &= v, \\ b_2 &= sv + tu, \\ b_n &= (s^2 + t)b_{n-2} + stb_{n-3} \text{ for all } n \geqslant 3. \end{aligned}$$

Demonstrate that the generating functions of both the sequences are the same.

Solution We have

$$\begin{aligned} A(x) &= a_0 + a_1 x + \sum_{n\geqslant 2} a_n x^n \\ &= u + vx + \sum_{n\geqslant 2} (sa_{n-1} + ta_{n-2})x^n \\ &= u + vx + sx \sum_{n\geqslant 2} a_{n-1}x^{n-1} + tx^2 \sum_{n\geqslant 2} a_{n-2}x^{n-2} \\ &= u + vx + sx\big(A(x) - u\big) + tx^2 A(x), \end{aligned}$$

that is,

$$A(x) = \frac{u + (v - su)x}{1 - sx - tx^2}.$$

On the other hand, we have

$$\begin{aligned} B(x) &= b_0 + b_1 x + b_2 x^2 + \sum_{n \geqslant 3} b_n x^n \\ &= u + vx + (sv + tu)x^2 + \sum_{n \geqslant 3} \Big((s^2 + t) b_{n-2} + st b_{n-3} \Big) x^n \\ &= u + vx + (sv + tu)x^2 + (s^2 + t)x \sum_{n \geqslant 3} b_{n-2} x^{n-2} + stx^2 \sum_{n \geqslant 3} b_{n-3} x^{n-3} \\ &= u + vx + (sv + tu)x^2 + (s^2 + t)x^2 \big(B(x) - u \big) + stx^3 B(x). \end{aligned}$$

This simplifies to

$$B(x) = \frac{u + vx + (sv + tu - s^2 u - tu)x^2}{1 - (s^2 + t)x^2 - stx^3} = \frac{u + vx + (sv - s^2 u)x^2}{1 - (s^2 + t)x^2 - stx^3}.$$

Now, look at the numerator and the denominator of $A(x)$. We have

$$(1 + sx)(u + (v - su)x) = u + (v - su)x + sux + s(v - su)x^2 = u + vx + (sv - s^2 u)x^2,$$

whereas

$$(1 + sx)(1 - sx - tx^2) = 1 - sx - tx^2 + sx - s^2x^2 - stx^3 = 1 - (s^2 + t)x^2 - stx^3.$$

We can therefore cancel $1 + sx$ from the numerator and the denominator of $B(x)$ to conclude that $A(x) = B(x)$.

9. (a) For $n \in \mathbb{N}$, denote by $\sigma(n)$ the sum of all positive integral divisors of n. We also take $\sigma(0) = 0$. Find the generating function of the sequence $\sigma(0), \sigma(1), \sigma(2), \ldots, \sigma(n), \ldots$.

Solution $\displaystyle\sum_{n \geqslant 1} (nx^n + nx^{2n} + nx^{3n} + \cdots) = \sum_{n \geqslant 1} \frac{nx^n}{1 - x^n}.$

(b) If $u, v \in \mathbb{N}$ are coprime, prove that $\sigma(uv) = \sigma(u)\sigma(v)$. Hence deduce a closed-form expression for $\sigma(n)$ with n having the prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$.

Solution We have $\sigma(p^e) = 1 + p + p^2 + \cdots + p^e = \dfrac{p^{e+1} - 1}{p - 1}$. By the multiplicative property, we therefore have

$$\sigma(n) = \prod_{i=1}^{t} \left(\frac{p_i^{e_i + 1} - 1}{p_i - 1} \right).$$

10. Using generating functions, solve the following mutually defined recurrences.

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 2, \\ b_0 &= 3, \\ a_n &= a_{n-1} + b_n \text{ for } n \geqslant 1, \\ b_n &= b_{n-1} + a_{n-2} \text{ for } n \geqslant 2. \end{aligned}$$

Solution We have

$$A(x) = a_0 + \sum_{n \geqslant 1} a_n x^n = 1 + \sum_{n \geqslant 1} (a_{n-1} + b_n)x^n = 1 + xA(x) + (B(x) - b_0) = xA(x) + B(x) - 2,$$

that is,

$$(1-x)A(x) - B(x) = -2.$$

We have $b_1 = a_1 - a_0 = 1$, and so

$$\begin{aligned}
B(x) &= b_0 + b_1 x + \sum_{n\geqslant 2} b_n x^n \\
&= 3 + x + \sum_{n\geqslant 2} (b_{n-1} + a_{n-2}) x^n \\
&= 3 + x + x(B(x) - b_0) + x^2 A(x) \\
&= 3 - 2x + xB(x) + x^2 A(x),
\end{aligned}$$

that is,

$$-x^2 A(x) + (1-x)B(x) = 3 - 2x.$$

Solve for $A(x)$ and $B(x)$ from these two linear equations.

$$\left((1-x)^2 - x^2\right)A(x) = -2(1-x) + (3-2x),$$

that is, $A(x) = \frac{1}{1-2x}$. Consequently, $B(x) = (1-x)A(x) + 2 = \frac{1-x}{1-2x} + 2 = \frac{1/2}{1-2x} + 5/2$. It follows that $a_n = 2^n$ for all $n \geqslant 0$, whereas $b_n = \begin{cases} 3 & \text{if } n = 0, \\ 2^{n-1} & \text{if } n \geqslant 1. \end{cases}$

11. Let the two-variable sequence $a_{m,n}$ be recursively defined as follows.

$$a_{m,n} = \begin{cases} 1 & \text{if } m = 0 \text{ or } n = 0, \\ a_{m-1,n} + a_{m,n-1} & \text{if } m \geqslant 1 \text{ and } n \geqslant 1. \end{cases}$$

Find the generating function $A(x,y) = \sum_{m,n\geqslant 0} a_{m,n} x^m y^n$. From this, derive a closed-form formula for $a_{m,n}$.

Solution We have

$$\begin{aligned}
A(x,y) &= \sum_{m,n\geqslant 0} a_{m,n} x^m y^n \\
&= 1 + \sum_{m\geqslant 1} x^m + \sum_{n\geqslant 1} y^n + \sum_{m,n\geqslant 1} (a_{m-1,n} + a_{m,n-1}) x^m y^n \\
&= 1 + \frac{x}{1-x} + \frac{y}{1-y} + \sum_{m,n\geqslant 1} a_{m-1,n} x^m y^n + \sum_{m,n\geqslant 1} a_{m,n-1} x^m y^n.
\end{aligned}$$

Replacing $m-1$ by m in the first sum gives

$$\sum_{m,n\geqslant 1} a_{m-1,n} x^m y^n = x \sum_{\substack{m\geqslant 0 \\ n\geqslant 1}} a_{m,n} x^m y^n = x\left[\sum_{\substack{m\geqslant 0 \\ n\geqslant 0}} a_{m,n} x^m y^n\right] - x\left[\sum_{m\geqslant 0} x^m\right] = xA(x,y) - \frac{x}{1-x}.$$

Likewise, the second sum is $yA(x,y) - \frac{y}{1-y}$. Using these expressions gives

$$A(x,y) = \frac{1}{1-x-y} = 1 + (x+y) + (x+y)^2 + (x+y)^3 + \cdots + (x+y)^{m+n} + \cdots.$$

This gives $a_{m,n} = \binom{m+n}{m} = \binom{m+n}{n}$.

Additional Exercises

12. Find the generating functions of the following sequences.

(a) $1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$
(b) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$
(c) $1, 3, 5, 7, 9, 11, 13, \ldots$
(d) $2, 4, 8, 14, 22, 32, 44, \ldots$
(e) $1, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0, \ldots$
(f) $1, 2, 0, 3, 4, 0, 5, 6, 0, 7, 8, 0, \ldots$
(g) $1/1, 1/2, 1/3, 1/4, 1/5, \ldots$
(h) $H_0, H_1, H_2, H_3, \ldots$ (where H_n is the n-th harmonic number)

13. Let $A(x)$ be the generating function for the sequence $a_0, a_1, a_2, \ldots$. Express the generating functions of the following sequences in terms of $A(x)$.

(a) $a_0, 2a_1, 3a_2, 4a_3, 5a_4, \ldots$
(b) $a_0, a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots$
(c) $a_0, a_0 - a_1, a_0 - a_1 + a_2, a_0 - a_1 + a_2 - a_3, \ldots$
(d) $a_0, a_1, a_2 - a_0, a_3 - a_1, a_4 - a_2 + a_0, a_5 - a_3 + a_1, a_6 - a_4 + a_2 - a_0, a_7 - a_5 + a_3 - a_1, \ldots$
(e) $a_0 + a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots$
(f) $a_0 + a_1, a_2 + a_3, a_4 + a_5, a_6 + a_7, \ldots$

14. Let V be a non-negative integer-valued random variable with $\Pr[V = n] = p_n$. The probability generating function $P(x)$ of V satisfies the identity $P'(x) = \xi + P(x)$ for some real constant ξ. Deduce a closed-form expression for $P(x)$. From this expression, derive the allowed values of ξ.

15. Let $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ be the sequence generated by $\sum_{r \in \mathbb{N}} \frac{x^r}{1 - x^r}$. Denote by p_n the parity of a_n, that is, $p_n = a_n \pmod 2$, that is, $p_n = \begin{cases} 0 & \text{if } a_n \text{ is even,} \\ 1 & \text{if } a_n \text{ is odd.} \end{cases}$ Determine all $n \in \mathbb{N}$, for which $p_n = 1$. Justify.

16. Let $a_n = \sum_{i=n}^{\infty} \frac{2^i}{i!}$ for all integers $n \geqslant 0$.

(a) Find a closed-form expression for the generating function $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$ of the sequence $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$.

(b) Use the expression for $A(x)$ in Part (a) to prove that $\sum_{n=0}^{\infty} a_n = 3e^2$.

17. Let $m \geqslant 1$ be an integer constant. Let $b_n^{(m)}$ denote the number of ordered partitions (that is, compositions) of the integer $n \geqslant 0$ such that no summand is larger than m.

(a) Prove that the generating function of $b_n^{(m)}$ is

$$B^{(m)}(x) = \frac{1 - x}{1 - 2x + x^{m+1}}.$$

(b) From the formula of Part (a), deduce that $b_n^{(2)} = F_{n+1}$, where $F_0, F_1, F_2, \ldots$ is the sequence of Fibonacci numbers.

18. Let l_n be the number of lines printed by the call $f(n)$ for some integer $n \geqslant 0$.

```
void f ( int n )
{
    int i, j;
    printf("Hi\n");
    if (n == 0) return;
    for (i=0; i<=n-1; ++i)
        for (j=0; j<=i; ++j)
            f(j);
}
```

(a) Let $L(x) = l_0 + l_1 x + l_2 x^2 + \cdots + l_n x^n + \cdots$ be the generating function of the sequence $l_0, l_1, l_2, \ldots$. Prove that $L(x) = \dfrac{1-x}{1-3x+x^2}$.

(b) Derive an explicit formula for l_n from the generating function $L(x)$.

19. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = n$ with integer-valued variables satisfying $x_1 \geqslant -1$, $x_2 \geqslant -2$, $x_3 \geqslant 3$, and $x_4 \geqslant 4$.

20. How many bit strings of length n are there in which 1's always occur in contiguous pairs? You should consider strings of the form 0011011110, but not of the form 0110111110, because the last 1 is not paired.

21. Use generating functions to prove that every positive integer has a unique binary representation (without leading zero bits).

22. Let $A(x)$ be the generating function of the sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers. Prove that $1/A(x)$ is the generating function of a sequence if and only if $a_0 \neq 0$.

23. **(a)** Find the probability generating function of the binomial distribution $\Pr[B_{n,p} = r] = \binom{n}{r} p^r (1-p)^{n-r}$ for $r = 0, 1, 2, \ldots, n$. Hence deduce the expectation $\mathrm{E}[B_{n,p}]$.

(b) Find the probability generating function of the uniform distribution $\Pr[U_{a,b} = r] = \dfrac{1}{b-a+1}$ for $r = a, a+1, a+2, \ldots, b$ (where $a, b \in \mathbb{Z}$ with $a \leqslant b$). Hence deduce the expectation $\mathrm{E}[U_{a,b}]$.

24. Prove the following identities involving the Stirling numbers $S(n,k)$ of the second kind. Take $S(0,0) = 1$, and $S(n,k) = 0$ for $n < k$.

(a) $\displaystyle\sum_{n \in \mathbb{N}_0} S(n,k) x^n = \prod_{r=1}^{k} \frac{x}{1-rx}$.

(b) $\displaystyle\sum_{n \in \mathbb{N}_0} S(n,k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$.

25. Generating functions with multiple variables are sometimes used. Suppose that r elements are chosen from $\{1, 2, 3, \ldots, n\}$ such that the sum of the chosen elements is s. We want to count how many such collections are possible. Argue that this count is the coefficient of $x^r y^s$ of $(1+xy)(1+xy^2)(1+xy^3)\cdots(1+xy^n)$. Find the two-variable generating function of these counts if the r elements are chosen from $\{1, 2, 3, \ldots, n\}$ with repetitions allowed.

26. Let a_n, $n \geqslant 0$, be the sequence satisfying

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_n &= a_{n-1} + \sum_{k=1}^{n-2} a_k a_{n-1-k} \quad \text{for } n \geqslant 2. \end{aligned}$$

Prove that the generating function for this sequence is $\dfrac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}$. Solve for a_n.

27. Solve the following recurrence relation using generating functions: $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3} + 1$ for $n \geqslant 3$.